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APPROXIMATION AND SHAPE-PRESERVING PROPERTIES OF Q-STANCU OPERATOR

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Abstract. We introduce the definition of q-Stancu operator and investigate its approximation and shape-preserving property. With the help of the sign changes of $f(x)$ and $L_n = f(f,q;x)$ the shape-preserving property of q-Stancu operator is obtained.

Key words: *q-Stancu operator, shape-preserving property, sign change*

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1 Introduction

Suppose $q > 0$. For $k = 0, 1, 2, \cdots$, the q-integer $[k]$ and q-factorial $[k]!$ are defined as

$$[k] = \begin{cases} \frac{1-q^k}{1-q}, q \neq 1, \\ k, \quad q = 1; \end{cases}$$

$$[k]! = \begin{cases} [k][k-1]\dots[1], k \geq 1, \\ 1, \qquad k = 0. \end{cases}$$

For integers $n, k, n \geq k \geq 0$, q-binomial coeffcients are defined naturally as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$$

We present the definition of q-Stancu operator as follows.

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Definition 1. *Suppose s is an integer and $0 \leq s < {-}fracn2, q > 0, n > 0$. For $f \in C[0,1]$, the operator*

$$L_n(f,q;x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) b_{n,k,s}(q;x), \tag{1.1}$$

is called q-Stancu operator, where

$$b_{n,k,s}(q;x) = \begin{cases} (1-q^{n-k-s}x)p_{n-s,k}(q;x), & 0 \leq k < s, \\ (1-q^{n-k-s}x)p_{n-s,k}(q;x) + q^{n-k}xp_{n-s,k-s}(q;x), & s < k \leq n-s, \\ q^{n-k}xp_{n-s,k-s}(q;x), & n-s < k \leq n, \end{cases}$$

$p_{n-s,k}(q;x), p_{n-s,k-s}(q;x)$ are the basis functions of q-Bernstein operator,

$$p_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{l=0}^{n-k-1} (1-q^l x)$$

It is not difficult to notice that on one hand for $s = 0$ or $s = 1$, q-Stancu operator is just the q-Bernstein operator which was introduced first by G.M. Phillips in 1997, on the other hand for $q = 1$, q-Stancu operator recoveres the Stancu operator. The q-Berstein operator possesses many remarkable properties which have made it an intensive area, seen [1-8]. While the study of Stancu operator is also a focus of many authors since 1981, after D.D. Stancu has defined this operator, see [9-12]. Both q-Bernstein operator and Stancu operator are some generalizations of the classical Bernstein operator which are specific cases of q-Bernstein operator when $q = 1$ or Stancu operator when $s = 0, s = 1$.

It is worth mentioning that the q-Stancu operator we defined here difier essentially from that in [13]. The q-Stancu operator in [13] just generalizes the control points of the Stancu operator based on the q-integers leaving alone the basis functions.While in our definition of q-Stancu operator both the control points and the basis functions are the q-analogue of those in Stancu operators. As a result, it is not a strange thing that these two q-Stancu operators behave quite difierent property, especially in the approximation problem.

By means of direct computations, we can get the following representation of q-Stancu operator:

$$L_n(f,q;x) = \sum_{k=0}^{n-s} \left\{ (1-q^{n-k-s}x) f\left(\frac{[k]}{[n]}\right) + q^{n-k-s}xf\left(\frac{[k+s]}{[n]}\right) \right\} p_{n-s,k}(q;x). \tag{1.2}$$

To process our study we need to give some essential properties of q-Stancu operator.

Proposition 1. *q-Stancu operator is a positive linear operator for $0 < q < 1$, while not for $q > 1$.*

Proposition 2. $L_n(1,q;x)=1, L_n(t,q;x)=x,$

$$L_n(t^2,q;x)=x^2+\left(\frac{[1]}{[n]}+\frac{q^{n-s}[s]^2-q^{n-s}[s]}{[n]^2}\right)x(1-x).$$

Proposition 3. *For* $f(x)\in C[0,1]$, $L_n(f,q;0)=f(0)$, $L_n(f,q;1)=f(1)$.

In the following the shape-preserving properties as well as the approximation properties of q-Stancu operators are considered when $0<q<1$.

By some elaborate computation, we get another vital representation of q-Stancu operator. The corresponding representation of Stancu operator has been ignored all the time, but one can see the efiect of this representation clearly.

Lemma 1. *Suppose* $0<q\le 1$ *and* s *is an integer such that* $0<s<\frac{n}{2}$. *For* $f\in C[0,1]$, *we have*

$$L_n(f,q;x)=\sum_{k=0}^{n-s+1}\left\{\frac{[n-s+1-k]}{[n-s+1]}f\left(\frac{[k]}{[n]}\right)+\frac{q^{n-s+1-k}[k]}{[n-s+1]}f\left(\frac{[s-1+k]}{[n]}\right)\right\}p_{n-s+1,k}(q;x). \tag{1.3}$$

Note: the representation is disabled when $s=0$.

2 Approximation Theorem

For $0<q<1$, similar to the q-Bernstein operator $B_n(\cdot,q)$, the q-Stancu operator $L_n(\cdot,q)$ for continuous functions is convergent uniformly to the function itself and to certain limit, under some necessary condition for $s\in\mathbf{N}$. The limit function is defined as:

Definition 2. *For any nonnegative integer n, and* $f(x)\in C[0,1]$,

$$B_\infty(f,q;x)=\begin{cases}\sum\limits_{k=0}^{\infty}f(1-q^k)p_{\infty,k}(q;x), & 0\le x<1,\\ f(1), & x=1,\end{cases} \tag{2.1}$$

here $p_{\infty,k}(q;x)=\dfrac{x^k}{(1-q)^k[k]!}\prod\limits_{s=0}^{\infty}(1-q^s x)$.

In detail, we have the following theorem.

Theorem 1. *Let* $f(x)\in C[0,1]$ *and* s *is an integer such that* $0\le s<\frac{n}{2}$, *then we have*

$$\|L_n(f,q,x)-B_\infty(f,q;x)\|_C\le\left(4-\frac{4\ln(1-q)}{q(1-q)}\right)\omega(f,q^{n-s+1}). \tag{2.2}$$

It can be seen from this theorem that for fixed integer s or $s=s(n), n-s(n)\to\infty$, then $\lim\limits_{n\to\infty}\|L_n(f,q;x)-B_\infty(f,q;x)\|_C=0$ holds for all $0<q<1$. This result[9] has some slightly difierence from corresponding result of Stancu operator

For Stancu operator, the index $s=s(n)$ should satisfy $s=o(n)$ as $n\to\infty$ in order to index quarantce the convergence of the relevant Stancu polynomials. While for q-Stancu operator it only needs $n-s(n)\to\infty$. Hereby for $s=s(n)=\frac{n-1}{2},\frac{n}{3},\frac{n}{4},\cdots$, we still have $\lim\limits_{n\to\infty}\|L_n(f,q;x)-B_\infty(f,q;x)\|_C=0$, but for Stancu operator it is not right.

Proof. Based on Proposition 2 and the linear property of the limit $B_\infty(\cdot,q)$ (see[3]), we can assume $f(0)=f(1)=0$ without loss of generality.

Then we have for all $x\in[0,1]$,

$$\begin{aligned}
&\left|L_n(f,q,x)-B_\infty(f,q;x)\right|\\
&=\left|\sum_{k=0}^{n-s+1}\left\{\tfrac{[n-s+1-k]}{[n-s+1]}f\left(\tfrac{[k]}{[n]}\right)+\tfrac{q^{n-s+1-k}[k]}{[n-s+1]}f\left(\tfrac{[s-1+k]}{[n]}\right)\right\}p_{n-s+1,k}(q;x)-\sum_{k=0}^{\infty}f(1-q^k)p_{\infty,k}(q;x)\right|\\
&\le\left|\sum_{k=0}^{n-s+1}\{\tfrac{[n-s+1-k]}{[n-s+1]}(f\left(\tfrac{[k]}{[n]}\right)-f(1-q^k))+\tfrac{q^{n-s+1-k}[k]}{[n-s+1]}(f\left(\tfrac{[s-1+k]}{[n]}\right)\right.\\
&\quad\left.-f(1-q^k))\}\,p_{n-s+1}(q;x)\right|+\left|\sum_{k=0}^{n-s+1}(f(1-q^k)-f(1))(p_{n-s+1,k}(q;x)-p_{\infty,k}(q;x))\right|\\
&\quad+\left|\sum_{k=n-s+2}^{\infty}(f(1-q^k)-f(1))p_{\infty,k}(q;x)\right|:=I_1+I_2+I_3
\end{aligned}$$

From the proof of Theorem 1 in [4], we know

$$I_2\le\frac{-4\ln(1-q)}{q(1-q)}\omega(f,q^{n-s+1}),\quad I_3\le\omega(f,q^{n-s+1}).$$

Since for $0<\delta\le\eta\le1$, $\dfrac{\omega(f,\eta)}{\eta}\le2\,\dfrac{\omega(f,\delta)}{\delta}$ (see[14]),we have

$$\begin{aligned}
I_1&\le\sum_{k=0}^{n-s+1}\left\{\tfrac{[n-s+1-k]}{[n-s+1]}\omega(f,\tfrac{[k]}{[n]}q^n)+\tfrac{q^{n-s+k}[k]}{[n-s+1]}\omega(f,\tfrac{[s-1]}{[n]}q^k+\tfrac{[k]}{[n]}q^n)\right\}p_{n-s+1,k}(q;x)\\
&\le\sum_{k=0}^{n-s+1}\omega(f,\tfrac{[k]}{[n]}q^n)p_{n-s+1,k}(q;x)+\sum_{k=0}^{n-s+1}\tfrac{q^{n-s+1}[k]}{[n-s+1]}\tfrac{[s-1]}{[n]}\frac{\omega(f,\frac{[s-1]}{[n]}q^k)}{\frac{[s-1]}{[n]}q^k}p_{n-s+1,k}(q;x)\\
&\le\omega(f,q^n)+\sum_{k=0}^{n-s+1}\tfrac{q^{n-s+1}[k]}{[n-s+1]}\tfrac{[s-1]}{[n]}\frac{2\omega(f,\frac{[s-1]}{[n]}q^{n-s+1})}{\frac{[s-1]}{[n]}q^{n-s+1}}p_{n-s+1,k}(q;x)\\
&\le\omega(f,q^n)+2\omega(f,\tfrac{[s-1]}{[n]}q^{n-s+1})\sum_{k=0}^{n-s+1}\tfrac{[k]}{[n-s+1]}p_{n-s+1,k}(q;x)\\
&\le\omega(f,q^n)+2x\omega(f,\tfrac{[s-1]}{[n]}q^{n-s+1}).
\end{aligned}$$

Combining the estimates for I_1,I_2,I_3 we complete the proof of Theorem 1.

More properties of this q-Stancu operator will be investigated in the next section.

3 Shape-preserving Property

The shape-preserving property is important to the study of both q-Bernstein and Stancu operator. Of course we consider this problem for the q-Stancu operator only.

The above Lemma 1 suggests that for any convex function on [0,1] the inequality

$$L_n(f,q;x) \geq B_{n-s+1}(f,q;x). \tag{3.1}$$

holds.

As we see the complexity of the derivative of q-Bernstein operator in the study of shape-preserving property the following theorem plays an important role.

Let v be any finite-dimensional vector. We use $S^-(v)$ for its strict sign change, namely, the times of the sign change from the first component to the last one. Thus for the vector $f = (f(x_0), \cdots, f(x_m))$,

$$S^-(f) = \sup_{0 \leq x_0 < \cdots < x_m \leq 1; m \in N} S^-(f(x_0), \cdots, f(x_m))$$

means the sign change of f on $\{x_0, \cdots, x_m\} \subset [0,1]$.

Theorem A[2]**.** *Suppose* $0 < q \leq 1$. *For* $f \in C[0,1]$, *we have*

$$S^-(B_n(f,q)) \leq S^-(f).$$

However, the following figure shows clearly that Theorem A can no longer hold for Stancu and q-Stancu operator.

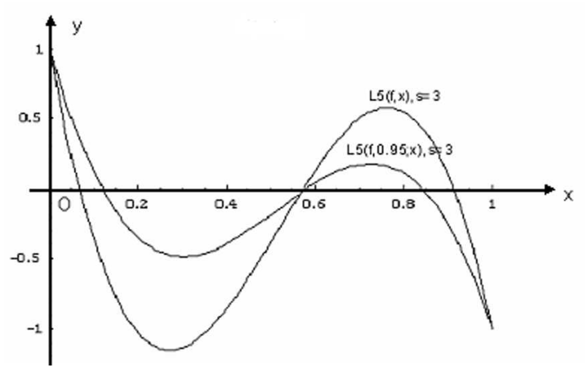

Fig. 1

Remark 1. In Figure-1, one curve is $L_5(f,x)$ for $s = 3$, while the other is $L_5(f,0.95;x)$ for $s = 3$, here the continuous function $f(x)$ is a linear spline joining up the points $(0,1),(0.2,1)$, $(0.4,14),(0.6,-17),(0.8,-1),(1,-1)$.

Evidently, $S^-(f) = 1 \leq 3 = S^-(L_5(f)) = S^-(L_5(f,0.95))$. However, we still get the shape-preserving theorem for q-Stancu operator:

Theorem 2. *Let $0<q\le 1$,s be an integer satisfying $0\le s<\frac{n}{2}$ and $f(x)$ be a continuous and increasing (decreasing) function on $[0,1]$, then $L_n(f,q;x)$ is increasing (decreasing) on $[0,1]$.*

Proof. We consider the increasing function f at first. For $s=0$, one can know from [2] that the result of Theorem 1 holds. In the following we consider the case of $s>0$. For $0<q\le 1$,

$$(p_{n-s+1,0}(q;x),p_{n-s+1,1}(q;x),\cdots,p_{n-s+1,n-s+1}(q;x))$$

is totally positive (see[2]). This means for any sequence satisfying $0\le x_0,<x_1<\cdots<x_m\le 1$,

The corresponding matrix $T=\left\{p_{n-s+1,j}(x_i)|i=0,1,\cdots,m;j=0,1,\cdots,n-s+1\right\}$is totally positive.

Then by virtue of Theorem 3.3 in [2] we conclude that

$$S^-(L_n(f,q;x))\le S^-(f(0),a_{n,1},\cdots,a_{n,n-s},f(1)), \tag{3.2}$$

where

$$a_{n,k}=\frac{[n-s+1-k]}{[n-s+1]}f\left(\frac{[k]}{[n]}\right)+\frac{q^{n-s-k+1}[k]}{[n-s+1]}f\left(\frac{[s-1+k]}{[n]}\right),k=1,2,\cdots,n-s.$$

By the continuity of $f(x)$, we see for $k=1,\cdots,n-s$ there exist $\xi_{n,k}\in\left(\frac{[k]}{[n]},\frac{[n-1+k]}{[n]}\right)$, such that $a_{n,k}=f(\xi_{n,k})$.

This together with the monotony of $f(x)$, implies

$$\begin{aligned}a_{n,k} &= \frac{[n-s-k]}{[n-s+1]}f\left(\frac{[k]}{[n]}\right)+\frac{q^{n-s-k}[1]}{[n-s+1]}f\left(\frac{[k]}{[n]}\right)+\frac{q^{n-s-k}[k]}{1-q^{n-s+1}}f\left(\frac{[s-1+k]}{[n]}\right)\\ &\le \frac{[n-s-k]}{[n-s+1]}f\left(\frac{[k+1]}{[n]}\right)+\frac{q^{n-s-k}[1]}{[n-s+1]}f\left(\frac{[s+k]}{[n]}\right)+\frac{q^{n-s-k+1}[k]}{[n-s+1]}f\left(\frac{[s+k]}{[n]}\right)\\ &= a_{n,k+1}\end{aligned}$$

Therefore $\xi_{n,k}\le\xi_{n,k+1}$, for $k=1,\cdots,n-s-1$.

Consequently we have

$$\begin{aligned}&S^-(L_n(f,q;x))\le S^-(f(0),a_{n,1},\cdots,a_{n,n-s},f(1))\\ &=S^-(f(0),f(\xi_{n,1}),\cdots,f(\xi_{n,n-s}),f(1))\le S^-(f)\end{aligned} \tag{3.3}$$

Since $f(x)$ is increasing on $[0,1]$,for any constant c,we have $S^-(f-c)\le 1$ .Otherwise, there exist a constant c_0 and $0<\eta_1<\eta_2<\eta_3<1$,such that

$$f(\eta_1)<c_0,f(\eta_2)>c_0,f(\eta_3)<c_0,$$

which are paradoxical with the increasing property of $f(x)$.

Therefore, for any constant c, the following holds

$$S^-(L_n(f,q)-c) = S^-(L_n(f-c,q)) \leq S^-(f-c) \leq 1. \tag{3.4}$$

Suppose $L_n(f,q)$ is not increasing on $[0,1]$, then with the help of Proposition 3, we get $L_n(f,q;0) = f(0) \leq f(1) = L_n(f,q;1)$. So we can assume without loss of generality that there exist $\zeta_1, \zeta_2, \zeta_3$ satisfying $0 \leq \zeta_1 < \zeta_2 < \zeta_3 \leq 1$,such that

$$L_n(f,q;\zeta_1) < L_n(f,q;\zeta_2) \quad \text{and} \quad L_n(f,q;\zeta_2) > L_n(f,q;\zeta_3).$$

Thus for any constant c such that $\max\{L_n(f,q;\zeta_3), L_n(f,q;\zeta_1)\} < c < L_n(f,q;\zeta_2)$, the relation

$$S^-(L_n(f,q)-c) = S^-(L_n(f-c,q)) \geq 2 \tag{3.5}$$

holds, which is in contradiction with (3.4). Therefore $L_n(f,q;x)$ is increasing on $[0,1]$.

For the decreasing case we can prove the theorem in the same way. Theorem 1 is proved.

For the convex-preserving property, we now can only prove the result in the case $0 \leq s \leq 2$. However, we believe the following theorem seems also to be true based on the Figure-2.

Theorem 3. *Let $0 < q < 1, 0 \leq s \leq 2, f(x)$ is a continuous and convex (concave) function on $[0,1]$, then $L_n(f,q)$ is also convex(concave) and $L_n(f,q;x) \leq f(x)(L_n(f,q;x) \geq f(x))$.*

Proof. For $s = 0,1$ Theorem 3 holds, which is similar to the case of q-Bernstein operator. So we only focus on the case $s = 2$. Since f is convex , for any linear function $l(x)$, $S^-(f-l) \leq 2$. Otherwise, there exist a linear function $l_0(x)$ and $0 < \eta_1 < \eta_2 < \eta_3 < \eta_4 < 1$ such that

$$S^-(f(\eta_1)-l_0(\eta_1), f(\eta_2)-l_0(\eta_2), f(\eta_3)-l_0(\eta_3), f(\eta_4)-l_0(\eta_4)) = 3$$

From the convex property of $f(x)$, we know $f(x)-l_0(x)$ is still a convex function , so $f(\eta_1)-l_0(\eta_1) > 0$.

Therefore,

$$k_{f-l_0}(\eta_1,\eta_2) < 0, k_{f-l_0}(\eta_2,\eta_3) > 0, k_{f-l_0}(\eta_3,\eta_4) < 0 \tag{3.6}$$

here we use $k_{f-l_0}(x_0,x_1)$ to denote the slope of the line between $(x_0, f(x_0)-l_0(x_0))$ and $(x_1, f(x_1)-l_0(x_1))$.

The above statement is inconsistent with the convex property of $f(x)-l_0(x)$.

On the other hand, since s, $\xi_{n,k}, k = 1,\cdots,n-s$ satisfy

$$0 < \xi_{n,1} < \xi_{n,2} < \cdots < \xi_{n,n-s} < 1,$$

We see for any continuous function $f(x)$, $S^-(L_n(f,q;x)) \leq S^-(f)$.

This together with Proposition 2 implies for any linear function $l(x)$ the relation

$$S^{-}(L_n(f,q)-l) = S^{-}(L_n(f-l,q)) \leq S^{-}(f-l) \leq 2 \tag{3.7}$$

holds.

Suppose $L_n(f,q;x)$ is not convex on $[0,1]$, then from $f(x)$ is convex on $[0,1]$ we conclude that for any $x \in [0,1]$,

$$f(x)-((1-x)f(0)+xf(1)) \leq 0.$$

This combining with Proposition 1-3 implies for all $x \in [0,1]$,

$$L_n(f(t)-((1-t)f(0)+tf(1)),q;x)$$

$$= L_n(f,q;x)-((1-x)f(0)+xf(1))$$

$$= L_n(f,q;x)-((1-x)L_n(f,q;0)+xL_n(f,q;1)) \leq 0. \tag{3.8}$$

The above result shows $f(x)$ is not concave on $[0,1]$. Consequently, there exist $0 < \zeta_1 < \zeta_2 < 1$ such that there exist $\theta_2 < \theta_3$ on $[\zeta_1, \zeta_2]$ fulfilling

$$L_{L_n(f,q)}(\zeta_1,\zeta_2)(\theta_2) > L_n(f,q;\theta_2), \tag{3.9}$$

$$L_{L_n(f,q)}(\zeta_1,\zeta_2)(\theta_3) < L_n(f,q;\theta_3) \tag{3.10}$$

and exist $0 < \theta_1 < \zeta_1, \zeta_2 < \theta_4 < 1$ (the existence can be insured by the modification of ζ_1 and ζ_2) satisfying

$$L_{L_n(f,q)}(\zeta_1,\zeta_2)(\theta_1) < L_n(f,q;\theta_1), \tag{3.11}$$

$$L_{L_n(f,q)}(\zeta_1,\zeta_2)(\theta_4) < L_n(f,q;\theta_4). \tag{3.12}$$

We use $L_{L_n(f,q)}(\zeta_1,\zeta_2)(x)$ to denote the linear function joining the two points $(\zeta_1, L_n(f,q;\zeta_1))$ and $(\zeta_2, L_n(f,q;\zeta_2))$.

Then let $l_0(x) = L_{L_n(f,q)}(\zeta_1,\zeta_2)(x)$ we have

$$\begin{aligned} S^{-}(L_n(f,q)-l_0) \quad \geq \quad & S^{-}(L_n(f,q;\theta_1)-l_0(\theta_1), L_n(f,q;\theta_2) \\ & -l_0(\theta_2), L_n(f,q;\theta_3)-l_0(\theta_3), L_n(f,q;\theta_4)-l_0(\theta_4)) = 3. \end{aligned}$$

The above inequalities are in contradiction with (3.7). Hence $L_n(f,q)$ is convex on $[0,1]$.

Using the Jessen inequality of convex function and Proposition 2, we get

$$\begin{aligned}
L_n(f,q;x) &= \sum_{k=0}^{n-s}\left\{(1-q^{n-k-s}x)f\left(\frac{[k]}{[n]}\right)+q^{n-k-s}xf\left(\frac{[k+s]}{[n]}\right)\right\}p_{n-s,k}(q;x)\\
&\geq \sum_{k=0}^{n-s} f((1-q^{n-k-s}x)f\left(\frac{[k]}{[n]}\right)+q^{n-k-s}xf\left(\frac{[k+s]}{[n]}\right))p_{n-s,k}(q;x)\\
&\geq f(\sum_{k=0}^{n-s}\left\{(1-q^{n-k-s}x)\frac{[k]}{[n]}+q^{n-k-s}x\frac{[k+s]}{[n]}\right\}p_{n-s,k}(q;x))\\
&= f(x).
\end{aligned}$$

For the case of concave functions , we can prove the theorem in the same way. The proof of Theorem 3 is complete.

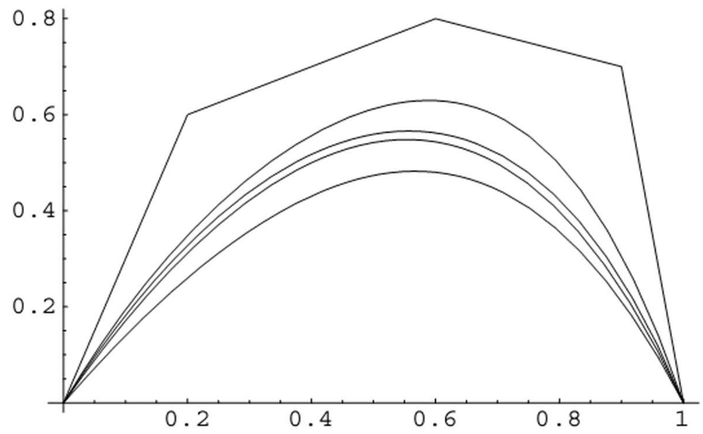

Fig. 2

Remark 2. The function $f(x)$ is the linear spline joining the points $(0,0),(0.2,0.6),(0.6,0.8)$, $(0.9,0.7)$ and $(1,0)$. The others are $L_{15}(f,0.7;x)$ for $s=3, L_{11}(f,0.7;x)$ for $s=5, L_7(f,0.7;x)$ for $s=3$ and $L_{20}(f,0.5;x)$ for $s=3$ from top to bottom.

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