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ON EXTREMAL PROPERTIES FOR THE POLAR DERIVATIVE OF POLYNOMIALS

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Abstract. If $p(z)$ is a polynomial of degree n having all its zeros on $|z| = k, k \leq 1$, then it is proved[5] that

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{k^{n-1} + k^n} \max_{|z|=1} |p(z)|.$$

In this paper, we generalize the above inequality by extending it to the polar derivative of a polynomial of the type $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \leq \mu \leq n$. We also obtain certain new inequalities concerning the maximum modulus of a polynomial with restricted zeros.

Key words: *polynomial, zeros, inequality, polar derivative*

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1 Introduction

If $p(z)$ is a polynomial of degree n and $p'(z)$ its derivative, then according to a famous result known as Bernstein's inequality (for reference see [2]), we have

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1.1}$$

The result is sharp and the equality in (1.1) holds for $p(z) = \lambda z^n$, where $|\lambda| = 1$.

For the class of polynomials not vanishing in $|z| < k$, $k \geq 1$, Malik[8] proved

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.2}$$

The result is sharp and the extremal polynomial is $p(z) = (z+k)^n$.

While seeking for an inequality analogous to (1.2) for polynomials not vanishing in $|z| < k$, $k \leq 1$, Govil[5] proved the following

Theorem A. *If $p(z)=\sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros on $|z|=k$, $k\le 1$, then*

$$\max_{|z|=1}|p'(z)|\le \frac{n}{k^{n-1}+k^n}\max_{|z|=1}|p(z)|. \tag{1.3}$$

Let α be a complex number. If $p(z)$ is a polynomial of degree n, then the polar derivative of $p(z)$ with respect to the point α, denoted by $D_\alpha p(z)$, is defined by

$$D_\alpha p(z)=np(z)+(\alpha-z)p'(z). \tag{1.4}$$

Clearly $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\left[\frac{D_\alpha p(z)}{\alpha}\right]=p'(z). \tag{1.5}$$

In this paper, we first prove the following result which is an extension of Theorem A due to Govil[5] to the polar derivative of a polynomial of the type $p(z)=c_n z^n+\sum_{j=\mu}^{n} c_{n-j}z^{n-j}$, $1\le\mu\le n$.

Theorem 1. *If $p(z)=c_n z^n+\sum_{j=\mu}^{n} c_{n-j}z^{n-j}$, $1\le\mu<n$, is a polynomial of degree n having all its zeros on $|z|=k$, $k\le 1$, then for every real or complex number α with $|\alpha|\ge k$, we have*

$$\max_{|z|=1}|D_\alpha p(z)|\le \frac{n(|\alpha|+k^\mu)}{k^{n-2\mu+1}+k^{n-\mu+1}}\max_{|z|=1}|p(z)|. \tag{1.6}$$

Instead of proving Theorem 1 we prove the following theorem which gives a better bound over the above theorem. More precisely, we prove.

Theorem 2. *If $p(z)=c_n z^n+\sum_{j=\mu}^{n} c_{n-j}z^{n-j}$, $1\le\mu<n$, is a polynomial of degree n having all its zeros on $|z|=k$, $k\le 1$, then for every real or complex number α with $|\alpha|\ge k$, we have*

$$\max_{|z|=1}|D_\alpha p(z)|\le \frac{n(|\alpha|+S_\mu)}{k^{n-2\mu+1}+k^{n-\mu+1}}\max_{|z|=1}|p(z)|, \tag{1.7}$$

where

$$S_\mu=\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{n|c_n|k^{\mu-1}+\mu|c_{n-\mu}|}. \tag{1.8}$$

To prove that the bound obtained in the above theorem is better than the bound obtained in Theorem 1, we show that

$$S_\mu\le k^\mu \quad\text{or}\quad \frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{\mu|c_{n-\mu}|+n|c_n|k^{\mu-1}}\le k^\mu$$

which is equivalent to

$$n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}\le \mu|c_{n-\mu}|k^{\mu}+n|c_n|k^{2\mu-1}$$

which implies

$$n|c_n|(k^{2\mu}-k^{2\mu-1})\le \mu|c_{n-\mu}|(k^{\mu}-k^{\mu-1})$$

or

$$\frac{n}{\mu}\left|\frac{c_n}{c_{n-\mu}}\right|\ge\frac{1}{k^{\mu}}$$

which is always true (see Lemma 5).

Remark 1. If we take $\mu=1$ and on dividing both sides of the inequalities (1.6) and (1.7) by $|\alpha|$ and letting $|\alpha|\to\infty$, we obtain Theorem A due to Govil[5].

Dividing both sides of the inequality (1.7) by $|\alpha|$ and letting $|\alpha|\to\infty$, we get the following result due to Dewan and Hans[4].

Corollary 1. *If*

$$p(z)=c_nz^n+\sum_{j=\mu}^{n}c_{n-j}z^{n-j},\qquad 1\le\mu<n,$$

is a polynomial of degree n having all its zeros on $|z|=k$, $k\le 1$, *then*

$$\max_{|z|=1}|p'(z)|\le\frac{n}{k^{n-2\mu+1}+k^{n-\mu+1}}\max_{|z|=1}|p(z)|. \tag{1.9}$$

The following corollary immediately follows from Theorem 2 by taking $\mu=1$.

Corollary 2. *If* $p(z)=\sum_{j=0}^{n}c_jz^j$ *is a polynomial of degree n having all its zeros on* $|z|=k$, $k\le1$, *then for every real or complex number* α *with* $|\alpha|\ge k$, *we have*

$$\max_{|z|=1}|D_\alpha p(z)|\le\frac{n(|\alpha|+S_1)}{k^{n-1}+k^n}\max_{|z|=1}|p(z)|, \tag{1.10}$$

where

$$S_1=\left(\frac{n|c_n|k^2+|c_{n-1}|}{n|c_n|+|c_{n-1}|}\right). \tag{1.11}$$

We next prove the following interesting results for the maximum modulus of polynomials.

Theorem 3. *If* $p(z)=\sum_{j=0}^{n}c_jz^j$ *is a polynomial of degree n having all its zeros on* $|z|=k$, $k\le1$, *then for every real or complex number* α *with* $|\alpha|\ge k$ *and* $0\le r\le k\le R$, *we have*

$$\max_{|z|=R}|D_\alpha p(z)|\le\frac{n(|\alpha|+RS_1')(R^{2n-1}+kR^{2n-2})}{k^{n-1}Rr^n+k^nRr^{n-1}+k^nr^n+k^{n+1}r^{n-1}}\max_{|z|=r}|p(z)|, \tag{1.12}$$

where

$$S_1' = \frac{1}{R}\frac{n|c_n|k^2 + R|c_{n-1}|}{n|c_n|R + |c_{n-1}|}. \tag{1.13}$$

If we divide both sides of (1.12) by $|\alpha|$ and letting $|\alpha| \to \infty$, we obtain the following result.

Corollary 3. *If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k, k \le 1$, then for $0 \le r \le k \le R$, we have*

$$\max_{|z|=R}|p'(z)| \le \frac{n(R^{2n-1} + kR^{2n-2})}{k^{n-1}Rr^n + k^n Rr^{n-1} + k^n r^n + k^{n+1}r^{n-1}} \max_{|z|=r}|p(z)|. \tag{1.14}$$

By involving the coefficients c_0 and c_1 of $p(z) = \sum_{j=0}^{n} c_j z^j$, we prove the following generalization of Theorem 3.

Theorem 4. *If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$, then for every real or complex number α with $|\alpha| \ge k$ and $0 \le r \le k \le R$, we have*

$$\begin{aligned}&\max_{|z|=R}|D_\alpha p(z)| \\ &\le \frac{n(|\alpha| + RS_1')\{2k^2R^{2n-1}|c_1| + R^{2n-2}(R^2+k^2)n|c_0|\}}{\begin{aligned}&2(k^{n+1}Rr^n + k^{n+2}r^n)|c_1| + (k^n r^{n+1} \\ &+k^{n+2}r^{n-1} + k^{n-1}Rr^{n+1} + k^{n+1}Rr^{n-1})n|c_0|\end{aligned}} \max_{|z|=r}|p(z)|,\end{aligned} \tag{1.15}$$

where S_1' is the same as defined in Theorem 3.

The following corollary immediately follows by dividing both sides of the inequality (1.15) by $|\alpha|$ and letting $|\alpha| \to \infty$.

Corollary 4. *If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$, then for $0 \le r \le k \le R$, we have*

$$\begin{aligned}&\max_{|z|=R}|p'(z)| \\ &\le \frac{n\{2k^2R^{2n-1}|c_1| + R^{2n-2}(R^2+k^2)n|c_0|\}}{\begin{aligned}&2(k^{n+1}Rr^n + k^{n+2}r^n)|c_1| + (k^n r^{n+1} \\ &+k^{n+2}r^{n-1} + k^{n-1}Rr^{n+1}Rr^{n+1} + k^{n+1}Rr^{n-1})n|c_0|.\end{aligned}} \max_{|z|=r}|p(z)|.\end{aligned} \tag{1.16}$$

2 Lemmas

We need the following lemmas for the proof of these theorems.

Lemma 1. *If $p(z)$ is a polynomial of degree n, then for $|z|=1$*

$$|p'(z)|+|q'(z)|\leq n\max_{|z|=1}|p(z)|, \tag{2.1}$$

where here and throughout this paper $q(z)=z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$.

This is a special case of a result due to Govil and Rahman[6].

Lemma 2. *Let*

$$p(z)=c_nz^n+\sum_{\upsilon=\mu}^{n}c_{n-\upsilon}z^{n-\upsilon}, \qquad 1\leq\mu<n,$$

be a polynomial of degree n having no zero in the disk $|z|<k$, $k\leq 1$. Then for $|z|=1$

$$k^{n-\mu+1}\max_{|z|=1}|p'(z)|\leq\max_{|z|=1}|q'(z)|. \tag{2.2}$$

The above lemma is due to Dewan and Hans[4].

Lemma 3. *Let $p(z)=c_0+\sum_{\upsilon=\mu}^{n}c_\upsilon z^\upsilon$, $1\leq\mu\leq n$ be a polynomial of degree n having no zero in the* disk $|z|<k$, $k\geq 1$. *Then for $|z|=1$*

$$k^{\mu}|p'(z)|\leq|q'(z)|. \tag{2.3}$$

The above lemma is due to Chan and Malik[3].

Lemma 4. *Let*

$$p(z)=c_nz^n+\sum_{v=\mu}^{n}c_{n-v}z^{n-v}, \qquad 1\leq\mu\leq n,$$

be a polynomial of degree n having all its zeros on $|z|=k$, $k\leq 1$. Then for $|z|=1$

$$k^{\mu}|p'(z)|\geq|q'(z)|. \tag{2.4}$$

Proof of Lemma 4. If $p(z)$ has all its zeros on $|z|=k$, $k\leq 1$, then $q(z)$ has all its zeros on $|z|=\frac{1}{k}$, $\frac{1}{k}\geq 1$. Now applying Lemma 3 to the polynomial $q(z)$, we get the desired result.

Lemma 5. *If*

$$p(z)=c_nz^n+\sum_{v=\mu}^{n}c_{n-v}z^{n-v}, \qquad 1\leq\mu\leq n,$$

be a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$. Then for $|z|=1$

$$|q'(z)|\leq S_\mu|p'(z)|, \tag{2.5}$$

$$\frac{\mu}{n}\left|\frac{c_{n-\mu}}{c_n}\right|\leq k^{\mu} \tag{2.6}$$

and S_μ is the same as defined in Theorem 2.

The above lemma is due to Aziz and Rather[1].

Lemma 6. *If $p(z)=\sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon$ be a polynomial of degree n having all its zeros in the disk $|z|\geq k$, $k>0$, then for $r\leq k$ and $R\geq k$*

$$\frac{M(p,r)}{r^n+kr^{n-1}}\geq\frac{M(p,R)}{R^n+kR^{n-1}}. \tag{2.7}$$

The above lemma is due to Jain[7].

Lemma 7. *If*

$$p(z)=\sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon$$

be a polynomial of degree n having all its zeros in the disk $|z|\geq k$, $k>0$, then for $r\leq k$ and $R\geq k$

$$\frac{M(p,r)}{2k^2r^n|c_1|+r^{n-1}(r^2+k^2)n|c_0|}\geq\frac{M(p,R)}{2k^2R^n|c_1|+R^{n-1}(R^2+k^2)n|c_0|}. \tag{2.8}$$

The above lemma is due to Mir[9].

3 Proof of the Theorems

Proof of Theorem 1. The proof of Theorem 1 follows from the same lines as that of Theorem 2, but instead of using Lemma 5, we use Lemma 4. We omit the details.

Proof of Theorem 2. Let

$$q(z)=z^n\overline{p\left(\frac{1}{\bar{z}}\right)}.$$

Then it can be easily verified that

$$|q'(z)|=|np(z)-zp'(z)| \quad \text{for } |z|=1\,.$$

Now for every real or complex number α, we have

$$D_\alpha p(z)=np(z)+(\alpha-z)p'(z)\,.$$

This implies with the help of Lemma 5 that

$$\begin{aligned} |D_\alpha p(z)| &\leq |\alpha p'(z)|+|np(z)-zp'(z)| \\ &= |\alpha||p'(z)|+|q'(z)| \\ &\leq (|\alpha|+S_\mu)|p'(z)|\,. \end{aligned} \tag{3.1}$$

Let z_0 be a point on $|z|=1$, such that $|q'(z_0)| = \max_{|z|=1}|q'(z)|$, then by Lemma 1, we get

$$|p'(z_0)| + \max_{|z|=1}|q'(z)| \le n\max_{|z|=1}|p(z)|. \tag{3.2}$$

Combining the inequality (3.2) with Lemma 4, we have

$$\left(\frac{1}{k^{\mu}}\right)|q'(z_0)| + \max_{|z|=1}|q'(z)| \le n\max_{|z|=1}|p(z)|,$$

which is equivalent to

$$\left(\frac{1}{k^{\mu}}+1\right)\max_{|z|=1}|q'(z)| \le n\max_{|z|=1}|p(z)|. \tag{3.3}$$

The above inequality when combined with Lemma 2, gives

$$\left(\frac{1}{k^{\mu}}+1\right)k^{n-\mu+1}\max_{|z|=1}|p'(z)| \le n\max_{|z|=1}|p(z)|,$$

which implies

$$\max_{|z|=1}|p'(z)| \le \frac{n}{k^{n-2\mu+1}+k^{n-\mu+1}}\max_{|z|=1}|p(z)|. \tag{3.4}$$

On combining the inequalities (3.1) and (3.4), we get the desired result.

Proof of Theorem 3. Let $0 \le r \le k \le R$. Since $p(z)$ has all its zero on $|z|=k, k\le 1$, then the polynomial $p(Rz)$ has all its zeros on $|z| = \frac{k}{R}, \frac{k}{R} \le 1$, therefore applying Corollary 2 to the polynomial $p(Rz)$ with $|\alpha| \ge k$, we get

$$\max_{|z|=1}|D_{\frac{\alpha}{R}}p(Rz)| \le \frac{n(\frac{|\alpha|}{R}+S_1')}{\frac{k^n}{R^n}+\frac{k^{n-1}}{R^{n-1}}}\max_{|z|=1}|p(Rz)|$$

or

$$\max_{|z|=1}\left|np(Rz)+\left(\frac{\alpha}{R}-z\right)Rp'(Rz)\right| \le \frac{n(\frac{|\alpha|}{R}+S_1')}{\frac{k^n}{R^n}+\frac{k^{n-1}}{R^{n-1}}}\max_{|z|=R}|p(z)|,$$

which is equivalent to

$$\max_{|z|=R}|D_{\alpha}p(z)| \le \frac{nR^{n-1}(|\alpha|+RS_1')}{k^{n-1}R+k^n}\max_{|z|=R}|p(z)|.$$

For $0 \le r \le k \le R$, the above inequality in conjunction with Lemma 6 yields

$$\max_{|z|=R}|D_{\alpha}p(z)| \le \frac{nR^{n-1}(|\alpha|+RS_1')}{k^{n-1}R+k^n} \times \frac{R^n+kR^{n-1}}{r^n+kr^{n-1}}\max_{|z|=r}|p(z)|,$$

from which Theorem 3 follows.

Proof of Theorem 4. The proof follows along the same lines as that of Theorem 3 but instead of using Lemma 6 we use Lemma 7.

Remark 2. For $\mu = n$, Theorems 1 and 2 hold, if the polynomial satisfies the condition $|c_0| \le k|c_n|$.

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