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Characterizations of Null Holomorphic Sectional Curvature of GCR-Lightlike Submanifolds of Indefinite Nearly Kähler Manifolds

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Abstract. We obtain the expressions for sectional curvature, holomorphic sectional curvature and holomorphic bisectional curvature of a GCR-lightlike submanifold of an indefinite nearly Kähler manifold and obtain characterization theorems for holomorphic sectional and holomorphic bisectional curvature. We also establish a condition for a GCR-lightlike submanifold of an indefinite complex space form to be a null holomorphically flat.

Key Words: Indefinite nearly Kähler manifold, GCR-lightlike submanifold, holomorphic sectional curvature, holomorphic bisectional curvature.

AMS Subject Classifications: 53C15, 53C40, 53C50

1 Introduction

Due to the growing importance of lightlike submanifolds in mathematical physics and relativity [5] and the significant applications of CR structures in relativity [3, 4], Duggal and Bejancu [5] introduced the notion of CR-lightlike submanifolds of indefinite Kähler manifolds. Contrary to the classical theory of CR-submanifolds, CR-lightlike submanifolds do not include complex and totally real lightlike submanifolds as subcases. Therefore Duggal and Sahin [7] introduced SCR-lightlike submanifolds of indefinite Kähler manifold which contain complex and totally real subcases but do not include CR and

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SCR cases. Therefore Duggal and Sahin [8] introduced *GCR*-lightlike submanifolds of indefinite Kähler manifolds, which behaves as an umbrella of complex, totally real, screen real and *CR*-lightlike submanifolds and further studied by [11–13]. Husain and Deshmukh [10] studied *CR* submanifolds of nearly Kähler manifolds. Recently, Sangeet et al. [14] introduced *GCR*-lightlike submanifolds of indefinite nearly Kähler manifolds and obtained their existence in indefinite nearly Kähler manifolds of constant holomorphic sectional curvature c and of constant type α. In present paper, we obtain the expressions for sectional curvature, holomorphic sectional curvature and holomorphic bisectional curvature of a *GCR*-lightlike submanifold of an indefinite nearly Kähler manifold and obtain characterization theorems for holomorphic sectional and holomorphic bisectional curvature.

2 Lightlike submanifolds

Let $(\bar{M},\bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m,n\geq 1$, $1\leq q\leq m+n-1$ and (M,g) be an m-dimensional submanifold of $\bar{M}$ and g be the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$, for detail see [5]. For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $RadT_xM=T_xM\cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping $RadTM: x\in M\longrightarrow RadT_xM$, defines a smooth distribution on M of rank $r>0$ then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $RadTM$ is called the radical distribution on M. Screen distribution $S(TM)$ is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM therefore

$$TM=RadTM\perp S(TM) \tag{2.1}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $RadTM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}|_M$ and to $RadTM$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$tr(TM)=ltr(TM)\perp S(TM^{\perp}), \tag{2.2a}$$

$$T\bar{M}|_M=TM\oplus tr(TM)=(RadTM\oplus ltr(TM))\perp S(TM)\perp S(TM^{\perp}). \tag{2.2b}$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as $\{\xi_1,\cdots,\xi_r,W_{r+1},\cdots,W_n,N_1,\cdots,N_r,X_{r+1},\cdots,X_m\}$, where $\{\xi_1,\cdots,\xi_r\}$, $\{N_1,\cdots,N_r\}$ are local lightlike bases of $\Gamma(RadTM|_u)$, $\Gamma(ltr(TM)|_u)$ and $\{W_{r+1},\cdots,W_n\}$, $\{X_{r+1},\cdots,X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})|_u)$ and $\Gamma(S(TM)|_u)$ respectively. For these quasi-orthonormal fields of frames, we have

Theorem 2.1 (see [5]). *Let (M,g) be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M},\bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $RadTM$ in $S(TM^{\perp})^{\perp}$ and* *a basis of $ltr(TM)|_{\mathrm{u}}$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}|_{\mathrm{u}}$, where* u *is a coordinate neighborhood of M such that*

$$\bar{g}(N_i,\xi_j)=\delta_{ij},\quad \bar{g}(N_i,N_j)=0 \quad \text{for any} \ \ i,j\in\{1,2,\cdots,r\},$$

where $\{\xi_1,\cdots,\xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ then according to the decomposition (2.2b), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y=\nabla_X Y+h(X,Y),\quad \bar{\nabla}_X U=-A_U X+\nabla^{\perp}_X U, \tag{2.3}$$

for any $X,Y\in\Gamma(TM)$ and $U\in\Gamma(tr(TM))$, where $\{\nabla_X Y,A_U X\}$ and $\{h(X,Y),\nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, A_U is a linear a operator on M and known as shape operator.

According to (2.2a) considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ respectively, then (2.3) become

$$\bar{\nabla}_X Y=\nabla_X Y+h^l(X,Y)+h^s(X,Y),\quad \bar{\nabla}_X U=-A_U X+D^l_X U+D^s_X U, \tag{2.4}$$

where we put $h^l(X,Y)=L(h(X,Y))$, $h^s(X,Y)=S(h(X,Y))$, $D^l_X U=L(\nabla^{\perp}_X U)$, $D^s_X U=S(\nabla^{\perp}_X U)$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$\bar{\nabla}_X N=-A_N X+\nabla^l_X N+D^s(X,N),\quad \bar{\nabla}_X W=-A_W X+\nabla^s_X W+D^l(X,W), \tag{2.5}$$

where $X\in\Gamma(TM)$, $N\in\Gamma(ltr(TM))$ and $W\in\Gamma(S(TM^{\perp}))$. Using (2.4) and (2.5) we obtain

$$\bar{g}(h^s(X,Y),W)+\bar{g}(Y,D^l(X,W))=g(A_W X,Y), \tag{2.6}$$

for any $W\in\Gamma(S(TM^{\perp}))$. Let P be the projection morphism of TM on $S(TM)$ then using (2.1), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$\nabla_X PY=\nabla^*_X PY+h^*(X,PY),\quad \nabla_X\xi=-A^*_{\xi}X+\nabla^{*t}_X\xi, \tag{2.7}$$

for any $X,Y\in\Gamma(TM)$ and $\xi\in\Gamma(RadTM)$, where $\{\nabla^*_X PY,A^*_{\xi}X\}$ and $\{h^*(X,PY),\nabla^{*t}_X\xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(RadTM)$ respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $RadTM$ respectively. h^* and A^* are $\Gamma(RadTM)$-valued and $\Gamma(S(TM))$-valued bilinear forms and are called as second fundamental forms of distributions $S(TM)$ and $RadTM$ respectively.

Using (2.4) and (2.7), we obtain

$$\bar{g}(h^l(X,PY),\xi)=g(A^*_{\xi}X,PY),\quad \bar{g}(h^*(X,PY),N)=\bar{g}(A_N X,PY), \tag{2.8}$$

for any $X,Y\in\Gamma(TM)$, $\xi\in\Gamma(Rad(TM))$ and $N\in\Gamma(ltr(TM))$.

In general, the induced connection ∇ on M is not a metric connection. Since $\bar{\nabla}$ is a metric connection, by using (2.4), we get

$$(\nabla_X g)(Y,Z)=\bar{g}(h^l(X,Y),Z)+\bar{g}(h^l(X,Z),Y).$$

However, it is important to note that ∇^* is a metric connection on $S(TM)$.

Denote by $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇ respectively then by straightforward calculations (see [5]), we have

$$\begin{aligned}\bar{R}(X,Y)Z=&R(X,Y)Z+A_{h^l(X,Z)}Y-A_{h^l(Y,Z)}X+A_{h^s(X,Z)}Y\\&-A_{h^s(Y,Z)}X+(\nabla_X h^l)(Y,Z)-(\nabla_Y h^l)(X,Z)\\&+D^l(X,h^s(Y,Z))-D^l(Y,h^s(X,Z))+(\nabla_X h^s)(Y,Z)\\&-(\nabla_Y h^s)(X,Z)+D^s(X,h^l(Y,Z))-D^s(Y,h^l(X,Z)),\end{aligned} \tag{2.9}$$

where

$$(\nabla_X h^s)(Y,Z)=\nabla^s_X h^s(Y,Z)-h^s(\nabla_X Y,Z)-h^s(Y,\nabla_X Z), \tag{2.10a}$$
$$(\nabla_X h^l)(Y,Z)=\nabla^l_X h^l(Y,Z)-h^l(\nabla_X Y,Z)-h^l(Y,\nabla_X Z). \tag{2.10b}$$

Then Codazzi equation is given respectively by

$$\begin{aligned}(\bar{R}(X,Y)Z)^{\perp}=&(\nabla_X h^l)(Y,Z)-(\nabla_Y h^l)(X,Z)+D^l(X,h^s(Y,Z))-D^l(Y,h^s(X,Z))\\&+(\nabla_X h^s)(Y,Z)-(\nabla_Y h^s)(X,Z)+D^s(X,h^l(Y,Z))-D^s(Y,h^l(X,Z)).\end{aligned} \tag{2.11}$$

Gray [9], defined nearly Kähler manifolds as

Definition 2.1. Let $(\bar{M},\bar{J},\bar{g})$ be an indefinite almost Hermitian manifold and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then $\bar{M}$ is called an indefinite nearly Kähler manifold if

$$(\bar{\nabla}_X\bar{J})Y+(\bar{\nabla}_Y\bar{J})X=0,\quad \forall X,Y\in\Gamma(T\bar{M}), \tag{2.12}$$

or equivalently

$$(\bar{\nabla}_X\bar{J})X=0,\quad \forall X\in\Gamma(T\bar{M}). \tag{2.13}$$

It is well known that every Kähler manifold is a nearly Kähler manifold but converse is not true. S^6 with its canonical almost complex structure is a nearly Kähler manifold but not a Kähler manifold. Due to rich geometric and topological properties, the study of nearly Kähler manifolds is as important as that of Kähler manifolds. Therefore we studied the geometry of CR, SCR and GCR-lightlike submanifolds of an indefinite nearly Kähler manifolds in [14].

Nearly Kähler manifold of constant holomorphic curvature c is denoted by $\bar{M}(c)$ and its curvature tensor field $\bar{R}$ is given by, [15]

$$\begin{aligned}\bar{R}(X,Y,Z,W)=&\frac{c}{4}\{\bar{g}(X,W)\bar{g}(Y,Z)-\bar{g}(X,Z)\bar{g}(Y,W)+\bar{g}(X,\bar{J}W)\bar{g}(Y,\bar{J}Z)\\&-\bar{g}(X,\bar{J}Z)\bar{g}(Y,\bar{J}W)-2\bar{g}(X,\bar{J}Y)\bar{g}(Z,\bar{J}W)\}\\&+\frac{1}{4}\{\bar{g}((\bar{\nabla}_X\bar{J})(W),(\bar{\nabla}_Y\bar{J})(Z))-\bar{g}((\bar{\nabla}_X\bar{J})(Z),(\bar{\nabla}_Y\bar{J})(W))\\&-2\bar{g}((\bar{\nabla}_X\bar{J})(Y),(\bar{\nabla}_Z\bar{J})(W))\}\end{aligned} \tag{2.14}$$

and the sectional curvature is given by

$$\bar{R}(X,Y,X,Y)=\frac{c}{4}\{\bar{g}(X,Y)^2-\bar{g}(X,X)\bar{g}(Y,Y)-3\bar{g}(X,\bar{J}Y)^2\}-\frac{3}{4}\|(\bar{\nabla}_X\bar{J})(Y)\|^2.$$

A nearly Kähler manifold is said to be of constant type α [9], if there exists a real valued C^∞ function α on $\bar{M}$ such that

$$\|(\bar{\nabla}_X\bar{J})(Y)\|^2=\alpha\{\|X\|^2\|Y\|^2-\bar{g}(X,Y)^2-\bar{g}(X,\bar{J}Y)^2\}. \tag{2.15}$$

3 Generalized Cauchy-Riemann lightlike submanifolds

In this section, we briefly recall generalized Cauchy-Riemann (GCR)-lightlike submanifold of an indefinite nearly Kähler manifold $(\bar{M},\bar{g},\bar{J})$, for detail see [14].

Definition 3.1 (see [14]). Let $(M,g,S(TM))$ be a real lightlike submanifold of an indefinite nearly Kähler manifold $(\bar{M},\bar{g},\bar{J})$ then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied
(A) There exist two subbundles D_1 and D_2 of $Rad(TM)$ such that

$$Rad(TM)=D_1\oplus D_2,\quad \bar{J}(D_1)=D_1,\quad \bar{J}(D_2)\subset S(TM).$$

(B) There exist two subbundles D_0 and D' of $S(TM)$ such that

$$S(TM)=\{\bar{J}D_2\oplus D'\}\perp D_0,\quad \bar{J}(D_0)=D_0,\quad \bar{J}(D')=L_1\perp L_2,$$

where D_0 is a non degenerate distribution on M, L_1 and L_2 are vector subbundles of $ltr(TM)$ and $S(TM)^\perp$ respectively.

Then the tangent bundle TM of M is decomposed as

$$TM=D\perp D',\quad D=Rad(TM)\oplus D_0\oplus JD_2.$$

M is called a proper GCR-lightlike submanifold if $D_1\neq\{0\}$, $D_2\neq\{0\}$, $D_0\neq\{0\}$ and $L_2\neq\{0\}$.

Let Q, P_1 and P_2 be the projections on D, $\bar{J}(L_1)=M_1$ and $\bar{J}(L_2)=M_2$, respectively. Then for any $X\in\Gamma(TM)$, we have $X=QX+P_1X+P_2X$, applying $\bar{J}$ both sides, we obtain

$$\bar{J}X=TX+wP_1X+wP_2X, \tag{3.1}$$

and we can write the Eq. (3.1) as

$$\bar{J}X=TX+wX, \tag{3.2}$$

where TX and wX are the tangential and transversal components of $\bar{J}X$, respectively. Similarly

$$\bar{J}V=BV+CV, \tag{3.3}$$

for any $V\in\Gamma(tr(TM))$, where BV and CV are the sections of TM and $tr(TM)$ respectively. Applying $\bar{J}$ to (3.2) and (3.3), we get $T^2=-I-B\omega$, and $C^2=-I-\omega B$. Using nearly Kählerian property of $\bar{\nabla}$ with (2.5), we have the following lemma.

Lemma 3.1 (see [14]). *Let M be a GCR-lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}$. Then we have*

$$(\nabla_X T)Y+(\nabla_Y T)X=A_{wY}X+A_{wX}Y+2Bh(X,Y) \tag{3.4}$$

and

$$(\nabla^t_X w)Y+(\nabla^t_Y w)X=2Ch(X,Y)-h(X,TY)-h(TX,Y),$$

for any $X,Y\in\Gamma(TM)$, where

$$(\nabla_X T)Y=\nabla_X TY-T\nabla_X Y \quad \text{and} \quad (\nabla^t_X w)Y=\nabla^t_X wY-w\nabla_X Y.$$

4 Holomorphic sectional curvature of a GCR-lightlike submanifold

Let $\bar{M}$ be an indefinite nearly Kähler manifold of constant holomorphic curvature c the using (2.9) and (2.14) for any X, Y, Z, W vector fields on TM, we obtain

$$\begin{aligned} g(R(X,Y)Z,W)=&\frac{c}{4}\{\bar{g}(X,W)\bar{g}(Y,Z)-\bar{g}(X,Z)\bar{g}(Y,W)+\bar{g}(X,\bar{J}W)\bar{g}(Y,\bar{J}Z)\\ &-\bar{g}(X,\bar{J}Z)\bar{g}(Y,\bar{J}W)-2\bar{g}(X,\bar{J}Y)\bar{g}(Z,\bar{J}W)\}\\ &+\frac{1}{4}\{\bar{g}((\bar{\nabla}_X\bar{J})(W),(\bar{\nabla}_Y\bar{J})(Z))-\bar{g}((\bar{\nabla}_X\bar{J})(Z),(\bar{\nabla}_Y\bar{J})(W))\\ &-2\bar{g}((\bar{\nabla}_X\bar{J})(Y),(\bar{\nabla}_Z\bar{J})(W))\}-g(A_{h^l(X,Z)}Y,W)\\ &+g(A_{h^l(Y,Z)}X,W)-g(A_{h^s(X,Z)}Y,W)+g(A_{h^s(Y,Z)}X,W)\\ &-\bar{g}((\nabla_X h^l)(Y,Z),W)+\bar{g}((\nabla_Y h^l)(X,Z),W)\\ &-\bar{g}(D^l(X,h^s(Y,Z)),W)+\bar{g}(D^l(Y,h^s(X,Z)),W). \end{aligned} \tag{4.1}$$

Using (2.6) in (4.1), we obtain

$$\begin{aligned} g(R(X,Y)Z,W) =& \frac{c}{4}\{\bar{g}(X,W)\bar{g}(Y,Z)-\bar{g}(X,Z)\bar{g}(Y,W)+\bar{g}(X,\bar{J}W)\bar{g}(Y,\bar{J}Z) \\ &-\bar{g}(X,\bar{J}Z)\bar{g}(Y,\bar{J}W)-2\bar{g}(X,\bar{J}Y)\bar{g}(Z,\bar{J}W)\} \\ &+\frac{1}{4}\{\bar{g}((\bar{\nabla}_X\bar{J})(W),(\bar{\nabla}_Y\bar{J})(Z))-\bar{g}((\bar{\nabla}_X\bar{J})(Z),(\bar{\nabla}_Y\bar{J})(W)) \\ &-2\bar{g}((\bar{\nabla}_X\bar{J})(Y),(\bar{\nabla}_Z\bar{J})(W))\}-g(A_{h^l(X,Z)}Y,W) \\ &+g(A_{h^l(Y,Z)}X,W)-\bar{g}(h^s(Y,W),h^s(X,Z))+\bar{g}(h^s(X,W),h^s(Y,Z)) \\ &-\bar{g}((\nabla_X h^l)(Y,Z),W)+\bar{g}((\nabla_Y h^l)(X,Z),W). \end{aligned} \tag{4.2}$$

Then the sectional curvature $K_M(X,Y)=g(R(X,Y)Y,X)$ of M determined by orthonormal vectors X and Y of $\Gamma(D_0\oplus M_2)$ and given by

$$\begin{aligned} K_M(X,Y)=&\frac{c}{4}\{1+3g(X,\bar{J}Y)^2\}+\frac{3}{4}\|(\bar{\nabla}_X\bar{J})(Y)\|^2-g(A_{h^l(X,Y)}Y,X) \\ &+g(A_{h^l(Y,Y)}X,X)-\bar{g}(h^s(Y,X),h^s(X,Y))+\bar{g}(h^s(X,X),h^s(Y,Y)). \end{aligned} \tag{4.3}$$

Corollary 4.1. Let M be a GCR-lightlike submanifold of an indefinite nearly Kähler manifold of constant holomorphic sectional curvature c. Then sectional curvature of M is given by

$$K_M(X,Y)=\frac{c}{4}\{1+3g(X,\bar{J}Y)^2\}+\frac{3}{4}\|(\bar{\nabla}_X\bar{J})(Y)\|^2,$$

if

(i) M_2 defines a totally geodesic foliation in $\bar{M}$.

(ii) D_0 defines a totally geodesic foliation in $\bar{M}$.

(iii) M is totally geodesic in $\bar{M}$.

Definition 4.1. The holomorphic sectional curvature $H(X)=g(R(X,\bar{J}X)\bar{J}X,X)$ of M determined by a unit vector $X\in\Gamma(D_0)$ is the sectional curvature of a plane section $\{X,\bar{J}X\}$.

Then using (2.8) and (4.3), for a unit vector field $X\in\Gamma(D_0)$, we get

$$\begin{aligned} H(X)=c&-\bar{g}(h^l(X,\bar{J}X),h^*(\bar{J}X,X))+\bar{g}(h^l(\bar{J}X,\bar{J}X),h^*(X,X)) \\ &-\bar{g}(h^s(\bar{J}X,X),h^s(X,\bar{J}X))+\bar{g}(h^s(X,X),h^s(\bar{J}X,\bar{J}X)) \\ &+\frac{3}{4}\|(\bar{\nabla}_X\bar{J})(\bar{J}X)\|^2. \end{aligned} \tag{4.4}$$

Since for a nearly Kähler manifold, we know that $(\bar{\nabla}_X\bar{J})(\bar{J}X)=-\bar{J}(\bar{\nabla}_X\bar{J})(X)=0$, therefore from (4.4), we obtain

$$\begin{aligned} H(X)=c&-\bar{g}(h^l(X,\bar{J}X),h^*(\bar{J}X,X))+\bar{g}(h^l(\bar{J}X,\bar{J}X),h^*(X,X)) \\ &-\bar{g}(h^s(\bar{J}X,X),h^s(X,\bar{J}X))+\bar{g}(h^s(X,X),h^s(\bar{J}X,\bar{J}X)). \end{aligned} \tag{4.5}$$

Theorem 4.1 (see [14]). *Let M be a GCR-lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}$ then the distribution D is integrable if and only if $h(X,\bar{J}Y)=h(Y,\bar{J}X)$, for any $X,Y\in\Gamma(D)$.*

Theorem 4.2. *Let M be a GCR-lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}(c)$ with constant holomorphic sectional curvature c and the distribution D_0 is integrable then $H(X)\leq c$ for any unit vector field $X\in\Gamma(D_0)$.*

Proof. Since D_0 is integrable therefore using the Theorem 4.1, we have $h(\bar{J}X,\bar{J}X)=-h(X,X)$, for any unit vector field $X\in\Gamma(D_0)$. Therefore from (4.5), we obtain

$$\begin{aligned}H(X)=c-\bar{g}(h^l(X,\bar{J}X),h^*(\bar{J}X,X))-\bar{g}(h^l(X,X),h^*(X,X))&\\ -\|h^s(X,\bar{J}X)\|^2-\|h^s(X,X)\|^2\leq c.&\end{aligned}\tag{4.6}$$

So, we complete the proof. □

Definition 4.2. A GCR-lightlike submanifold M of an indefinite nearly Kähler manifold $\bar{M}$ is said to be D- totally geodesic (resp. D'-totally geodesic) if and only if $h(X,Y)=0$ for any $X,Y\in\Gamma(D_0)$ (resp. $X,Y\in\Gamma(D')$).

Lemma 4.1. *Let M be a GCR-lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}$. If the distribution D_0 defines a totally geodesic foliation in $\bar{M}$ then M is D_0-geodesic.*

Proof. To show M is D_0-geodesic we have to prove

$$\bar{g}(h^l(X,Y),\xi)=\bar{g}(h^s(X,Y),W)=0,$$

for any $X,Y\in\Gamma(D_0)$, $\xi\in\Gamma(Rad(TM))$ and $W\in\Gamma(S(TM^\perp))$. Since D_0 defines totally geodesic foliation in $\bar{M}$ therefore we obtain

$$\bar{g}(h^l(X,Y),\xi)=\bar{g}(\bar{\nabla}_XY,\xi)=0$$

and

$$\bar{g}(h^s(X,Y),W)=\bar{g}(\bar{\nabla}_XY,W)=0.$$

Hence the assertion follows. □

Theorem 4.3. *Let M be a GCR-lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}$ with constant holomorphic sectional curvature c. If D_0 defines a totally geodesic foliation in $\bar{M}$ then $H(X)=c$, for any unit vector field $X\in\Gamma(D_0)$.*

Proof. The assertion follows directly using the Lemma 4.1 in (4.5). □

Theorem 4.4. *Let M be a GCR-lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}$ of constant type α and of constant holomorphic sectional curvature c. If M is M_2-totally geodesic then*

$$K_M(X,Y)=\frac{1}{4}(c+3\alpha),$$

where $K_M(X,Y)$ is the sectional curvature of the plane section $X\wedge Y$ in $M_2\subset D'$.

Proof. Let plane section $X\wedge Y$ is spanned by the orthonormal unit vectors $X,Y\in\Gamma(M_2)\subset\Gamma(D')$, then using (2.8) in (4.3), we get

$$\begin{aligned}K_M(X,Y)=&\frac{c}{4}-\bar{g}(h^l(X,Y),h^*(X,Y))+\bar{g}(h^l(Y,Y),h^*(X,X))-\bar{g}(h^s(X,Y),h^s(X,Y))\\&+\bar{g}(h^s(X,X),h^s(Y,Y))+\frac{3}{4}\|(\bar{\nabla}_X\bar{J})(Y)\|^2.\end{aligned}\tag{4.7}$$

Since $\bar{M}$ is of constant type α, using (2.15) we obtain

$$\begin{aligned}K_M(X,Y)=&\frac{1}{4}(c+3\alpha)-\bar{g}(h^l(X,Y),h^*(X,Y))+\bar{g}(h^l(Y,Y),h^*(X,X))\\&-\bar{g}(h^s(X,Y),h^s(X,Y))+\bar{g}(h^s(X,X),h^s(Y,Y)).\end{aligned}\tag{4.8}$$

Using the hypothesis that M is M_2-totally geodesic in (4.8), the assertion follows. ☐

Definition 4.3. The holomorphic bisectional curvature for the pair of unit vector fields $\{X,Y\}$ on $\bar{M}$ is given by

$$\bar{H}(X,Y)=\bar{g}(\bar{R}(X,\bar{J}X)Y,\bar{J}Y).$$

Definition 4.4. A GCR-lightlike submanifold M of an indefinite nearly Kähler manifold $\bar{M}$ is said to be mixed geodesic if and only if $h(X,Y)=0$ for any $X\in\Gamma(D)$ and $Y\in\Gamma(D')$.

Theorem 4.5. *Let M be a mixed geodesic GCR-lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}$ with D_0 as a parallel distribution with respect to ∇ on M. Then $\bar{H}(X,Z)=0$, for any $X\in\Gamma(D_0)$ and $Z\in\Gamma(M_2)$.*

Proof. Let $X,Y\in\Gamma(D_0)$ and $Z\in\Gamma(M_2)$ then using the hypothesis that the distribution D_0 is parallel with respect to ∇ on M, we have

$$g(T\nabla_X Z,Y)=-\bar{g}(\bar{\nabla}_X Z,TY)=g(Z,\nabla_X TY)=0.$$

Hence the non degeneracy of the distribution D_0 implies that, $T\nabla_X Z=0$, that is

$$\nabla_X Z\in\Gamma(D'),\tag{4.9}$$

for any $Z\in\Gamma(M_2)$. Now replacing Y by $\bar{J}X$ respectively in (2.11) and then taking inner product with $\bar{J}Z$, for any $X\in\Gamma(D_0)$ and $Z\in\Gamma(M_2)$. Then by virtue of (2.10b), we get

$$\begin{aligned}\bar{H}(X,Z)=&-\bar{g}(h^s(\nabla_X\bar{J}X,Z),\bar{J}Z)+\bar{g}(\nabla^s_X(h^s(\bar{J}X,Z)),\bar{J}Z)-\bar{g}(h^s(\bar{J}X,\nabla_X Z),\bar{J}Z)\\&-\bar{g}(\nabla^s_{\bar{J}X}(h^s(X,Z)),\bar{J}Z)+\bar{g}(h^s(\nabla_{\bar{J}X}X,Z),\bar{J}Z)+\bar{g}(h^s(X,\nabla_{\bar{J}X}Z),\bar{J}Z)\\&+\bar{g}(D^s(X,h^l(\bar{J}X,Z)),\bar{J}Z)-\bar{g}(D^s(\bar{J}X,h^l(X,Z)),\bar{J}Z).\end{aligned}$$

Hence using that M is mixed totally geodesic with (4.9), the assertion follows. ☐

Lemma 4.2 (see [15]). *If $\bar{M}$ is a nearly Kähler manifold, then*

$$(\bar{\nabla}_X\bar{J})Y+(\bar{\nabla}_{\bar{J}X}\bar{J})\bar{J}Y=0, \quad N(X,Y)=-4\bar{J}((\bar{\nabla}_X\bar{J})(Y)),$$

for any $X,Y\in\Gamma(T(\bar{M}))$. where $N(X,Y)$ is the Nijenhuis tensor and given by

$$N(X,Y)=[\bar{J}X,\bar{J}Y]-\bar{J}[X,\bar{J}Y]-\bar{J}[\bar{J}X,Y]-[X,Y].$$

Theorem 4.6. *Consider an indefinite nearly Kähler manifold $\bar{M}$ of constant holomorphic sectional curvature c. In order that it may admit a mixed geodesic GCR-lightlike submanifold M with parallel distribution D_0, it is necessary that $c\geq 0$.*

Proof. Using the Theorem 4.5, we have $\bar{H}(X,Z)=0$ for any $X\in\Gamma(D_0)$ and $Z\in\Gamma(M_2)$. Using this result with the Lemma 4.2 and the fact that $(\bar{\nabla}_X\bar{J})(\bar{J}Y)=-\bar{J}(\bar{\nabla}_X\bar{J})(Y)$ in (2.14) we obtain

$$c\|X\|^2\|Z\|^2=\|(\bar{\nabla}_X\bar{J})(Z)\|^2,$$

this implies that $c\geq 0$. □

Theorem 4.7. *Let M be a GCR-lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}$. If D' defines a totally geodesic foliation in $\bar{M}$ then $\bar{g}(h^s(D',D_0),\bar{J}D')=0$.*

Proof. Let D' defines a totally geodesic foliation in $\bar{M}$ this implies that $\bar{\nabla}_XY=\nabla_XY\in\Gamma(D')$ and $h(X,Y)=0$ for any $X,Y\in\Gamma(D')$. Therefore using (3.4), we obtain $A_{wY}X+A_{wX}Y=-2Bh(X,Y)=0$. Let $Z\in\Gamma(D_0)$ then using (2.4), (2.6), (2.12) we get

$$\begin{aligned}0=&g(A_{wP_1Y}X,Z)+g(A_{wP_2Y}X,Z)+g(A_{wP_1X}Y,Z)+g(A_{wP_2X}Y,Z)\\=&-\bar{g}(\bar{\nabla}_X\bar{J}P_1Y,Z)+\bar{g}(h^s(X,Z),wP_2Y)-\bar{g}(\bar{\nabla}_Y\bar{J}P_1X,Z)+\bar{g}(h^s(Y,Z),wP_2X)\\=&\bar{g}(h^s(X,Z),wP_2Y)+\bar{g}(h^s(Y,Z),wP_2X)-\bar{g}(\bar{\nabla}_X\bar{J}P_1Y+\bar{\nabla}_Y\bar{J}P_1X,Z)\\=&\bar{g}(h^s(X,Z),wP_2Y)+\bar{g}(h^s(Y,Z),wP_2X)-\bar{g}(\bar{\nabla}_XY+\bar{\nabla}_YX,\bar{J}Z)\\=&\bar{g}(h^s(X,Z),wP_2Y)+\bar{g}(h^s(Y,Z),wP_2X)\\=&\bar{g}(h^s(D',D_0),\bar{J}D')+\bar{g}(h^s(D',D_0),\bar{J}D').\end{aligned}$$

Thus the assertion follows. □

Definition 4.5 (see [6]). A lightlike submanifold (M,g) of a semi-Riemannian manifold $(\bar{M},\bar{g})$ is said to be a totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $H\in\Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that $h(X,Y)=H\bar{g}(X,Y)$, for $X,Y\in\Gamma(TM)$. Using (2.5), it is clear that M is a totally umbilical, if and only if, on each coordinate neighborhood u there exist smooth vector fields $H^l\in\Gamma(ltr(TM))$ and $H^s\in\Gamma(S(TM^\perp))$ such that

$$h^l(X,Y)=H^lg(X,Y), \quad h^s(X,Y)=H^sg(X,Y), \quad D^l(X,W)=0,$$

for $X,Y\in\Gamma(TM)$ and $W\in\Gamma(S(TM^\perp))$. M is called totally geodesic if $H=0$, that is, if $h(X,Y)=0$.

Theorem 4.8 (see [14]). *Let M be a totally umbilical proper GCR-lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}$. If D_0 defines a totally geodesic foliation in M then the induced connection ∇ is a metric connection. Moreover, $h^s=0$.*

Theorem 4.9. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite nearly Kähler manifold of constant holomorphic sectional curvature $c\neq 0$ with the distribution D_0 defining a totally geodesic foliation in M. Then M is of constant curvature if and only if $\bar{M}$ is of constant type c.*

Proof. Let $X,Y\in\Gamma(D_0\oplus M_2)$ be two orthonormal vectors such that $g(X,Y)=g(X,\bar{J}Y)=0$. Since M is a totally umbilical GCR-lightlike submanifold with the distribution D_0 defining a totally geodesic foliation in M therefore using (4.3) and (4.5), the sectional curvature and holomorphic sectional curvature of M are given, respectively, by

$$K_M(X,Y)=\frac{c}{4}+\frac{3}{4}\|(\nabla_X\bar{J})Y\|^2+\|H^s\|^2$$

and

$$H(X)=c+\|H^s\|^2.$$

It follows that if $\bar{M}$ is of constant type c, then $K_M(X,Y)=c+\|H^s\|^2$. Hence M is a space of constant curvature c. $\square$

5 Null holomorphically flat GCR-lightlike submanifold

Let $x\in\bar{M}$ and U be a null vector of $T_x\bar{M}$. A plane π of $T_x\bar{M}$ is called a null plane directed by U if it contains U, $\bar{g}_x(U,V)=0$, for any $V\in\pi$ and there exists $V_0\in\pi$ such that $\bar{g}_x(V_0,V_0)\neq 0$. Following Beem-Ehrlich [1], the null sectional curvature of π with respect to U and $\bar{\nabla}$, as a real number is defined as

$$\bar{K}_U(\pi)=\frac{\bar{g}_x(\bar{R}(V,U)U,V)}{\bar{g}_x(V,V)},$$

where V is an arbitrary non-null vector in π.

Consider $u\in M$ and a null plane π of T_uM directed by $\xi_u\in Rad(TM)$ then the null sectional sectional curvature of π with respect to ξ_u and ∇, as a real number is defined as

$$K_{\xi_u}(\pi)=\frac{g_u(R(V_u,\xi_u)\xi_u,V_u)}{g_u(V_u,V_u)},$$

where V_u is an arbitrary non-null vector in π.

Let M be a GCR-lightlike submanifold of an indefinite nearly Kähler manifold of constant holomorphic sectional curvature c then using (4.2), the null sectional sectional curvature of π with respect to ξ is given by

$$\begin{aligned}K_\xi(\pi)=&\frac{1}{g(V,V)}\{g(A_{h^l(\xi,\xi)}V,V)-g(A_{h^l(V,\xi)}\xi,V)+\bar{g}(h^s(V,V),h^s(\xi,\xi))\\&-\bar{g}(h^s(\xi,V),h^s(V,\xi))-\bar{g}((\nabla_V h^l)(\xi,\xi),V)+\bar{g}((\nabla_\xi h^l)(V,\xi),V)\}\\&-\frac{3}{4}\bar{g}((\bar{\nabla}_V J)\xi,(\bar{\nabla}_\xi J)V).\end{aligned}\tag{5.1}$$

Then using (2.8), we obtain

$$\begin{aligned}K_\xi(\pi)=&\frac{1}{g(V,V)}\{g(h^*(V,V),h^l(\xi,\xi))-g(h^*(\xi,V),h^l(V,\xi))+\bar{g}(h^s(V,V),h^s(\xi,\xi))\\&-\bar{g}(h^s(\xi,V),h^s(V,\xi))-\bar{g}((\nabla_V h^l)(\xi,\xi),V)+\bar{g}((\nabla_\xi h^l)(V,\xi),V)\}\\&-\frac{3}{4}\bar{g}((\bar{\nabla}_V J)\xi,(\bar{\nabla}_\xi J)V).\end{aligned}\tag{5.2}$$

We know that a plane π is called holomorphic if it remains invariant under the action of the almost complex structure $\bar{J}$, that is, if $\pi=\{Z,\bar{J}Z\}$. The sectional curvature associated with the holomorphic plane is called the holomorphic sectional curvature, denoted by $\bar{H}(\pi)$ and given by $\bar{H}(\pi)=\bar{R}(Z,\bar{J}Z,Z,\bar{J}Z)/\bar{g}(Z,Z)^2$. The holomorphic plane $\pi=\{Z,\bar{J}Z\}$ is called null or degenerate if and only if Z is a null vector. A manifold $(\bar{M},\bar{g},\bar{J})$ is called null holomorphically flat if the curvature tensor $\bar{R}$ satisfies, (see [2])

$$\bar{R}(Z,\bar{J}Z,Z,\bar{J}Z)=0,$$

for all null vectors Z. Put $\bar{g}(\bar{R}(X,Y)Z,W)=\bar{R}(X,Y,Z,W)$, then from (5.2), we obtain

$$\begin{aligned}R(\xi,\bar{J}\xi,\xi,\bar{J}\xi)=&g(h^*(\xi,\xi),h^l(\bar{J}\xi,\bar{J}\xi))-g(h^*(\bar{J}\xi,\xi),h^l(\xi,\bar{J}\xi))\\&+\bar{g}(h^s(\xi,\xi),h^s(\bar{J}\xi,\bar{J}\xi))-\bar{g}(h^s(\bar{J}\xi,\xi),h^s(\xi,\bar{J}\xi))\\&-\bar{g}((\nabla_\xi h^l)(\bar{J}\xi,\bar{J}\xi),\xi)+\bar{g}((\nabla_{\bar{J}\xi}h^l)(\xi,\bar{J}\xi),\xi)\\&-\frac{3}{4}\bar{g}((\bar{\nabla}_\xi J)J\xi,(\bar{\nabla}_{J\xi}J)\xi).\end{aligned}\tag{5.3}$$

Since $(\bar{\nabla}_X J)Y=-J(\bar{\nabla}_X J)Y$ therefore using (2.13), we have

$$\bar{g}((\bar{\nabla}_\xi J)J\xi,(\bar{\nabla}_{J\xi}J)\xi)=-\bar{g}(J(\bar{\nabla}_\xi J)\xi,(\bar{\nabla}_{J\xi}J)\xi)=0.$$

Thus (5.3) becomes

$$\begin{aligned}R(\xi,\bar{J}\xi,\xi,\bar{J}\xi)=&g(h^*(\xi,\xi),h^l(\bar{J}\xi,\bar{J}\xi))-g(h^*(\bar{J}\xi,\xi),h^l(\xi,\bar{J}\xi))\\&+\bar{g}(h^s(\xi,\xi),h^s(\bar{J}\xi,\bar{J}\xi))-\bar{g}(h^s(\bar{J}\xi,\xi),h^s(\xi,\bar{J}\xi))\\&-\bar{g}((\nabla_\xi h^l)(\bar{J}\xi,\bar{J}\xi),\xi)+\bar{g}((\nabla_{\bar{J}\xi}h^l)(\xi,\bar{J}\xi),\xi).\end{aligned}\tag{5.4}$$

Let M be a totally umbilical lightlike submanifold then, we have $h(\bar{J}\xi,\bar{J}\xi)=Hg(\bar{J}\xi,\bar{J}\xi)=Hg(\xi,\xi)=0$ and $h(\xi,\bar{J}\xi)=Hg(\xi,\bar{J}\xi)=0$, for any $\xi\in\Gamma(Rad(TM))$. Thus from (5.4), we have the following theorem.

Theorem 5.1. *Let M be a GCR-lightlike submanifold of an indefinite nearly Kähler manifold of constant holomorphic sectional curvature c. If M is totally umbilical lightlike submanifold then M is null holomorphically flat.*

Moreover, from (5.4) it is clear that the expression of $R(\xi,\bar{J}\xi,\xi,\bar{J}\xi)$ is expressed in terms of screen second fundamental forms of M, thus GCR-lightlike submanifold M of an indefinite nearly Kähler manifold of constant holomorphic sectional curvature c is null holomorphically flat if M is totally geodesic.

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