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SECOND ORDER DECOUPLED IMPLICIT/EXPLICIT METHOD OF THE PRIMITIVE EQUATIONS OF THE OCEAN I: TIME DISCRETIZATION

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Abstract. In this article, we propose the time discretization of the second order decoupled implicit/explicit method of the 3D primitive equations of the ocean in the case of the Dirichlet boundary conditions on the side. We deduce the second order optimal error estimates on the L^2 and H^1 norms of the time discrete velocity and density and the L^2 norm of the time discrete pressure under the restriction of the time step $0 < \tau \leq \beta$ for some positive constant β. Also, we deduce some stability results on the time discrete solution under the same restriction on the time step.

Key words. Primitive equations of the ocean, stability, optimal error estimate, second order decoupled implicit/explicit method.

1. Introduction

Given a smooth bounded domain $\omega \subset R^2$ and the cylindrical domain $\Omega = \omega \times (-d, 0) \subset R^3$, consider in Ω the following 3D viscous primitive equations of the ocean:

$$u_t + L_1 u + (u \cdot \nabla)u + w\partial_z u + \nabla P + f\vec{k} \times u = F_1, \tag{1.1}$$

$$\theta_t + L_2\theta + (u \cdot \nabla)\theta + w\partial_z\theta - \sigma w = F_2 \tag{1.2}$$

$$\nabla \cdot u + \partial_z w = 0, \tag{1.3}$$

$$\partial_z P + \gamma\theta = 0. \tag{1.4}$$

The unknowns for the 3D viscous PEs are the fluid velocity flied $(u, w) = (u_1, u_2, w) \in R^3$ with $u = (u_1, u_2)$ being horizontal, the density θ and the pressure P. Here $f = f_0(\beta + y)$ is the given coriolis rotation frequency with β-plane approximation, F_1 and F_2 are two given functions and $\vec{k}$ is vertical unit vector and $\sigma > 0$ and $\gamma > 0$ are given constant. The elliptic operators L_1 and L_2 are given respectively as the following:

$$L_i = -\nu_i\Delta - \mu_i\partial_z^2,\ i = 1,\ 2.$$

Here the positive constants $\nu_1,\ \mu_1$ are the horizontal and vertical viscosity coefficients; while the positive constants $\nu_2,\ \mu_2$ are the horizontal and vertical thermal diffusivity coefficients and

$$u_t = \frac{\partial u}{\partial t},\ \theta_t = \frac{\partial \theta}{\partial t},\ \nabla = (\partial_x, \partial_y),\ \Delta = \partial_{xx} + \partial_{yy},\ \partial_{x_i} = \frac{\partial}{\partial x_i},\ \partial_{x_i x_i} = \partial_{x_i}^2,$$

with $i = 1, 2, 3$ and $(x_1, x_2, x_3) = (x, y, z)$.

We partition the boundary of Ω into the following three parts:

$$\Gamma_u = \{(x, y, z) \in \bar{\Omega}; z = 0\},\ \Gamma_b = \{(x, y, z) \in \bar{\Omega}; z = -d\},$$

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$$\Gamma_s = \{(x, y, z) \in \bar{\Omega}; (x, y) \in \partial\omega,\ -d \le z \le 0\}.$$

We consider the following homogenous boundary conditions of the 3D viscous PEs as in [3, 13, 19]:

$$w|_{\Gamma_u \cup \Gamma_b} = 0. \tag{1.5}$$

$$\partial_z u|_{\Gamma_u \cup \Gamma_b} = 0,\ u \cdot n|_{\Gamma_s} = 0,\ \frac{\partial u}{\partial n} \times n|_{\Gamma_s} = 0; \tag{1.6-1}$$

$$\text{or } u_z|_{\Gamma_u \cup \Gamma_b} = 0,\ u|_{\Gamma_s} = 0; \tag{1.6-2}$$

$$\partial_z \theta|_{\Gamma_b} = (\partial_z \theta + \alpha\theta)|_{\Gamma_u} = 0,\ \frac{\partial \theta}{\partial n}|_{\Gamma_s} = 0. \tag{1.7}$$

Here n is the normal vector of Γ_s, α is a positive constant. Also, the initial conditions of $u(x, y, z, t)$ and $\theta(x, y, z, t)$ should be given by

$$u(x, y, z, 0) = u_0(x, y, z),\ \theta(x, y, z, 0) = \theta_0(x, y, z). \tag{1.8}$$

Using the Dirichlet boundary condition (1.5) of w on $\Gamma_u \cap \Gamma_b$ and (1.3)-(1.4), we have

$$w(x, y, z, t) = -\int_{-d}^{z} \nabla \cdot u(x, y, \xi, t)d\xi,\ \int_{-d}^{0} \nabla \cdot u(x, y, \xi, t)d\xi = 0,$$

$$P(x, y, z, t) = p(x, y, t) - \gamma \int_{-d}^{z} \theta(x, y, \xi, t)d\xi.$$

With the above statements, one obtains the initial boundary value problem of the 3D viscous PEs:

$$u_t + L_1 u + \nabla p(x, y, t) - \gamma \int_{-d}^{z} \nabla\theta(x, y, \xi, t)d\xi + f\vec{k} \times u$$

$$+ (u \cdot \nabla)u - (\int_{-d}^{z} \nabla \cdot u(x, y, \xi, t)d\xi)\partial_z u = F_1, \tag{1.9}$$

$$\theta_t + L_2\theta + \sigma \int_{-d}^{z} \nabla \cdot u(x, y, \xi, t)d\xi + (u \cdot \nabla)\theta - (\int_{-d}^{z} \nabla \cdot u(x, y, \xi, t)d\xi)\partial_z \theta$$

$$= F_2, \tag{1.10}$$

$$\nabla \cdot \bar{u} = 0, \tag{1.11}$$

together with the boundary condition (1.6)-(1.7) and the initial condition (1.8), where

$$\bar{\phi}(x, y) = \frac{1}{d}\int_{-d}^{0} \phi(x, y, z)dz,\ \tilde{\phi} = \phi - \bar{\phi},$$

for any function $\phi(x, y, z)$ in Ω.

Remark. Recall [3, 13], $F_1 = 0$ and $\gamma = 1$ in (1.9), $\gamma = 1$ in (1.4) and $\sigma = 0$ in (1.2) and (1.10), the boundary condition on u is (1.6-1). While in [19], the boundary condition on u is (1.6-2).

The 3D viscous PEs are very important research subjects in the field of geophysical fluid dynamics, at both the theoretical and numerical levels. There are some well-known difficulties associated with this fundamental equation for 3D oceanic model since their strong nonlinearity.The Mathematical study of the PEs originates in a series of articles, by Lions, Temam and Wang in the early 1990s: see, for instance, [16, 17, 18], where the mathematical formulation of the PEs, which resembles that of the Navier-Stokes equations, was established. Also, the asymptotic analysis and the finite dimensional behavior of the 3D viscous PEs in thin domain as the depth of the domain goes to zero were studied in [11, 12]. For a more extensive discussion and review on this subject, the reader is referred to the recent articles [2, 3, 13, 7, 21, 22]. Furthermore, a two-grid finite difference method for the primitive equations of the ocean was proposed by Medjo and Temam in [19]. In particular, in the recent paper [7], we have discussed the H^2-regularity of the solution and some its time derivatives for the 3D viscous PEs, where main results in this paper are included in the following theorem.

Theorem 1.1. Assume that the initial data (u_0, θ_0) is the H^2-regularity, and

$$(\partial_z F_1, \partial_z F_2),\ (\partial_t^i F_1, \partial_t^i F_2) \in L^\infty(R^+; L^2(\Omega)^2) \times L^\infty(R^+; L^2(\Omega)),\ i = 0, 1, \cdots, m.$$

Then, the solution (u, p, θ) of the 3D viscous PEs satisfies the following bounds:

$$\begin{aligned} &\sigma^{2m-2}(t)[\|\partial_t^m u(t)\|_{L^2}^2 + \|\partial_t^m \theta(t)\|_{L^2}^2 + \sigma(t)[\|\partial_t^m u(t)\|_{H^1}^2 + \|\partial_t^m \theta(t)\|_{H^1}^2 \\ &\quad + \|\partial_t^{m-1} u(t)\|_{H^2}^2 + \|\partial_t^{m-1} \theta(t)\|_{H^2}^2 + \|\partial_t^{m-1} p(t)\|_{H^1}^2] \le \kappa, \end{aligned} \tag{1.12}$$

$$\begin{aligned} &\int_0^t e^{\alpha_2(s-t)} \sigma^{2m-2}(s)[\|\partial_t^m u\|_{H^1}^2 + \|\partial_t^m \theta\|_{H^1}^2 \\ &\quad + \sigma(s)(\|\partial_t^m u\|_{H^2}^2 + \|\partial_t^m \theta\|_{H^2}^2 + \|\partial_t^m p\|_{H^1}^2)]ds \le \kappa, \end{aligned} \tag{1.13}$$

$$\int_0^t e^{\alpha_2(s-t)} \sigma^{2m-1}(s)(\|\partial_t^{m+1} u\|_{L^2}^2 + \|\partial_t^{m+1} \theta\|_{L^2}^2)ds \le \kappa, \tag{1.14}$$

for all $t \ge 0$ and $m = 1, 2, 3$, where $\alpha_2 > 0$ is a fixed constant and κ is a general positive constant depending on the data $(\omega, \alpha, \sigma, \gamma, d, \nu_1, \mu_1, \nu_2, \mu_2, f)$ and $(F_1, F_2, u_0, \theta_0)$, which can take the different value at its different occurrences.

In very recent papers [8, 9], we have considered the first order decoupled method of problem (1.9)-(1.11) in the finite time interval $[0, T]$ with the boundary conditions (1.6)-(1.7) and the initial condition (1.8), the time variable is discreted by the first order decoupled semi-implicit scheme and the spatial variable is discreted by $P_1(P_1) - P_1 - P_1(P_1)$ finite element for the velocity, pressure and density. Here we have discussed the stability of the first order decoupled semi-implicit scheme with respect to the time discretization and the optimal error estimates with respect to the time and spatial discretizations of the numerical solution.

In this paper we analyze the time discretization of the second order decoupled implicit/explicit method for problem (1.9)-(1.11) in the finite time interval $[0, T]$ with the boundary conditions (1.6)-(1.7) and the initial condition (1.8), the spatial variable remaining continuous. Setting τ be time step size and $t_n = n\tau$ and $T = N\tau$, we consider the time discrete approximation (u^n, p^n, θ^n) of $(u(t_n), \frac{1}{\tau}\int_{t_{n-1}}^{t_n} p(t)dt, \theta(t_n))$. In our second order decoupled implicit/explicit method, the linear terms are discreted by the Crank-Nicolson scheme with (u^n, p^n) and θ^n being uncoupled and the nonlinear terms are discreted by the Admas-Bathforth scheme, where the Crank-Nicolson/Admas-Bathforth scheme were applied to solving the 2D and 3D Navier-Stokes equations [6, 23]. Splitting $u^n = \bar{u}^n + \tilde{u}^n$, we reduce the time discrete solution (u^n, p^n, θ^n) of the 3D primitive equations of the ocean into the solution $(\bar{u}^n, p^n)$ defined by the Stokes equations in the 2D domain ω and the solution $(\tilde{u}^n, \theta^n)$ determined by the linear equations in the 3D domain Ω, respectively. The L^2 and H^1 lower order error estimates of the time discrete solution (u^n, θ^n) are provided by the induction method and the Gronwall lemma and the L^2 and H^1 second order error estimate of (u^n, p^n, θ^n) are proven by using the negative norm technique and the Gronwall lemma under the restriction of time step $0 < \tau \le \beta$ for some positive constants β. Also, we deduce some stability results on the time discrete solution under the same restriction on the time step.

The main results of this paper are stated as follows.

Theorem 1.2. Under the assumptions (**A1**)-(**A2**), if $0 < \tau \leq \beta$ for some positive constant β , then (u^n, p^n, θ^n) satisfies the following stability and convergence results:

$$\tau \sum_{n=1}^{m} [\|\hat{u}^n\|_{H^2}^2 + \|\hat{\theta}^n\|_{H^2}^2] + \|u^m\|_{H^1}^2 + \|\theta^m\|_{H^1}^2 \leq \kappa, \tag{1.15}$$

$$\tau \sum_{n=1}^{m} [\|\hat{u}^n\|_{H^3}^2 + \|p^n\|_{H^2(\omega)}^2 + \|\hat{\theta}^n\|_{H^3}^2 + \|d_t u^n\|_{H^1}^2 + \|d_t \theta^n\|_{H^1}^2] + \|u^m\|_{H^2}^2 + \|p^m\|_{H^1(\omega)}^2 + \|\theta^m\|_{H^2}^2 \leq \kappa, \tag{1.16}$$

$$\tau^2 \sum_{n=1}^{m} [\|d_t u^n\|_{H^2}^2 + \|d_t \theta^n\|_{H^2}^2] + \tau \|u^m\|_{H^3}^2 + \tau \|p^m\|_{H^2(\omega)}^2 + \tau \|\theta^m\|_{H^3}^2 \leq \kappa, \tag{1.17}$$

$$\begin{aligned} &\tau \sum_{n=1}^{m} \sigma^3(t_n) [\|d_t(u(t_m) - u^m)\|_{L^2}^2 + \|d_t(\theta(t_m) - \theta^m)\|_{L^2}^2] \\ &+\tau \sum_{n=1}^{m} \sigma^3(t_n) \|\frac{1}{\tau} \int_{t_{n-1}}^{t_n} p(t) dt - p^n\|_{L^2(\omega)}^2 + \sigma^2(t_m) \|u(t_m) - u^m\|_{L^2}^2 \\ &+\sigma^2(t_m) \|\theta(t_m) - \theta^m\|_{L^2}^2 + \sigma^3(t_m) \|u(t_m) - u^m\|_{H^1}^2 \\ &+\sigma^3(t_m) \|\theta(t_m) - \theta^m\|_{H^1}^2 \leq \kappa \tau^4. \end{aligned} \tag{1.18}$$

where, $\sigma(t) = \min\{1, t\}$. Here and after, κ is used to denote a general positive constant depending on the data $(\omega, \alpha, \sigma, \gamma, d, \nu_1, \mu_1, \nu_2, \mu_2, f, F_1, F_2, u_0, \theta_0, T)$, which can take the different value at its different occurrences.

This paper is organized as follows. In §2, some basic mathematical setting and some important inequalities are recalled and some basic lemmas and some estimates of the nonlinear terms are provided. In §3, the time discretization of the second order decoupled implicit/explicit method is proposed for the 3D primitive equations of the ocean and the error estimates of the first time discrete solution are provided. In §4, the lower order error estimates of the time discrete solution (u^n, p^n, θ^n) are obtained. In §5, the second order error estimates of the time discrete solution (u^n, p^n, θ^n) are obtained, namely, Theorem 1.2 is proven.

2. Preliminaries

For the 3D domain Ω and 2D domain ω and $m \geq 0,\ p \geq 1$, we introduce the standard Sobolev spaces $H^m(\Omega)$ and $H^m(\omega)$ or $H^m(\Omega)^2$ and $H^m(\omega)^2$ with the norms $\|\cdot\|_{H^m}$ and $\|\cdot\|_{H^m(\omega)}$ and semi-norms $|\cdot|_{H^m}$ and $|\cdot|_{H^m(\omega)}$, respectively. For some detail cases of the Sobolev spaces, the reader can refer to Adams [1]. Set

$$H_2 = L^2(\Omega),\ X_2 = V_2 = H^1(\Omega),\ H_1 = \{v \in L^2(\Omega)^2; \mathrm{div}\ \bar{v} = 0,\ v \cdot n|_{\Gamma_s} = 0\},$$
$$X_1 = \{v \in H^1(\Omega)^2;\ v \cdot n|_{\Gamma_s} = 0\},$$

in the case of (2.6-1) and

$$X_1 = \{v \in H^1(\Omega)^2;\ v|_{\Gamma_s} = 0\},$$

in the case of (2.6-2), and $V_1 = X_1 \cap H_1$. And we introduce the Sobolev space X_0 in the 2D domain by

$$X_0 = \bar{X}_1 = \{\bar{v} \in H^1(\omega)^2; \bar{v} = d^{-1} \int_{-d}^{0} v(x, y, z) dz,\ v \in X_1\},\ V_0 = \{\bar{v} \in X_0; \nabla \cdot \bar{v} = 0\}.$$

We also use $(\cdot, \cdot)_\Omega$ and $(\cdot, \cdot)_\omega$ to denote the inner product in $L^2(\Omega)$ and $L^2(\omega)$ or $L^2(\Omega)^2$ and $L^2(\omega)^2$ or $L^2(\Omega)^4$ and $L^2(\omega)^4$, respectively.

We denote by A_1 the Stokes-type operator associated with the primitive equations(see [3, 13, 14]), that is $A_1 = PL_1$, where P is the L^2-orthogonal projection from $L^2(\Omega)^2$ to H_1. Also, we write $A_2 = L_2$. Therefore, we define the bilinear forms $a_i : X_i \times X_i \to R,\ i = 1, 2$ as follows:

$$a_1(u, v) = \nu_1(\nabla u, \nabla v)_\Omega + \mu_1(u_z, v_z)_\Omega = (L_1^{\frac{1}{2}} u, L_1^{\frac{1}{2}} v)_\Omega,$$

$$a_2(\theta, \eta) = \nu_2(\nabla\theta, \nabla\eta)_\Omega + \mu_2(\theta_z, \eta_z)_\Omega + \alpha(\theta(z = 0), \eta(z = 0))_\omega = (A_2^{\frac{1}{2}}\theta, A_2^{\frac{1}{2}}\eta)_\Omega,$$

where

$$a_1(u, v) = (A_1^{\frac{1}{2}} u, A_1^{\frac{1}{2}} v)_\Omega \ \forall u,\ v \in V_1.$$

Define

$$D(A_i) = \{\phi \in H_i; A_i\phi \in H_i\},\ i = 1, 2,$$

with the norm $\|A_i \cdot \|_{L^2}$.

We have the following Poincar$\acute{e}$ inequalities[1, 3, 13]:

$$\gamma_0\|u\|^2_{L^2} \le \nu_1\|\nabla u\|^2_{L^2}\ \forall u \in X_1; \tag{2.1}$$

$$\gamma_0\|\theta\|^2_{L^2} \le \mu_2(\|\partial_z\theta\|^2_{L^2} + \alpha\|\theta(z = 0)\|^2_{L^2(\omega)})\ \forall\theta \in X_2, \tag{2.2}$$

for some positive constant γ_0 depending on Ω or (ω, d). Here and after, we shall use the letters c and C(with or without subscripts) to denote the general positive constants depending on the data $(\omega, \alpha, \sigma, \gamma, d, \nu_1, \mu_1, \nu_2, \mu_2)$, which can take the different value at their different occurrences.

Also, we need a further assumption on the regularity results of the solution of the Stokes-type system associated with the primitive equations of ocean and the modified Poisson equation when the domain ω is sufficient smooth.

(**A1**). For a given $g_1 \in H^k(\Omega)^2$, the steady modified Stokes-type system

$$-\nu_1\Delta v - \mu_1\partial_{zz}v + \nabla q(x, y) = g_1 \text{ in } \Omega,\ \text{div } \bar{v}(x, y) = 0 \text{ in } \omega$$

admits a unique solution $(v, q) \in H^{2+k}(\Omega)^2 \times L^2_0(\omega) \cap H^{1+k}(\omega)$ for the boundary conditions $\partial_z v|_{\Gamma_u\cup\Gamma_b} = 0$ and $v \cdot n|_{\Gamma_s} = 0,\ \frac{\partial v}{\partial n} \times n|_{\Gamma_s} = 0$ or $v|_{\Gamma_s} = 0$ such that

$$\|v\|^2_{H^{2+k}} + \|q\|^2_{H^{1+k}(\omega)} \le c\|g_1\|^2_{H^k}, \tag{2.3}$$

for $k = 0,\ 1,\ 2$ and for a given $g_2 \in H^k(\Omega)$, the elliptic equation

$$-\nu_2\Delta\phi - \mu_2\partial_{zz}\phi = g_2 \text{ in } \Omega$$

admits a unique solution $\phi \in H^{2+k}(\Omega)$ for the boundary conditions $\partial_z\phi|_{\Gamma_b} = (\partial_z\phi + \alpha\phi)|_{\Gamma_u} = 0$, and $\frac{\partial\phi}{\partial n}|_{\Gamma_s} = 0$ such that

$$\|\phi\|_{H^{2+k}} \le c\|g_2\|^2_{H^k}, \tag{2.4}$$

for $k = 0,\ 1, 2$. In the case $k = 0,\ 1$, the assumption (**A1**) can be referred in [8] by He.

We usually make the following assumption about the prescribed data for problem (1.6)-(1.11):

(**A2**) The initial data $(u_0, \theta_0) \in D(A_1)\times D(A_2)$ and $(F_1, F_2) \in L^\infty([0, T]; L^2(\Omega)^2)\times L^\infty([0, T]; L^2(\Omega))$ such that for some positive constant C_0,

$$\begin{aligned}&\|A_1u_0\|^2_{L^2} + \|A_2\theta_0\|^2_{L^2} + \sup_{0\le t\le T}\{\|F_1(t)\|^2_{H^2} + \|F_2(t)\|^2_{H^2}\}\\ &+ \sup_{0\le t\le T}\{\|F_{1t}(t)\|^2_{H^2} + \|F_{2t}(t)\|^2_{H^2} + \|F_{1tt}(t)\|^2_{H^2} + \|F_{2tt}(t)\|^2_{H^2}\} \le C_0.\end{aligned} \tag{2.5}$$

Also, we recall the following important inequality(see [3, ?, 7])

$$
\begin{aligned}
&c_1\|\phi\|_{H^1}^2 \le \|L_i^{\frac{1}{2}}\phi\|_{L^2}^2 \le c_0\|\phi\|_{H^1}^2,\ \phi\in X_i,\\
&c_1\|\phi\|_{H^2}^2 \le \|A_i\phi\|_{L^2}^2 \le c_0\|\phi\|_{H^2}^2,\ \phi\in D(A_i),\\
&c_1\|\phi\|_{H^3}^2 \le \|A_i^{\frac{3}{2}}\phi\|_{L^2}^2 \le c_0\|\phi\|_{H^3}^2,\ \phi\in D(A_i^{\frac{3}{2}}),\\
&\|P\phi\|_{H^i}^2 \le c\|\phi\|_{H_i}^2,\ i=0,\ 1,\ 2,\ \phi\in H^i(\Omega)^2 \qquad (2.6)
\end{aligned}
$$

for $i=1,\ 2$ and

$$
\begin{aligned}
&\int_\omega\int_{-d}^0|\nabla u(x,y,\xi)|d\xi\int_{-d}^0|\phi_z||w|dzdxdy\\
&\le c_0\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\phi_z\|_{L^2}^{\frac{1}{2}}\|\nabla\phi_z\|_{L^2}^{\frac{1}{2}}\|w\|_{L^2}, \qquad (2.7)
\end{aligned}
$$

for $u\in D(A_1),\ (\phi,w)\in D(A_1)\times L^2(\Omega)^2$ or $(\phi,w)\in D(A_2)\times L^2(\Omega)$ and the following Sobolev and Ladyzhenskaya inequalities [1, 3, 4, 5, 15]:

$$
\|\phi\|_{L^3(\Omega)} \le c\|\phi\|_{L^2}^{\frac{1}{2}}\|\phi\|_{H^1}^{\frac{1}{2}},\ \|\phi\|_{L^6}+\|\phi\|_{L^4(\partial\Omega)}\le c_0\|\phi\|_{H^1}, \qquad (2.8)
$$

for all $\phi\in X_1$ or X_2, where the norm $\|\cdot\|_{L^q}$ denotes $\|\cdot\|_{L^q(\Omega)^2}$ or $\|\cdot\|_{L^q(\Omega)}$ and

$$
\begin{aligned}
&\|\phi\|_{L^4(\omega)}\le c\|\phi\|^{\frac{1}{2}}\|\phi\|_{H^1}^{\frac{1}{2}}\ \forall\phi\in H^1(\omega),\ \|\phi\|_{L^\infty(\omega)}\le c\|\phi\|^{\frac{1}{2}}\|\phi\|_{H^2}^{\frac{1}{2}}\ \forall\phi\in H^2(\omega),\\
&\|\phi\|_{L^\infty(\Omega)}+\|\nabla\phi\|_{L^3(\Omega)}+\|\partial_z\phi\|_{L^3(\Omega)}\le c_0\|A_i^{\frac{1}{2}}\phi\|_{L^2}^{\frac{1}{2}}\|A_i\phi\|_{L^2}^{\frac{1}{2}}, \qquad (2.9)
\end{aligned}
$$

for all $\phi\in H^2(\Omega)^2\cap V_i$ with $i=1,\ 2$.

It is easy to see [3] that

$$
\begin{aligned}
&(\int_{-d}^z\nabla\cdot v(x,y,\xi)d\xi,\phi)_\Omega-(\int_{-d}^z\nabla\phi(x,y,\xi)d\xi,v)_\Omega\\
&=(\int_{-d}^0\nabla\cdot v(x,y,\xi)d\xi,\int_{-d}^0\phi(x,y,\xi)d\xi)_\omega\ \forall v\in X_1,\ \phi\in X_2, \qquad (2.10)
\end{aligned}
$$

$$
f\vec{k}\times v\cdot v|v|^{4k}=0\ \forall v\in L^2(\Omega)^2,\ k=0,1. \qquad (2.11)
$$

Moreover, we define the trilinear forms:

$$
b(v,\phi,\psi)=((v\cdot\nabla)\phi,\psi)_\Omega-((\int_{-d}^z\nabla\cdot vd\xi)\partial_z\phi,\psi)_\Omega,
$$

for all $v\in X_1,\ (\phi,\psi)\in H^1(\Omega)^2\times H^1(\Omega)^2$ or $(\phi,\psi)\in H^1(\Omega)\times H^1(\Omega)$. It is easy to see that

$$
b(v,\phi,\phi|\phi|^{4k})=-\frac{1}{2+4k}(\int_{-d}^0\nabla\cdot vd\xi,|\phi(z=0)|^{2+4k})_\omega, \qquad (2.12)
$$

for all $v\in X_1$, $\phi\in H^1(\Omega)$ or $H^1(\Omega)^2$ and $k=0,\ 1$.

With the above statements and the boundary conditions (1.6)-(1.7), we deduce the weak form: find $(u,p,\theta)(t)\in X_1\times M_0\times X_2$ with $t\ge 0$ such that $(v,q,\eta)\in X_1\times M_0\times X_2$

$$
\begin{aligned}
&(u_t,v)_\Omega+a_1(u,v)+b(u,u,v)+(f\vec{k}\times u,v)_\Omega-\gamma(\int_{-d}^z\nabla\theta(x,y,\xi,t)d\xi,\phi)_\Omega\\
&-(\nabla\cdot v,p)_\Omega+(\nabla\cdot u,q)_\Omega=(F_1,v)_\Omega, \qquad (2.13)
\end{aligned}
$$

$$
(\theta_t,\phi)_\Omega+a_2(\theta,\phi)+b(u,\theta,\phi)+\sigma(\int_{-d}^z\nabla\cdot u(x,y,\xi,t)d\xi,\phi)_\Omega=(F_2,\phi)_\Omega, \qquad (2.14)
$$

together with the initial condition (1.8). In order to analyze the stability and convergence of the numerical solution, we need the following uniform Gronwall lemma [20].

Lemma 2.1. Let C_0 is a positive constant and $a_n,\ b_n,\ d_n$ be three positive series satisfying

$$a_m + \tau \sum_{n=1}^{m} b_n \le \tau \sum_{n=0}^{m-1} d_n a_n + C_0,\ m \ge 1, \tag{2.15}$$

Then

$$a_m + \tau \sum_{n=1}^{m} b_n \le C_0 \exp(\tau \sum_{n=0}^{m-1} d_n)\ \forall m \ge 1. \tag{2.16}$$

Lemma 2.2. Let $\Omega_1 \subset R^{m_1}$ and $\Omega_2 \subset R^{m_2}$ be two measurable sets, where m_1 and m_2 are positive integers. Suppose that $f(\xi,\eta)$ is measurable over $\Omega_1 \times \Omega_2$. Then

$$[\int_{\Omega_1} (\int_{\Omega_2} |f(\xi,\eta)| d\eta)^p d\xi]^{\frac{1}{p}} \le \int_{\Omega_2} (\int_{\Omega_1} |f(\xi,\eta)|^p d\xi)^{\frac{1}{p}} d\eta. \tag{2.17}$$

Theorem 2.1. Assume that assumptions (**A1**)-(**A2**) hold. Then, the solution (u,p,θ) of the 3D viscous PEs satisfies the following bounds:

$$\begin{aligned}\sigma^{2m-2}(t)[\|\partial_t^m u(t)\|_{L^2}^2 + \|\partial_t^m \theta(t)\|_{L^2}^2 + \sigma(t)(\|\partial_t^m u(t)\|_{H^1}^2 + \|\partial_t^m \theta(t)\|_{H^1}^2 \\ + \|\partial_t^{m-1} u(t)\|_{H^2}^2 + \|\partial_t^{m-1} \theta(t)\|_{H^2}^2 + \|\partial_t^{m-1} p(t)\|_{H^1(\omega)}^2)] \le \kappa,\end{aligned} \tag{2.18}$$

$$\begin{aligned}\int_0^t \sigma^{2m-2}(s)[\|\partial_t^m u\|_{H^1}^2 + \|\partial_t^m \theta\|_{H^1}^2] ds + \int_0^t \sigma^{2m-1}(s) \|\partial_t^m u\|_{H^2}^2 ds \\ + \int_0^t \sigma^{2m-2}(s)[\|\partial_t^m \theta\|_{H^2}^2 + \|\partial_t^m p\|_{H^1(\omega)}^2)] ds \le \kappa,\end{aligned} \tag{2.19}$$

$$\int_0^t \sigma^{2m-1}(s)(\|\partial_t^{m+1} u\|_{L^2}^2 + \|\partial_t^{m+1} \theta\|_{L^2}^2) ds \le \kappa, \tag{2.20}$$

for all $0 \le t \le T$ and $m = 1,2,3$, and

$$\sigma(t)[\|u(t)\|_{H^3}^2 + \|\theta(t)\|_{H^3}^2 + \|p(t)\|_{H^2(\omega)}^2] \le \kappa, \tag{2.21}$$

$$\sigma^3(t)[\|\partial_t u(t)\|_{H^3}^2 + \|\partial_t \theta(t)\|_{H^3}^2 + \|\partial_t p(t)\|_{H^2(\omega)}^2] \le \kappa, \tag{2.22}$$

$$\int_0^T [\|u(t)\|_{H^3}^2 + \|\theta(t)\|_{H^3}^2 + \|p(t)\|_{H^2(\omega)}^2] dt \le \kappa, \tag{2.23}$$

$$\int_0^T \sigma^2(t)[\|\partial_t u(t)\|_{H^3}^2 + \|\partial_t \theta(t)\|_{H^3}^2 + \|\partial_t p(t)\|_{H^2(\omega)}^2] dt \le \kappa, \tag{2.24}$$

$$\int_0^T \sigma^3(t)[\|\partial_t u(t)\|_{H^4}^2 + \|\partial_t \theta(t)\|_{H^4}^2 + \|\partial_t p(t)\|_{H^3(\omega)}^2] dt \le \kappa, \tag{2.25}$$

and

$$\sigma^3(t)[\|\partial_t u(t)\|_{H^3}^2 + \|\partial_t\theta(t)\|_{H^3}^2 + \|\partial_t p(t)\|_{H^2(\omega)}^2] \le \kappa, \tag{2.26}$$

$$\sigma^3(t)[\|\partial_t^2 u(t)\|_{H^3}^2 + \|\partial_t^2\theta(t)\|_{H^3}^2 + \|\partial_t^2 p(t)\|_{H^2(\omega)}^2] \le \kappa, \tag{2.27}$$

$$\int_0^T \sigma^2(t)[\|\partial_t u(t)\|_{H^3}^2 + \|\partial_t\theta(t)\|_{H^3}^2 + \|\partial_t p(t)\|_{H^2(\omega)}^2]dt \le \kappa, \tag{2.28}$$

$$\int_0^T \sigma^4(t)[\|\partial_t^2 u(t)\|_{H^3}^2 + \|\partial_t^2\theta(t)\|_{H^3}^2 + \|\partial_t^2 p(t)\|_{H^2(\omega)}^2]dt \le \kappa, \tag{2.29}$$

$$\int_0^T \sigma^5(t)[\|\partial_t^2 u(t)\|_{H^4}^2 + \|\partial_t^2\theta(t)\|_{H^4}^2 + \|\partial_t^2 p(t)\|_{H^3(\omega)}^2]dt \le \kappa, \tag{2.30}$$

for all $0 \le t \le T$.

Proof. Under the assumption (**A2**), we define $F_1|_{[T,\infty)} = 0$, $F_2|_{[T,\infty)} = 0$. Then, we deduce (2.18)-(2.20) by Theorem 1.1. Moreover, by the assumption (**A1**) and (1.9)-(1.11), we deduce

$$\begin{aligned}
&\|u(t)\|_{H^3} + \|p(t)\|_{H^2(\omega)}\\
&\le c\|\partial_t u(t)\|_{H^1} + c\|F_1(t)\|_{H^1} + c\|\theta(t)\|_{H^2} + c\|u(t)\|_{H^1} + c\|u(t)\|_{H^2}^{\frac{3}{2}}\|u(t)\|_{H^3}^{\frac{1}{2}}\\
&\le \frac{1}{2}\|u(t)\|_{H^3} + c\|\partial_t u(t)\|_{H^1} + c\|F_1(t)\|_{H^1} + c\|\theta(t)\|_{H^2} + c\|u(t)\|_{H^1} + c\|u(t)\|_{H^2}^3,\\
&\|\theta(t)\|_{H^3} \le c\|\partial_t\theta(t)\|_{H^1} + c\|F_2(t)\|_{H^1} + c\|u(t)\|_{H^2} + c\|u(t)\|_{H^2}\|\theta(t)\|_{H^2}^{\frac{1}{2}}\|\theta(t)\|_{H^3}^{\frac{1}{2}}\\
&\le \frac{1}{2}\|\theta(t)\|_{H^3} + c\|\partial_t\theta(t)\|_{H^1} + c\|F_2(t)\|_{H^1} + c\|u(t)\|_{H^2} + c\|u(t)\|_{H^2}^2\|\theta(t)\|_{H^2},
\end{aligned}$$

which and (2.18)-(2.19) with $m = 1$ yield (2.21) and (2.23).

Furthermore, we deduce from (1.9)-(1.11) that

$$\begin{aligned}
L_1\partial_t u + \nabla\partial_t p(x,y,t) = &-\partial_t^2 u + \gamma\int_{-d}^z \nabla\partial_t\theta(x,y,\xi,t)d\xi - f\vec{k}\times\partial_t u\\
&-(\partial_t u\cdot\nabla)u - (u\cdot\nabla)\partial_t u + (\int_{-d}^z \nabla\cdot\partial_t u(x,y,\xi,t)d\xi)\partial_z u\\
&+(\int_{-d}^z \nabla\cdot u(x,y,\xi,t)d\xi)\partial_z\partial_t u + F_1,
\end{aligned}\tag{2.31}$$

$$\begin{aligned}
L_2\partial_t\theta = &-\partial_t^2\theta - \sigma\int_{-d}^z \nabla\cdot\partial_t u(x,y,\xi,t)d\xi - (\partial_t u\cdot\nabla)\theta - (u\cdot\nabla)\partial_t\theta\\
&+(\int_{-d}^z \nabla\cdot\partial_t u(x,y,\xi,t)d\xi)\partial_z\theta + (\int_{-d}^z \nabla\cdot u(x,y,\xi,t)d\xi)\partial_z\partial_t\theta + F_2,
\end{aligned}\tag{2.32}$$

$$\nabla\cdot\partial_t\bar{u} = 0. \tag{2.33}$$

Hence, by (2.1)-(2.2), (2.7)-(2.9), Lemma 2.2 and some simple calculations, we deduce from the assumption (**A1**) and (2.29)-(2.31) that

$$\begin{aligned}
\|\partial_t u(t)\|_{H^3} + \|\partial_t p(t)\|_{H^2(\omega)} &\le c\|\partial_t^2 u(t)\|_{H^1} + c\|\partial_t F_1(t)\|_{H^1} + c\|\partial_t\theta(t)\|_{H^2}\\
&\quad + c\|\partial_t u(t)\|_{H^1} + c\|\partial_t u(t)\|_{H^2}\|u(t)\|_{H^3},\\
\|\partial_t\theta(t)\|_{H^3} &\le c\|\partial_t^2\theta(t)\|_{H^1} + c\|\partial_t F_2(t)\|_{H^1} + c\|\partial_t u(t)\|_{H^2} + c\|\partial_t u(t)\|_{H^2}\|\theta(t)\|_{H^3}\\
&\quad + c\|u(t)\|_{H^3}\|\partial_t\theta(t)\|_{H^2},\\
\|\partial_t u(t)\|_{H^4} + \|\partial_t p(t)\|_{H^3(\omega)} &\le c\|\partial_t^2 u(t)\|_{H^2} + c\|\partial_t F_1(t)\|_{H^2} + c\|\partial_t\theta(t)\|_{H^3}\\
&\quad + c\|\partial_t u(t)\|_{H^2} + c\|\partial_t u(t)\|_{H^3}\|u(t)\|_{H^3},\\
\|\partial_t\theta(t)\|_{H^4} &\le c\|\partial_t^2\theta(t)\|_{H^2} + c\|\partial_t F_2(t)\|_{H^2} + c\|\partial_t u(t)\|_{H^3} + c\|\partial_t u(t)\|_{H^3}\|\theta(t)\|_{H^3}\\
&\quad + c\|u(t)\|_{H^3}\|\partial_t\theta(t)\|_{H^3},
\end{aligned}$$

which and (2.18)-(2.19) with $m = 1,\ 2, 3$ and (2.21) yield (2.22) and (2.24)-(2.25). Similarly, we can prove (2.26)-(2.30). The proof ends.

3. Second order decoupled implicit/explicit scheme

Setting τ be time step size, $t_n = n\tau,\ T = N\tau$ and $u^0 = u_0,\ \theta^0 = \theta_0$, then we can define the second order time discrete schemes of (1.9)-(1.11) as follows.

Second order decoupled implicit/explicit scheme: For each n and find (θ^n, u^n, p^n) satisfying

$$\begin{aligned}
&(d_t u^n, v)_\Omega + a_1(\hat{u}^n, v) - (\nabla\cdot v, p^n)_\Omega + (\nabla\cdot\hat{u}^n, q)_\Omega - \gamma\Big(\int_{-d}^{z}\nabla Z(\theta^n)d\xi, v\Big)_\Omega\\
&\qquad + (f\vec{k}\times Z(u^n), v)_\Omega + Z(b(u^n, u^n, v)) = (F_1^n, v)_\Omega, \qquad (3.1)\\
&(d_t\theta^n, \phi)_\Omega + a_2(\hat{\theta}^n, \phi) + \sigma\Big(\int_{-d}^{z}\nabla\cdot Z(u^n)d\xi, \phi\Big)_\Omega + Z(b(u^n, \theta^n, \phi))\\
&\qquad = (F_2^n, \phi)_\Omega, \qquad (3.2)
\end{aligned}$$

for $2 \le n \le N$ with $(v, q, \phi) \in X_1\times M\times X_2$, where $Z(\xi^n) = \frac{3}{2}\xi^{n-1} - \frac{1}{2}\xi^{n-2}, \hat{\xi}^n = \frac{1}{2}(\xi^n + \xi^{n-1})$ and

$$d_t u^n = \frac{1}{\tau}(u^n - u^{n-1}),\ d_t\theta^n = \frac{1}{\tau}(\theta^n - \theta^{n-1}),$$

$$F_1^n = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} F_1(t)dt,\ F_2^n = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} F_2(t)dt.$$

Here (u^1, p^1, θ^1) is defined by the first order implicit/explicit scheme:

$$\begin{aligned}
&(d_t u^1, v)_\Omega + a_1(u^1, v) - (\nabla\cdot v, p^1)_\Omega + (\nabla\cdot u^1, q)_\Omega - \gamma\Big(\int_{-d}^{z}\nabla\theta^0 d\xi, v\Big)_\Omega\\
&\qquad + (f\vec{k}\times u^0, v)_\Omega + b(u^0, u^0, v) = (F_1^1, v)_\Omega,\\
&(d_t\theta^1, \phi)_\Omega + a_2(\theta^1, \phi) + \sigma\Big(\int_{-d}^{z}\nabla\cdot u^0 d\xi, \phi\Big)_\Omega + b(u^0, \theta^0, \phi) = (F_2^1, \phi)_\Omega.
\end{aligned}$$

Here, given (u^{n-1}, θ^{n-1}), (u^n, p^n) can be solved by (3.1) without θ^n and θ^n can be solved by (3.2) without (u^n, p^n), respectively. Set

$$u^n = \bar{u}^n + \tilde{u}^n,\ \bar{u}^n = d^{-1}\int_{-d}^{0} u^n(x, y, z)dz,\ \tilde{u}^n = u^n - \bar{u}^n,$$

then (3.1) can be rewritten as follows: for each n and (u^{n-1}, θ^{n-1}), find $(\bar{u}^n, p^n, \tilde{u}^n)$ such that

$$
\begin{aligned}
&(d_t\bar{u}^n, \bar{v})_\Omega + a_1(\hat{\bar{u}}^n, \bar{v}) - (\nabla\cdot\bar{v}, p^n) + (\nabla\cdot\hat{\bar{u}}^n, q)_\Omega - \gamma(\int_{-d}^{z}\nabla Z(\theta^n)d\xi, \bar{v})_\Omega \\
&\qquad + (f\vec{k}\times Z(\bar{u}^n), \bar{v})_\Omega + Z(b(u^n, u^n, \bar{v})) = (\bar{F}_1^n, \bar{v})_\Omega, \qquad (3.3)\\
&(d_t\tilde{u}^n, \tilde{v})_\Omega + a_1(\hat{\tilde{u}}^n, \tilde{v}) - \gamma(\int_{-d}^{z}\nabla Z(\theta^n)d\xi, \tilde{v})_\Omega + (f\vec{k}\times Z(\tilde{u}^n), \tilde{v})_\Omega \\
&\qquad + Z(b(u^n, u^n, \tilde{v})) = (\tilde{F}_1^n, \tilde{v})_\Omega, \qquad (3.4)
\end{aligned}
$$

for $2 \le n \le N$ with $v \in X_1$ and $q \in M$, together with the initial condition:

$$\bar{u}^0 = \bar{u}_0(x,y),\ \tilde{u}^0 = \tilde{u}_0(x,y,z).$$

Here $(\bar{u}^n, p^n)$ can be solved by the Stokes equations (3.3) in the 2D domain ω without $\tilde{u}^n$ and $\tilde{u}^n$ can be solved by the elliptic equations (3.4) in the 3D domain Ω without $\tilde{u}^n$. Hence, the scheme (3.2)-(3.4) is a second order decoupled implicit/explicit scheme with respect to θ^n, $(\bar{u}^n, p^n)$ and $\tilde{u}^n$.

From the definition of (u^1, p^1, θ^1), the assumptions (**A1**)-(**A2**) and (2.13)-(2.14), we deduce the following stability and convergence results.

Lemma 3.1. Suppose that the assumptions (**A1**)-(**A2**) hold, then there hold

$$
\begin{aligned}
\|A_1^{-1}e^1\|_{L^2}^2 + \tau\|A_1^{-\frac{1}{2}}e^1\|_{L^2}^2 + \|A_2^{-1}\varepsilon^1\|_{L^2}^2 + \tau\|A_2^{-\frac{1}{2}}\varepsilon^1\|_{L^2}^2 &\le \kappa\tau^4,\\
\|A_1^{-\frac{1}{2}}e^1\|_{L^2}^2 + \tau\|e^1\|_{L^2}^2 + \|A_2^{-\frac{1}{2}}\varepsilon^1\|_{L^2}^2 + \tau\|\varepsilon^1\|_{L^2}^2 &\le \kappa\tau^3,\\
\|e^1\|_{L^2}^2 + \tau\|A_1^{\frac{1}{2}}e^1\|_{L^2}^2 + \|\varepsilon^1\|_{L^2}^2 + \tau\|A_2^{\frac{1}{2}}\varepsilon^1\|_{L^2}^2 &\le \kappa\tau^2,\\
\|A_1^{\frac{1}{2}}e^1\|_{L^2}^2 + \tau\|A_1e^1\|_{L^2}^2 + \|A_2^{\frac{1}{2}}\varepsilon^1\|_{L^2}^2 + \tau\|A_2\varepsilon^1\|_{L^2}^2 &\le \kappa\tau,\\
\|A_1e^1\|_{L^2}^2 + \tau\|A_1^{\frac{3}{2}}e^1\|_{L^2}^2 + \|A_2\varepsilon^1\|_{L^2}^2 + \tau\|A_2^{\frac{3}{2}}\varepsilon^1\|_{L^2}^2 &\le \kappa,\\
\|r^1\|_{L^2}^2 &\le \kappa,
\end{aligned}
$$

where $(e^1, r^1, \varepsilon^1) = (u(t_1) - u^1, \frac{1}{\tau}\int_{t_0}^{t_1} p(t)dt - p^1, \theta(t_1) - \theta^1)$.

Proof. By (2.1)-(2.2), (2.6)-(2.9), Lemma 2.2 and some simple computations, Lemma 3.1 can be proved.

Finally, using (2.1)-(2.2, (2.6)-(2.12) and Lemma 2.2, we deduce the following important estimates of the trilinear form b, which are useful in proof of the error estimates of the time discrete solution (u^n, p^n, θ^n).

Lemma 3.2. There hold the following estimates for the trilinear form b:

$$
\begin{aligned}
&|b(u,v,v)| = 0 \ \forall u \in V_1,\ v \in X_i,\\
&|b(u,v,w)| \le c\|u\|_{L^2}\|A_i^{\frac{1}{2}}w\|_{L^2}\|v\|_{H^3}\ \forall u \in H_1, v \in H^3(\Omega)^{3-i}\cap X_i, w \in V_i,\\
&|b(u,w,v)| \le c\|u\|_{H^3}\|A_i^{\frac{1}{2}}w\|_{L^2}\|v\|_{L^2}\ \forall u \in H^3(\Omega)^2\cap H_1, v \in H_i, w \in V_i,\\
&|b(u,w,v)| \le c\|A_1^{\frac{1}{2}}u\|_{L^2}\|w\|_{H^3}\|v\|_{L^2}\ \forall u \in V_1, w \in H^3(\Omega)^{3-i}\cap X_i, v \in H_i,\\
&|b(u,w,v)| \le c\|A_1u\|_{L^2}\|w\|_{H^3}\|A_i^{-\frac{1}{2}}v\|_{L^2}\ \forall u \in D(A_1),\ w \in H^3(\Omega)^{3-i}\cap X_i,\ v \in H_i,\\
&|b(u,w,v)| \le c\|A_1^{\frac{1}{2}}u\|_{L^2}^{\frac{1}{2}}\|A_1u\|_{L^2}^{\frac{1}{2}}\|A_i^{\frac{1}{2}}w\|_{L^2}^{\frac{1}{2}}\|A_iw\|_{L^2}^{\frac{1}{2}}\|v\|_{L^2}\ \forall u \in D(A_1),\ w \in D(A_i),\ v \in H_i,\\
&|b(u,w,v)| \le c\|u\|_{H^3}\|A_iw\|_{L^2}\|A_i^{-\frac{1}{2}}v\|_{L^2}\ \forall u \in H^3(\Omega)^2\cap X_1,\ w \in D(A_i),\ v \in H_i,\\
&|b(u,w,v)| \le c\|A_1^{-\frac{1}{2}}u\|_{L^2}\|w\|_{H^3}\|A_iv\|_{L^2}\ \forall u \in H_1,\ w \in H^3(\Omega)^{3-i}\cap X_i,\ v \in D(A_i),\\
&|b(u,w,v)| \le c\|u\|_{H^3}\|w\|_{H^3}\|A_i^{-1}v\|_{L^2}\ \forall u \in H^3(\Omega)^2\cap X_1,\ w \in H^3(\Omega)^{3-i}\cap X_i,\ v \in H_i,
\end{aligned}
$$

where $i = 1, 2$.

4. Lower-order error estimates

In this section, we shall provide the lower-order error estimates of the numerical solution (u^n, p^n, θ^n).

To this end, we apply the integrate operator $\frac{1}{\tau}\int_{t_{n-1}}^{t_n} \cdot dt$ to (2.13) and (2.14) and using the following formulas:

$$
\hat{\psi}(t_n) - \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\psi(t)dt = \frac{1}{2\tau}\int_{t_{n-1}}^{t_n}(t-t_{n-1})(t_n-t)\psi_{tt}(t)dt, \tag{4.1}
$$

$$
\begin{aligned}
Z(\psi(t_n)) - \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\psi(t)dt &= \frac{1}{2\tau}\int_{t_{n-1}}^{t_n}(t-t_{n-1})(t_n-t)\psi_{tt}(t)dt\\
&+ \frac{1}{2}\int_{t_{n-1}}^{t_n}(t-t_n)\psi_{tt}(t)dt - \frac{1}{2}\int_{t_{n-2}}^{t_{n-1}}(t-t_{n-2})\psi_{tt}(t)dt,
\end{aligned} \tag{4.2}
$$

for each $\psi \in H^2(t_{n-1}, t_n)$, to obtain

$$
\begin{aligned}
&(d_tu(t_n), v)_\Omega + a_1(\hat{u}(t_n), v) + Z(b(u(t_n), u(t_n), v)) + (f\vec{k}\times Z(u(t_n)), v)_\Omega\\
&\qquad - \gamma(\int_{-d}^{z}\nabla Z(\theta(t_n))d\xi, v)_\Omega - (\nabla\cdot v, \frac{1}{\tau}\int_{t_{n-1}}^{t_n}p(t)dt)_\Omega + (\nabla\cdot u(t_n), q)_\Omega\\
&\qquad = (F_1^n, v)_\Omega + (E_1^n, v)_\Omega,
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
&(d_t\theta(t_n), \phi)_\Omega + a_2(\hat{\theta}(t_n), \phi) + Z(b(u(t_n), \theta(t_n), \phi)) + \sigma(\int_{-d}^{z}\nabla\cdot Z(u(t_n))d\xi, \phi)_\Omega\\
&\qquad = (F_2^n, \phi)_\Omega + (E_2^n, \phi)_\Omega,
\end{aligned} \tag{4.4}
$$

for $2 \le n \le N$ with $(v, q, \phi) \in X_1 \times M_0 \times X_2$, where

$$
\begin{aligned}
(E_1^n, v)_\Omega = &\frac{1}{2\tau}\int_{t_{n-1}}^{t_n}(t_n - t)(t - t_{n-1})a_1(u_{tt}(t), v)dt \\
&+ \frac{1}{2\tau}\int_{t_{n-1}}^{t_n}(t_n - t)(t - t_{n-1})b_{tt}(u(t), u(t), v)dt \\
&+ \frac{1}{2}\int_{t_{n-1}}^{t_n}(t - t_n)b_{tt}(u(t), u(t), v)dt - \frac{1}{2}\int_{t_{n-2}}^{t_{n-1}}(t - t_{n-2})b_{tt}(u(t), u(t), v)dt \\
&+ \frac{1}{2\tau}\int_{t_{n-1}}^{t_n}(t_n - t)(t - t_{n-1})(f\vec{k} \times u_{tt}(t), v)_\Omega dt \\
&+ \frac{1}{2}\int_{t_{n-1}}^{t_n}(t - t_n)(f\vec{k} \times u_{tt}(t), v)_\Omega dt - \frac{1}{2}\int_{t_{n-2}}^{t_{n-1}}(t - t_{n-2})(f\vec{k} \times u_{tt}(t), v)_\Omega dt \\
&- \frac{\gamma}{2\tau}\int_{t_{n-1}}^{t_n}(t_n - t)(t - t_{n-1})(\int_{-d}^{z}\nabla\theta_{tt}(t)d\xi, v)_\Omega dt \\
&- \frac{\gamma}{2}\int_{t_{n-1}}^{t_n}(t - t_n)(\int_{-d}^{z}\nabla\theta_{tt}(t)d\xi, v)_\Omega dt \\
&+ \frac{\gamma}{2}\int_{t_{n-1}}^{t_n}(t - t_{n-2})(\int_{-d}^{z}\nabla\theta_{tt}(t)d\xi, v)_\Omega dt, \qquad (4.5)
\end{aligned}
$$

$$
\begin{aligned}
(E_2^n, \phi)_\Omega = &\frac{1}{2\tau}\int_{t_{n-1}}^{t_n}(t_n - t)(t - t_{n-1})a_2(\theta_{tt}(t), \phi)dt \\
&+ \frac{1}{2\tau}\int_{t_{n-1}}^{t_n}(t_n - t)(t - t_{n-1})b_{tt}(u(t), \theta(t), \phi)dt \\
&+ \frac{1}{2}\int_{t_{n-1}}^{t_n}(t - t_n)b_{tt}(u(t), \theta(t), \phi)dt - \frac{1}{2}\int_{t_{n-2}}^{t_{n-1}}(t - t_{n-2})b_{tt}(u(t), \theta(t), \phi)dt \\
&+ \frac{\sigma}{2\tau}\int_{t_{n-1}}^{t_n}(t_n - t)(t - t_{n-1})(\int_{-d}^{z}\nabla \cdot u_{tt}(t)d\xi, \phi)_\Omega dt \\
&+ \frac{\sigma}{2}\int_{t_{n-1}}^{t_n}(t - t_n)(\int_{-d}^{z}\nabla \cdot u_{tt}(t)d\xi, \phi)_\Omega dt \\
&- \frac{\sigma}{2}\int_{t_{n-2}}^{t_{n-1}}(t - t_{n-2})(\int_{-d}^{z}\nabla \cdot u_{tt}(t)d\xi, \phi)_\Omega dt, \qquad (4.6)
\end{aligned}
$$

where

$$b_{tt}(u(t), u(t), v) = b(u_{tt}(t), u(t), v) + b(u(t), u_{tt}(t), v) + 2b(u_t(t), u_t(t), v),$$

$$b_{tt}(u(t), \theta(t), v) = b(u_{tt}(t), \theta(t), v) + b(u(t), \theta_{tt}(t), v) + 2b(u_t(t), \theta_t(t), v).$$

From Theorem 2.1 and (4.5)-(4.6), we can deduce the following estimates for the error pair (E_1^n, E_2^n).

Lemma 4.1. Under the assumptions of Theorem 2.1, the error pair (E_1^n, E_2^n) satisfies the following estimates:

$$
\begin{aligned}
\tau\sum_{n=2}^{m}(\|A_1^{-\frac{3}{2}}PE_1^n\|_{L^2}^2+\|A_2^{-\frac{3}{2}}E_2^n\|_{L^2}^2) &\le \kappa\tau^4,\\
\tau\sum_{n=2}^{m}\sigma^i(t_n)(\|A_1^{-1}PE_1^n\|_{L^2}^2+\|A_2^{-1}E_2^n\|_{L^2}^2) &\le \kappa\tau^{3+i},\ i=0,1,\\
\tau\sum_{n=2}^{m}\sigma^i(t_n)(\|A_1^{-\frac{1}{2}}PE_1^n\|_{L^2}^2+\|A_2^{-\frac{1}{2}}E_2^n\|_{L^2}^2) &\le \kappa\tau^{2+i},\ i=0,1,2,\\
\tau\sum_{n=2}^{m}\sigma^i(t_n)(\|E_1^n\|_{L^2}^2+\|E_2^n\|_{L^2}^2) &\le \kappa\tau^{1+i},\ i=0,1,2,3,\\
\tau\sum_{n=2}^{m}\sigma^i(t_n)(\|A_1^{\frac{1}{2}}PE_1^n\|_{L^2}^2+\|A_2^{\frac{1}{2}}E_2^n\|_{L^2}^2) &\le \kappa\tau^{i},\ i=0,1,2,3,4,\\
\tau^2\sum_{n=2}^{m}\sigma^i(t_n)(\|A_1PE_1^n\|_{L^2}^2+\|A_2E_2^n\|_{L^2}^2) &\le \kappa\tau^{i},\ i=0,1,2,3,4.
\end{aligned}
$$

By (2.1)-(2.2), (2.6) and Lemma 3.2 and some simple computations, we deduce

$$
\begin{aligned}
\tau\|A_1^{-\frac{3}{2}}PE_1^n\|_{L^2}^2 \le\ & c\tau^4(1+\max_{0\le t\le T}\|A_1u(t)\|_{L^2}^2)\\
&\times\int_{t_{n-2}}^{t_n}[\|A_1^{-\frac{1}{2}}u_{tt}(t)\|_{L^2}^2+\|A_2^{-\frac{1}{2}}\theta_{tt}\|_{L^2}^2]dt\\
&+c\tau^4\max_{0\le t\le T}\|u_t(t)\|_{L^2}^2\int_{t_{n-2}}^{t_n}\|A_1^{\frac{1}{2}}u_t(t)\|_{L^2}^2dt,\\
\tau\|A_2^{-\frac{3}{2}}E_2^n\|_{L^2}^2 \le\ & c\tau^4[1+\max_{0\le t\le T}(\|A_1u(t)\|_{L^2}^2+\|A_2\theta(t)\|_{L^2}^2)]\\
&\times\int_{t_{n-2}}^{t_n}[\|A_1^{-\frac{1}{2}}u_{tt}(t)\|_{L^2}^2+\|A_2^{-\frac{1}{2}}\theta_{tt}(t)\|_{L^2}^2]dt\\
&+c\tau^4\max_{0\le t\le T}\|\theta_t(t)\|_{L^2}^2\int_{t_{n-2}}^{t_n}\|A_1^{\frac{1}{2}}u_t(t)\|_{L^2}^2dt,
\end{aligned}\tag{4.7}
$$

$$
\begin{aligned}
\tau\sigma^i(t_n)\|A_1^{-1}PE_1^n\|_{L^2}^2 \le\ & c\tau^{3+i}(1+\max_{0\le t\le T}\|A_1u(t)\|_{L^2}^2)\\
&\times\int_{t_{n-2}}^{t_n}\sigma(t)[\|u_{tt}(t)\|_{L^2}^2+\|\theta_{tt}\|_{L^2}^2]dt\\
&+c\tau^{3+i}\max_{0\le t\le T}\sigma(t)\|A_1^{\frac{1}{2}}u_t(t)\|_{L^2}^2\int_{t_{n-2}}^{t_n}\|A_1^{\frac{1}{2}}u_t(t)\|_{L^2}^2dt,\\
\tau\sigma^i(t_n)\|A_2^{-1}E_2^n\|_{L^2}^2 \le\ & c\tau^{3+i}[1+\max_{0\le t\le T}(\|A_1u(t)\|_{L^2}^2+\|A_2\theta(t)\|_{L^2}^2)]\\
&\times\int_{t_{n-2}}^{t_n}\sigma(t)[\|u_{tt}(t)\|_{L^2}^2+\|\theta_{tt}(t)\|_{L^2}^2]dt\\
&+c\tau^{3+i}\max_{0\le t\le T}\sigma(t)\|A_2^{\frac{1}{2}}\theta_t(t)\|_{L^2}^2\int_{t_{n-2}}^{t_n}\|A_1^{\frac{1}{2}}u_t(t)\|_{L^2}^2dt,
\end{aligned}\tag{4.8}
$$

for $i=0,\ 1$ and

$$\begin{aligned}
\tau\sigma^i(t_n)\|A_1^{-\frac{1}{2}}PE_1^n\|_{L^2}^2 \le{}& c\tau^{2+i}(1+\max_{0\le t\le T}\|A_1u(t)\|_{L^2}^2)\\
&\times\int_{t_{n-2}}^{t_n}\sigma^2(t)[\|A_1^{\frac{1}{2}}u_{tt}(t)\|_{L^2}^2+\|A_2^{\frac{1}{2}}\theta_{tt}\|_{L^2}^2]dt\\
&+c\tau^{2+i}\max_{0\le t\le T}\sigma^2(t)\|A_1u_t(t)\|_{L^2}^2\int_{t_{n-2}}^{t_n}\|A_1^{\frac{1}{2}}u_t(t)\|_{L^2}^2dt,\\
\tau\sigma^i(t_n)\|A_2^{-\frac{1}{2}}E_2^n\|_{L^2}^2 \le{}& c\tau^{2+i}[1+\max_{0\le t\le T}(\|A_1u(t)\|_{L^2}^2+\|A_2\theta(t)\|_{L^2}^2)]\\
&\times\int_{t_{n-2}}^{t_n}\sigma^2(t)[\|A_1^{\frac{1}{2}}u_{tt}(t)\|_{L^2}^2+\|A_2^{\frac{1}{2}}\theta_{tt}(t)\|_{L^2}^2]dt\\
&+c\tau^{2+i}\max_{0\le t\le T}\sigma^2(t)\|A_2\theta_t(t)\|_{L^2}^2\int_{t_{n-2}}^{t_n}\|A_1^{\frac{1}{2}}u_t(t)\|_{L^2}^2dt, \qquad (4.9)
\end{aligned}$$

for $i=0,\ 1,\ 2$ and

$$\begin{aligned}
\tau\sigma^i(t_n)\|E_1^n\|_{L^2}^2 \le{}& c\tau^{1+i}(1+\max_{0\le t\le T}\|A_1u(t)\|_{L^2}^2)\\
&\times\int_{t_{n-2}}^{t_n}\sigma^3(t)[\|A_1u_{tt}(t)\|_{L^2}^2+\|A_2\theta_{tt}\|_{L^2}^2]dt\\
&+c\tau^2\max_{0\le t\le T}\sigma^2(t)\|A_1u_t(t)\|_{L^2}^2\int_{t_{n-2}}^{t_n}\sigma(t)\|A_1u_t(t)\|_{L^2}^2dt,\\
\tau\sigma^i(t_n)\|E_2^n\|_{L^2}^2 \le{}& c\tau^{1+i}[1+\max_{0\le t\le T}(\|A_1u(t)\|_{L^2}^2+\|A_2\theta(t)\|_{L^2}^2)]\\
&\times\int_{t_{n-2}}^{t_n}\sigma^3(t)[\|A_1u_{tt}(t)\|_{L^2}^2+\|A_2\theta_{tt}(t)\|_{L^2}^2]dt\\
&+c\tau^{1+i}\max_{0\le t\le T}\sigma^2(t)\|A_2\theta_t(t)\|_{L^2}^2\int_{t_{n-2}}^{t_n}\sigma(t)\|A_1u_t(t)\|_{L^2}^2dt, \qquad (4.10)
\end{aligned}$$

for $i=0,1,2,3,$ and

$$\begin{aligned}
\tau\sigma^i(t_n)\|A_1^{\frac{1}{2}}PE_1^n\|_{L^2}^2 \le{}& c\tau^{i}(1+\max_{0\le t\le T}\|A_1u(t)\|_{L^2}^2)\\
&\times\int_{t_{n-2}}^{t_n}\sigma^4(t)[\|u_{tt}(t)\|_{H^3}^2+\|\theta_{tt}\|_{H^3}^2]dt\\
&+c\tau^2\max_{0\le t\le T}\sigma^2(t)\|A_1u_t(t)\|_{L^2}^2\int_{t_{n-2}}^{t_n}\sigma^2(t)\|u_t(t)\|_{H^3}^2dt,\\
\tau\sigma^i(t_n)\|A_2^{\frac{1}{2}}E_2^n\|_{L^2}^2 \le{}& c\tau^{i}[1+\max_{0\le t\le T}(\|A_1u(t)\|_{L^2}^2+\|A_2\theta(t)\|_{L^2}^2)]\\
&\times\int_{t_{n-2}}^{t_n}\sigma^4(t)[\|u_{tt}(t)\|_{H^3}^2+\|\theta_{tt}(t)\|_{H^3}^2]dt\\
&+c\tau^{i}\max_{0\le t\le T}\sigma^2(t)\|A_2\theta_t(t)\|_{L^2}^2\int_{t_{n-2}}^{t_n}\sigma^2(t)\|u_t(t)\|_{H^3}^2dt, \qquad (4.11)
\end{aligned}$$

for $i = 0, 1, 2, 3, 4$, and

$$\begin{aligned}
&\tau^2\sigma^i(t_n)\|A_1PE_1^n\|_{L^2}^2 \le c\tau^i(1+\max_{0\le t\le T}\sigma(t)\|u(t)\|_{H^3}^2)\\
&\times\int_{t_{n-2}}^{t_n}\sigma^4(t)[\|u_{tt}(t)\|_{H^3}^2+\|\theta_{tt}\|_{H^3}^2]dt + c\tau^i\int_{t_{n-1}}^{t_n}\sigma^5(t)\|u_{tt}\|_{H^4}^2dt\\
&+c\tau^i\max_{0\le t\le T}\sigma^3(t)\|u_t(t)\|_{H^3}^2\int_{t_{n-2}}^{t_n}\sigma^2(t)\|u_t(t)\|_{H^3}^2dt,\\
&\tau\sigma^i(t_n)\|A_2E_2^n\|_{L^2}^2 \le c\tau^i[1+\max_{0\le t\le T}\sigma(t)(\|u(t)\|_{H^3}^2+\|\theta(t)\|_{H^3}^2)]\\
&\times\int_{t_{n-2}}^{t_n}\sigma^4(t)[\|u_{tt}(t)\|_{H^3}^2+\|\theta_{tt}(t)\|_{H^3}^2]dt + c\tau^i\int_{t_{n-1}}^{t_n}\sigma^5(t)\|\theta_{tt}\|_{H^4}^2dt\\
&+c\tau^i\max_{0\le t\le T}\sigma^3(t)\|\theta_t(t)\|_{H^3}^2\int_{t_{n-2}}^{t_n}\sigma^2(t)\|u_t(t)\|_{H^3}^2dt,
\end{aligned}\tag{4.12}$$

for $i = 0, 1, 2, 3, 4$.

Hence, Lemma 4.1 can be proven by (4.7)-(4.12) and Theorem 2.1.

Lemma 4.2. Under the assumptions of Theorem 2.1, (u^n, p^n, θ^n) satisfies the following error estimate:

$$\begin{aligned}
&\sum_{n=1}^m[\|A_1^{\frac{k+1}{2}}\hat{e}^n\|_{L^2}^2\tau+\|A_2^{\frac{k+1}{2}}\hat{\varepsilon}^n\|_{L^2}^2\tau+\frac12\|A_1^{\frac{k}{2}}(e^n-e^{n-1})\|_{L^2}^2+\frac12\|A_2^{\frac{k}{2}}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2]\\
&\qquad+\|A_1^{\frac{k}{2}}e^m\|_{L^2}^2+\|A_2^{\frac{k}{2}}\varepsilon^m\|_{L^2}^2+\frac{\tau}{2}\|A_1^{\frac{k+1}{2}}e^m\|_{L^2}^2+\frac{\tau}{2}\|A_2^{\frac{k+1}{2}}\varepsilon^m\|_{L^2}^2\tau\\
&\qquad\le\frac12\kappa_k\tau^{2-k}+C_1\tau\sum_{n=1}^m N_k^{n-1},
\end{aligned}\tag{4.13}$$

for some constants κ_k and C_1 with $1 \le m \le N,\ k = 0,\ 1,\ 2$, where $\sum_{n=2}^1 N_k^n = 0$,

$$(e^n, r^n, \varepsilon^n) = (u(t_n) - u^n, \frac{1}{\tau}\int_{t_{n-1}}^{t_n} p(t)dt - p^n, \theta(t_n) - \theta^n),$$

and

$$\begin{aligned}
N_0^{n-1} &= \tau(\|A_1^{\frac12}e^{n-1}\|_{L^2}^2\|A_1e^{n-1}\|_{L^2}^2+\|A_2^{\frac12}\varepsilon^{n-1}\|_{L^2}^2\|A_2\varepsilon^{n-1}\|_{L^2}^2)\\
&+\tau(\|A_1^{\frac12}e^{n-2}\|_{L^2}^2\|A_1e^{n-2}\|_{L^2}^2+\|A_2^{\frac12}\varepsilon^{n-2}\|_{L^2}^2\|A_2\varepsilon^{n-2}\|_{L^2}^2),
\end{aligned}\tag{4.14}$$

$$\begin{aligned}
N_1^{n-1} &= (\|A_1^{\frac12}e^{n-1}\|_{L^2}^2\|A_1e^{n-1}\|_{L^2}^2+\|A_2^{\frac12}\varepsilon^{n-1}\|_{L^2}^2\|A_2\varepsilon^{n-1}\|_{L^2}^2)\\
&+(\|A_1^{\frac12}e^{n-2}\|_{L^2}^2\|A_1e^{n-2}\|_{L^2}^2+\|A_2^{\frac12}\varepsilon^{n-2}\|_{L^2}^2\|A_2\varepsilon^{n-2}\|_{L^2}^2)\\
&+(\|A_1^{\frac12}e^{n-1}\|_{L^2}\|A_1e^{n-1}\|_{L^2}+\|A_1^{\frac12}e^{n-2}\|_{L^2}\|A_1e^{n-2}\|_{L^2})\|A_1e^{n-1}\|_{L^2}\\
&+(\|A_1^{\frac12}e^{n-1}\|_{L^2}\|A_1e^{n-1}\|_{L^2}+\|A_2^{\frac12}\varepsilon^{n-1}\|_{L^2}\|A_2\varepsilon^{n-1}\|_{L^2})\|A_2\varepsilon^{n-1}\|_{L^2}\\
&+(\|A_1^{\frac12}e^{n-2}\|_{L^2}\|A_1e^{n-2}\|_{L^2}+\|A_2^{\frac12}\varepsilon^{n-2}\|_{L^2}\|A_2\varepsilon^{n-2}\|_{L^2})\|A_2\varepsilon^{n-1}\|_{L^2},
\end{aligned}\tag{4.15}$$

and

$$\begin{aligned}
N_2^{n-1} &= \|A_1^{\frac{1}{2}}e^{n-1}\|_{L^2}\|A_1^{\frac{3}{2}}e^{n-1}\|_{L^2}\|A_1e^{n-1}\|_{L^2}^2 \\
&+ \|A_1^{\frac{1}{2}}e^{n-2}\|_{L^2}\|A_1^{\frac{3}{2}}e^{n-2}\|_{L^2}\|A_1e^{n-2}\|_{L^2}^2 \\
&+ \|A_1^{\frac{1}{2}}e^{n-1}\|_{L^2}\|A_1e^{n-1}\|_{L^2}\|A_2\varepsilon^{n-1}\|_{L^2}\|A_2^{\frac{3}{2}}\varepsilon^{n-1}\|_{L^2} \\
&+ \|A_1e^{n-1}\|_{L^2}\|A_1^{\frac{3}{2}}e^{n-1}\|_{L^2}\|A_2^{\frac{1}{2}}\varepsilon^{n-1}\|_{L^2}\|A_2\varepsilon^{n-1}\|_{L^2} \\
&+ \|A_1^{\frac{1}{2}}e^{n-2}\|_{L^2}\|A_1e^{n-2}\|_{L^2}\|A_2\varepsilon^{n-2}\|_{L^2}\|A_2^{\frac{3}{2}}\varepsilon^{n-2}\|_{L^2} \\
&+ \|A_1e^{n-2}\|_{L^2}\|A_1^{\frac{3}{2}}e^{n-2}\|_{L^2}\|A_2^{\frac{1}{2}}\varepsilon^{n-2}\|_{L^2}\|A_2\varepsilon^{n-2}\|_{L^2} \\
&+ \|A_1^{\frac{1}{2}}e^{n-1}\|_{L^2}^{\frac{1}{2}}\|A_1e^{n-1}\|_{L^2}\|A_1^{\frac{3}{2}}e^{n-1}\|_{L^2}^{\frac{3}{2}} \\
&+ \|A_1^{\frac{1}{2}}e^{n-2}\|_{L^2}^{\frac{1}{2}}\|A_1e^{n-2}\|_{L^2}\|A_1^{\frac{3}{2}}e^{n-2}\|_{L^2}^{\frac{1}{2}}\|A_1^{\frac{3}{2}}e^{n-1}\|_{L^2} \\
&+ \|A_1^{\frac{1}{2}}e^{n-1}\|_{L^2}^{\frac{1}{2}}\|A_1e^{n-1}\|_{L^2}^{\frac{1}{2}}\|A_2\varepsilon^{n-1}\|_{L^2}^{\frac{1}{2}}\|A_2^{\frac{3}{2}}\varepsilon^{n-1}\|_{L^2}^{\frac{1}{2}}\|A_2^{\frac{3}{2}}\varepsilon^{n-1}\|_{L^2} \\
&+ \|A_1e^{n-1}\|_{L^2}^{\frac{1}{2}}\|A_1^{\frac{3}{2}}e^{n-1}\|_{L^2}^{\frac{1}{2}}\|A_2^{\frac{1}{2}}\varepsilon^{n-1}\|_{L^2}^{\frac{1}{2}}\|A_2\varepsilon^{n-1}\|_{L^2}^{\frac{1}{2}}\|A_2^{\frac{3}{2}}\varepsilon^{n-1}\|_{L^2} \\
&+ \|A_1^{\frac{1}{2}}e^{n-2}\|_{L^2}^{\frac{1}{2}}\|A_1e^{n-2}\|_{L^2}^{\frac{1}{2}}\|A_2\varepsilon^{n-2}\|_{L^2}^{\frac{1}{2}}\|A_2^{\frac{3}{2}}\varepsilon^{n-2}\|_{L^2}^{\frac{1}{2}}\|A_2^{\frac{3}{2}}\varepsilon^{n-1}\|_{L^2} \\
&+ \|A_1e^{n-2}\|_{L^2}^{\frac{1}{2}}\|A_1^{\frac{3}{2}}e^{n-2}\|_{L^2}^{\frac{1}{2}}\|A_2^{\frac{1}{2}}\varepsilon^{n-2}\|_{L^2}^{\frac{1}{2}}\|A_2\varepsilon^{n-2}\|_{L^2}^{\frac{1}{2}}\|A_2^{\frac{3}{2}}\varepsilon^{n-2}\|_{L^2}, \qquad (4.16)
\end{aligned}$$

Proof. Now, subtracting (3.1) and (3.2) from (4.3) and (4.4), respectively, we deduce

$$\begin{aligned}
&(d_te^n, v)_\Omega + a_1(\hat{e}^n, v) + Z(b(e^n, u(t_n), v)) + Z(b(u(t_n), e^n, v)) - Z(b(e^n, e^n, v)) \\
&\quad + (f\vec{k}\times Z(e^n), v)_\Omega - \gamma(\int_{-d}^{z}\nabla Z(\varepsilon^n)d\xi, v)_\Omega - (\nabla\cdot v, r^n)_\Omega + (\nabla\cdot e^n, q)_\Omega \\
&\quad = (E_1^n, v)_\Omega, \qquad (4.17) \\
&(d_t\varepsilon^n, \phi)_\Omega + a_2(\hat{\varepsilon}^n, \phi) + Z(b(e^n, \theta(t_n), \phi)) + Z(b(u(t_n), \varepsilon^n, \phi)) - Z(b(e^n, e^n, \phi)) \\
&\quad + \sigma(\int_{-d}^{z}\nabla\cdot Z(e^n)d\xi, \phi)_\Omega = (E_2^n, \phi)_\Omega, \qquad (4.18)
\end{aligned}$$

for $2 \le n \le N$.

Taking $(v,q) = 2(A_1^k e^n, 0)\tau$ in (4.17) and $\phi = 2A_2^k \varepsilon^n \tau$ in (4.18), adding these two relations and using (2.10)-(2.12), we find

$$
\begin{aligned}
&[\|A_1^{\frac{k}{2}} e^n\|_{L^2}^2 - \|A_1^{\frac{k}{2}} e^{n-1}\|_{L^2}^2 + \|A_1^{\frac{k}{2}}(e^n - e^{n-1})\|_{L^2}^2] \\
&+ [\|A_2^{\frac{k}{2}} \varepsilon^n\|_{L^2}^2 - \|A_2^{\frac{k}{2}} \varepsilon^{n-1}\|_{L^2}^2 + \|A_2^{\frac{k}{2}}(\varepsilon^n - \varepsilon^{n-1})\|_{L^2}^2] \\
&+ \frac{\tau}{2}[\|A_1^{\frac{k+1}{2}} e^n\|_{L^2}^2 - \|A_1^{\frac{k+1}{2}} e^{n-1}\|_{L^2}^2 + 4\|A_1^{\frac{k+1}{2}} \hat{e}^n\|_{L^2}^2] \\
&+ \frac{\tau}{2}[\|A_2^{\frac{k+1}{2}} \varepsilon^n\|_{L^2}^2 - \|A_2^{\frac{k+1}{2}} \varepsilon^{n-1}\|_{L^2}^2 + 4\|A_2^{\frac{k+1}{2}} \hat{\varepsilon}^n\|_{L^2}^2] \\
&+3\tau[b(e^{n-1}, u(t_{n-1}), A_1^k e^n) + b(u(t_{n-1}), e^{n-1}, A_1^k e^n) - b(e^{n-1}, e^{n-1}, A_1^k e^n)] \\
&-\tau[b(e^{n-2}, u(t_{n-2}), A_1^k e^n) + b(u(t_{n-2}), e^{n-2}, A_1^k e^n) - b(e^{n-2}, e^{n-2}, A_1^k e^n)] \\
&+3\tau[b(e^{n-1}, \theta(t_{n-1}), A_2^k \varepsilon^n) + b(u(t_{n-1}), \varepsilon^{n-1}, A_2^k \varepsilon^n) - b(e^{n-1}, \varepsilon^{n-1}, A_2^k \varepsilon^n)] \\
&-\tau[b(e^{n-2}, \theta(t_{n-2}), A_2^k \varepsilon^n) + b(u(t_{n-2}), \varepsilon^{n-2}, A_2^k \varepsilon^n) - b(e^{n-2}, \varepsilon^{n-2}, A_2^k \varepsilon^n)] \\
&+ 2\tau(f\vec{k} \times Z(e^n), A_1^k e^n)_\Omega - 2\gamma\tau(\int_{-d}^{z} \nabla Z(\varepsilon^n) d\xi, A_1^k e^n)_\Omega \\
&+ 2\sigma\tau(\int_{-d}^{z} \nabla \cdot Z(e^n) d\xi, A_2^k \varepsilon^n)_\Omega \\
&= 2\tau(E_1^n, A_1^k e^n)_\Omega + 2\tau(E_2^n, A_2^k \varepsilon^n)_\Omega,
\end{aligned} \tag{4.19}
$$

for $k = 0,\ 1,\ 2$.

Using (2.1)-(2.2), (2.6) and Lemma 3.2, integration by parts and some simple calculations and noting $e^n = \hat{e}^n + \frac{1}{2}(e^n - e^{n-1}),\ \varepsilon^n = \hat{\varepsilon}^n + \frac{1}{2}(\varepsilon^n - \varepsilon^{n-1})$ and $e^n = 2\hat{e}^n + e^{n-1},\ \varepsilon^n = 2\hat{\varepsilon}^n + \varepsilon^{n-1}$, there hold the following estimates:

$$
\begin{aligned}
&|b(e^j, u(t_j), A_1^k e^n)|\tau + |b(u(t_j), e^j, A_1^k e^n)|\tau \\
&\le \frac{1}{96}\|A_1^{\frac{k+1}{2}} \hat{e}^n\|_{L^2}^2 \tau + \frac{1}{96}\|A_1^{\frac{k}{2}}(e^n - e^{n-1})\|_{L^2}^2 \\
&+ c\tau\|u(t_j)\|_{H^3}^2 (\|A_1^{\frac{k}{2}} e^j\|_{L^2}^2 + \frac{\tau}{2}\|A_1^{\frac{k+1}{2}} e^j\|_{L^2}^2), \\
&|b(e^j, \theta(t_j), A_2^k \varepsilon^n)|\tau \le \frac{1}{96}\|A_2^{\frac{k+1}{2}} \hat{\varepsilon}^n\|_{L^2}^2 \tau + \frac{1}{96}\|A_2^{\frac{k}{2}}(\varepsilon^n - \varepsilon^{n-1})\|_{L^2}^2 \\
&+ c\tau\|\theta(t_j)\|_{H^3}^2 (\|A_1^{\frac{k}{2}} e^j\|_{L^2}^2 + \frac{\tau}{2}\|A_1^{\frac{k+1}{2}} e^j\|_{L^2}^2), \\
&|b(u(t_j), \varepsilon^j, A_2^k \varepsilon^n)|\tau \le \frac{1}{96}\|A_2^{\frac{k+1}{2}} \hat{\varepsilon}^n\|_{L^2}^2 \tau + \frac{1}{96}\|A_2^{\frac{k}{2}}(\varepsilon^n - \varepsilon^{n-1})\|_{L^2}^2 \\
&+ c\tau\|u(t_j)\|_{H^3}^2 (\|A_2^{\frac{k}{2}} \varepsilon^j\|_{L^2}^2 + \frac{\tau}{2}\|A_2^{\frac{k+1}{2}} \varepsilon^j\|_{L^2}^2), \\
&2\gamma|(\int_{-d}^{z} \nabla Z(\varepsilon^n) d\xi, A_1^k e^n)_\Omega|\tau \le \frac{1}{32}\|A_1^{\frac{k+1}{2}} \hat{e}^n\|_{L^2}^2 \tau + \frac{1}{32}\|A_1^{\frac{k}{2}}(e^n - e^{n-1})\|_{L^2}^2 \\
&+ c[\|A_2^{\frac{k}{2}} \varepsilon^{n-1}\|_{L^2}^2 + \|A_2^{\frac{k}{2}} \varepsilon^{n-2}\|_{L^2}^2 + \frac{\tau}{2}(\|A_2^{\frac{k+1}{2}} \varepsilon^{n-1}\|_{L^2}^2 + \|A_2^{\frac{k+1}{2}} \varepsilon^{n-2}\|_{L^2}^2)]\tau,
\end{aligned}
$$

$$
\begin{aligned}
&2\sigma(\int_{-d}^{z}\nabla\cdot Z(e^n)d\xi, A_2^k\varepsilon^n)_\Omega\tau \le \frac{1}{32}\|A_2^{\frac{k+1}{2}}\hat{\varepsilon}^n\|_{L^2}^2\tau + \frac{1}{32}\|A_2^{\frac{k}{2}}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2\\
&+c[\|A_1^{\frac{k}{2}}e^{n-1}\|_{L^2}^2+\|A_1^{\frac{k}{2}}e^{n-2}\|_{L^2}^2+\frac{\tau}{2}(\|A_1^{\frac{k+1}{2}}e^{n-1}\|_{L^2}^2+\|A_1^{\frac{k+1}{2}}e^{n-2}\|_{L^2}^2)]\tau,\\
&2\tau|(f\vec{k}\times Z(e^n), A_1^k e^n)_\Omega| \le \frac{1}{32}\|A_1^{\frac{k+1}{2}}\hat{e}^n\|_{L^2}^2\tau+\frac{1}{32}\|A_1^{\frac{k}{2}}(e^n-e^{n-1})\|_{L^2}^2\\
&+c(\|A_1^{\frac{k}{2}}e^{n-1}\|_{L^2}^2+\|A_1^{\frac{k}{2}}e^{n-2}\|_{L^2}^2)\tau,\\
&2|(E_1^n, A_1^k e^n)_\Omega|\tau \le \frac{1}{32}\|A_1^{\frac{k+1}{2}}\hat{e}^n\|_{L^2}^2\tau+\frac{1}{32}\|A_1^{\frac{k}{2}}(e^n-e^{n-1})\|_{L^2}^2\\
&+c\|A_1^{\frac{k-1}{2}}PE_1^n\|_{L^2}^2\tau+c\|A_1^{\frac{k}{2}}PE_1^n\|_{L^2}^2\tau^2,\\
&2|(E_2^n, A_2^k\varepsilon^n)_\Omega|\tau \le \frac{1}{32}\|A_2^{\frac{k+1}{2}}\hat{\varepsilon}^n\|_{L^2}^2\tau+\frac{1}{32}\|A_2^{\frac{k}{2}}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2\\
&+c\|A_2^{\frac{k-1}{2}}E_2^n\|_{L^2}^2\tau+c\|A_2^{\frac{k}{2}}E_2^n\|_{L^2}^2\tau^2, \qquad (4.20)\\
&|b(e^j,e^j,A_1^k e^n)|\tau \le \frac{1}{96}\|e^n-e^{n-1}\|_{L^2}^2+\frac{1}{96}\|e^{n-1}-e^{n-2}\|_{L^2}^2\\
&+c\tau^2\|A_1^{\frac12}e^j\|_{L^2}^2\|A_1e^j\|_{L^2}^2,\\
&|b(e^j,\varepsilon^j,A_2^k\varepsilon^n)|\tau \le \frac{1}{96}\|\varepsilon^n-\varepsilon^{n-1}\|_{L^2}^2+\frac{1}{96}\|\varepsilon^{n-1}-\varepsilon^{n-2}\|_{L^2}^2\\
&+c\tau^2(\|A_1^{\frac12}e^j\|_{L^2}^2\|A_1e^j\|_{L^2}^2+\|A_2^{\frac12}\hat{\varepsilon}^j\|_{L^2}^2\|A_2\varepsilon^j\|_{L^2}^2), \qquad (4.21)
\end{aligned}
$$

for $k=0,\ j=n-1,\ n-2$ and

$$
\begin{aligned}
|b(e^j,e^j,A_1^k e^n)|\tau &\le c\tau\|A_1^{\frac12}e^j\|_{L^2}\|A_1e^j\|_{L^2}\|A_1e^n\|_{L^2}\\
&\le \frac{1}{96}\|A_1\hat{e}^n\|_{L^2}^2\tau+c\tau\|A_1^{\frac12}e^j\|_{L^2}^2\|A_1e^j\|_{L^2}^2+c\tau\|A_1^{\frac12}e^j\|_{L^2}\|A_1e^j\|_{L^2}^2,\\
|b(e^j,\varepsilon^j,A_2^k\varepsilon^n)|\tau &\le \frac{1}{96}\|A_2\hat{\varepsilon}^n\|_{L^2}^2\tau\\
&+c\tau(\|A_1^{\frac12}e^j\|_{L^2}^2\|A_1e^j\|_{L^2}^2+\|A_2^{\frac12}\varepsilon^j\|_{L^2}^2\|A_2\varepsilon^j\|_{L^2}^2)\\
&+c\tau(\|A_1^{\frac12}e^j\|_{L^2}\|A_1e^j\|_{L^2}+\|A_2^{\frac12}\varepsilon^j\|_{L^2}\|A_2\varepsilon^j\|_{L^2})\|A_2\varepsilon^{n-1}\|_{L^2}, \qquad (4.22)
\end{aligned}
$$

for $k=1,\ j=n-1,\ n-2$ and

$$
\begin{aligned}
|b(e^j,e^j,A_1^k e^n)|\tau &\le \frac{1}{96}\|A_1^{\frac32}\hat{e}^n\|_{L^2}^2\tau+c\tau\|A_1^{\frac12}e^j\|_{L^2}\|A_1^{\frac32}e^j\|_{L^2}\|A_1e^j\|_{L^2}^2\\
&+c\tau\|A_1^{\frac12}e^j\|_{L^2}^{\frac12}\|A_1^{\frac32}e^j\|_{L^2}^{\frac12}\|A_1e^j\|_{L^2}\|A_1^{\frac32}e^{n-1}\|_{L^2},\\
|b(e^j,\varepsilon^j,A_2^k\varepsilon^n)|\tau &\le \frac{1}{96}\|A_2^{\frac32}\hat{\varepsilon}^n\|_{L^2}^2\tau\\
&+c\tau\|A_1^{\frac12}e^j\|_{L^2}\|A_1e^j\|_{L^2}\|A_2\varepsilon^j\|_{L^2}\|A_2^{\frac32}\varepsilon^j\|_{L^2}\\
&+c\tau\|A_1e^j\|_{L^2}\|A_1^{\frac32}e^j\|_{L^2}\|A_2^{\frac12}\varepsilon^j\|_{L^2}\|A_2\varepsilon^j\|_{L^2}\\
&+c\tau\|A_1e^j\|_{L^2}^{\frac12}\|A_1^{\frac32}e^j\|_{L^2}^{\frac12}\|A_2^{\frac12}\varepsilon^j\|_{L^2}^{\frac12}\|A_2\varepsilon^j\|_{L^2}^{\frac12}\|A_2^{\frac32}\varepsilon^j\|_{L^2}\\
&+c\tau\|A_1^{\frac12}e^j\|_{L^2}^{\frac12}\|A_1e^j\|_{L^2}^{\frac12}\|A_2\varepsilon^j\|_{L^2}^{\frac12}\|A_2^{\frac32}\varepsilon^j\|_{L^2}^{\frac12}\|A_2^{\frac32}\varepsilon^{n-1}\|_{L^2}, \qquad (4.23)
\end{aligned}
$$

for $k=2$ and $j=n-1,\ n-2$.

Combining (4.19) with (4.20) and (4.21)-(4.23) yields

$$
\begin{aligned}
&[\|A_1^{\frac{k}{2}}e^n\|_{L^2}^2+\|A_2^{\frac{k}{2}}\varepsilon^n\|_{L^2}^2+\frac{\tau}{2}\|A_1^{\frac{k+1}{2}}e^n\|_{L^2}^2+\frac{\tau}{2}\|A_2^{\frac{k+1}{2}}\varepsilon^n\|_{L^2}^2]\\
&-[\|A_1^{\frac{k}{2}}e^{n-1}\|_{L^2}^2+\|A_2^{\frac{k}{2}}\varepsilon^{n-1}\|_{L^2}^2+\frac{1}{2}\|A_1^{\frac{k+1}{2}}e^{n-1}\|_{L^2}^2\tau+\frac{1}{2}\|A_2^{\frac{k+1}{2}}\varepsilon^{n-1}\|_{L^2}^2\tau]\\
&+\|A_1^{\frac{k+1}{2}}\hat{e}^n\|_{L^2}^2\tau+\|A_2^{\frac{k+1}{2}}\hat{\varepsilon}^n\|_{L^2}^2\tau+\frac{3}{4}[\|A_1^{\frac{k}{2}}(e^n-e^{n-1})\|_{L^2}^2+\|A_2^{\frac{k}{2}}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2]\\
&-\frac{1}{16}[\|A_1^{\frac{k}{2}}(e^{n-1}-e^{n-2})\|_{L^2}^2+\|A_2^{\frac{k}{2}}(\varepsilon^{n-1}-\varepsilon^{n-2})\|_{L^2}^2]\\
&\le\frac{\tau}{2}\sum_{j=n-2}^{n-1}d_j[\|A_1^{\frac{k}{2}}e^j\|_{L^2}^2+\|A_2^{\frac{k}{2}}\varepsilon^j\|_{L^2}^2+\frac{1}{2}\|A_1^{\frac{k+1}{2}}e^j\|_{L^2}^2\tau+\frac{1}{2}\|A_2^{\frac{k+1}{2}}\varepsilon^j\|_{L^2}^2\tau]\\
&+c\|A_1^{\frac{k-1}{2}}PE_1^n\|_{L^2}^2\tau+c\|A_1^{\frac{k}{2}}PE_1^n\|_{L^2}^2\tau^2+c\|A_2^{\frac{k-1}{2}}E_2^n\|_{L^2}^2\tau\\
&+c\|A_2^{\frac{k}{2}}E_2^n\|_{L^2}^2\tau^2+c\tau N_k^{n-1},
\end{aligned}\tag{4.24}
$$

where

$$d_{n-1}=c\tau(1+\|u(t_{n-1})\|_{H^3}^2+\|\theta(t_{n-1})\|_{H^3}^2).$$

Summing (4.24) from $n=2$ to m and using Lemma 3.1 and Lemma 4.1, we deduce

$$
\begin{aligned}
&\sum_{n=1}^{m}[\|A_1^{\frac{k+1}{2}}\hat{e}^n\|_{L^2}^2\tau+\|A_2^{\frac{k+1}{2}}\hat{\varepsilon}^n\|_{L^2}^2\tau+\frac{1}{2}\|A_1^{\frac{k}{2}}(e^n-e^{n-1})\|_{L^2}^2+\frac{1}{2}\|A_2^{\frac{k}{2}}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2]\\
&+\|A_1^{\frac{k}{2}}e^m\|_{L^2}^2+\|A_2^{\frac{k}{2}}\varepsilon^m\|_{L^2}^2+\frac{\tau}{2}\|A_1^{\frac{k+1}{2}}e^m\|_{L^2}^2+\frac{\tau}{2}\|A_2^{\frac{k+1}{2}}\varepsilon^m\|_{L^2}^2\tau\\
&\le C\tau^2+\tau\sum_{n=1}^{m-1}d_n(\|A_1^{\frac{k}{2}}e^n\|_{L^2}^2+\|A_2^{\frac{k}{2}}\varepsilon^n\|_{L^2}^2)+c\tau\sum_{n=2}^{m}N_k,
\end{aligned}\tag{4.25}
$$

for $1\le m\le N$. Applying Lemma 2.1 to (4.25) and using Theorem 2.1 and noting

$$
\begin{aligned}
\tau\sum_{n=1}^{m-1}d_n&=c\tau\sum_{n=1}^{m-1}[1+\|\frac{1}{\tau}\int_{t_{n-1}}^{t_n}(u(t_n)-u(t)+u(t))dt\|_{H^3}^2\\
&+\|\frac{1}{\tau}\int_{t_{n-1}}^{t_n}(\theta(t_n)-\theta(t)+\theta(t))dt\|_{H^3}^2]\le C,
\end{aligned}\tag{4.26}
$$

we deduce (4.13).

Theorem 4.1. Under the assumption of Theorem 2.1, if $0<\tau<1$ satisfies

$$
\begin{aligned}
&160C_1\kappa_1\tau\le1,\ 8C_1\kappa_1^2\tau\le\kappa_0,\ 40C_1\kappa_2\tau\le1,128C_1\kappa_1\sqrt{\kappa_0}\tau\le\sqrt{\kappa_2},32C_1\sqrt{\kappa_0\kappa_2}\tau\le1,\\
&64^4C_1^4(4\kappa_0+\kappa_1)\kappa_1^2T\tau\le\kappa_2,\ 32^4C_1^4(4\kappa_0+\kappa_1)\kappa_2T\tau\le1,
\end{aligned}\tag{4.27}
$$

then (u^n, p^n, θ^n) satisfies the following error estimates:

$$\sum_{n=1}^{m}[\|A_1^{\frac12}\hat{e}^n\|_{L^2}^2\tau+\|A_2^{\frac12}\hat{\varepsilon}^n\|_{L^2}^2\tau+\frac12\|e^n-e^{n-1}\|_{L^2}^2+\frac12\|\varepsilon^n-\varepsilon^{n-1}\|_{L^2}^2]$$
$$+\|e^m\|_{L^2}^2+\|\varepsilon^m\|_{L^2}^2+\frac12\|A_1^{\frac12}e^m\|_{L^2}^2\tau+\frac12\|A_2^{\frac12}\varepsilon^m\|_{L^2}^2\tau\le\kappa_0\tau^2, \tag{4.28}$$
$$\sum_{n=1}^{m}[\|A_1\hat{e}^n\|_{L^2}^2\tau+\|A_2\hat{\varepsilon}^n\|_{L^2}^2\tau+\frac12\|A_1^{\frac12}(e^n-e^{n-1})\|_{L^2}^2+\frac12\|A_2^{\frac12}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2]$$
$$+\|A_1^{\frac12}e^m\|_{L^2}^2+\|A_2^{\frac12}\varepsilon^m\|_{L^2}^2+\frac12\|A_1e^m\|_{L^2}^2\tau+\frac12\|A_2\varepsilon^m\|_{L^2}^2\tau\le\kappa_1\tau, \tag{4.29}$$
$$\sum_{n=1}^{m}[\|A_1^{\frac32}\hat{e}^n\|_{L^2}^2\tau+\|A_2^{\frac32}\hat{\varepsilon}^n\|_{L^2}^2\tau+\frac12\|A_1(e^n-e^{n-1})\|_{L^2}^2+\frac12\|A_2(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2]$$
$$+\|A_1e^m\|_{L^2}^2+\|A_2\varepsilon^m\|_{L^2}^2+\frac{\tau}{2}\|A_1^{\frac32}e^m\|_{L^2}^2+\frac{\tau}{2}\|A_2^{\frac32}\varepsilon^m\|_{L^2}^2\le\kappa_2, \tag{4.30}$$

for $0\le m\le N$.

Proof. Now, we will use the induction method to prove Theorem 4.1. It follows Lemma 3.1 that (4.28)-(4.30) hold for $m=0,1$. Assuming that (4.28)-(4.30) hold for $m=0,\ 1,\cdots,J$, we need to prove (4.28)-(4.30) hold for $m=J+1$.

Using (4.14)-(4.16), the convergence condition (4.27) and the induction assumption on (4.28)-(4.30) for $m=0,1,\cdots,J$, we obtain

$$C_1\tau\sum_{n=2}^{J+1}N_0^{n-1}\le 2C_1(4\kappa_0+\kappa_1)\kappa_1\tau^3,\ C_1\tau\sum_{n=2}^{J+1}N_1^{n-1}\le 10C_1(4\kappa_1+\kappa_2)\kappa_1\tau^2,$$
$$C_1\tau\sum_{n=2}^{J+1}N_2^{n-1}\le 4C_1(4\kappa_1+\kappa_2)\sqrt{\kappa_0\kappa_2}\tau+4(4\kappa_0+\kappa_1)^{\frac14}(2\kappa_1^{\frac12}+\kappa_2^{\frac12})(T\tau)^{\frac14}\kappa_2^{\frac34}.$$

Using the convergence condition (4.27) in the above inequalities and using Lemma 4.2 with $m=J+1$, we have showed that (4.28)-(4.30) hold for $m=J+1$. The proof ends.

Theorem 4.2. Under the assumption of Theorem 4.1, (u^m, p^m, θ^m) satisfies the following error estimate:

$$\tau\sum_{n=2}^{m}\sigma^i(t_n)(\|d_te^n\|_{L^2}^2+\|r^n\|_{L^2}^2)$$
$$\le c\tau\sum_{n=2}^{m}\sigma^i(t_n)\|A_1\hat{e}^n\|_{L^2}^2+c\tau\sum_{n=1}^{m-1}\sigma^i(t_n)(\|A_1^{\frac12}e^n\|_{L^2}^2+\|A_2^{\frac12}\varepsilon^n\|_{L^2}^2)$$
$$+c\tau\sum_{n=1}^{m-1}(\|u(t_n)\|_{H^3}^2+\|A_1e^n\|_{L^2}^2)\sigma^i(t_n)\|A_1^{\frac12}e^n\|_{L^2}^2+c\tau\sum_{n=2}^{m}\sigma^i(t_n)\|E_1^n\|_{L^2}^2, \tag{4.31}$$

and

$$\tau\sum_{n=2}^{m}\sigma^i(t_n)\|d_t\varepsilon^n\|_{L^2}^2 \le c\tau\sum_{n=2}^{m}\sigma^i(t_n)\|A_2\hat{\varepsilon}^n\|_{L^2}^2 + c\tau\sum_{n=2}^{m}\sigma^i(t_n)\|E_2^n\|_{L^2}^2$$
$$+ c\tau\sum_{n=1}^{m-1}(1+\|u(t_n)\|_{H^3}^2+\|\theta(t_n)\|_{H^3}^2)\sigma^i(t_n)(\|A_1^{\frac{1}{2}}e^n\|_{L^2}^2+\|A_2\varepsilon^n\|_{L^2}^2)$$
$$+ c\tau\sum_{n=1}^{m-1}\sigma^i(t_n)(\|A_1^{\frac{1}{2}}e^n\|_{L^2}^2+\|A_2^{\frac{1}{2}}\varepsilon^n\|_{L^2}^2)(\|A_1e^n\|_{L^2}^2+\|A_2\varepsilon^n\|_{L^2}^2). \tag{4.32}$$

Proof. Using Lemma 3.2 and some simple calculations, we deduce from (4.14) and (4.15) that

$$\|d_te^n\|_{L^2} \le \|A_1\hat{e}^n\|_{L^2} + c\sum_{j=n-2}^{n-1}(\|u(t_j)\|_{H^3}\|A_1^{\frac{1}{2}}e^j\|_{L^2}+\|A_1^{\frac{1}{2}}e^j\|_{L^2}\|A_1e^j\|_{L^2})$$
$$+ c\sum_{j=n-2}^{n-1}(\|A_1^{\frac{1}{2}}e^j\|_{L^2}+\|A_1^{\frac{1}{2}}\varepsilon^j\|_{L^2}) + c\|E_1^n\|_{L^2},$$
$$\|d_t\varepsilon^n\|_{L^2} \le \|A_2\hat{\varepsilon}^n\|_{L^2} + c\sum_{j=n-2}^{n-1}(\|u(t_j)\|_{H^3}\|A_2^{\frac{1}{2}}\varepsilon^j\|_{L^2}+\|\theta(t_j)\|_{H^3}\|A_1^{\frac{1}{2}}e^j\|_{L^2}$$
$$+ c\sum_{j=n-2}^{n-1}(\|A_1^{\frac{1}{2}}e^j\|_{L^2}\|A_1e^j\|_{L^2}+\|A_2^{\frac{1}{2}}\varepsilon^j\|_{L^2}\|A_2\varepsilon^j\|_{L^2}+\|A_1^{\frac{1}{2}}e^j\|_{L^2}) + c\|E_2^n\|_{L^2},$$

which yield

$$\|d_te^n\|_{L^2}^2 \le c\|A_1\hat{e}^n\|_{L^2}^2\tau + c\sum_{j=n-2}^{n-1}(1+\|u(t_j)\|_{H^3}^2)\|A_1^{\frac{1}{2}}e^j\|_{L^2}^2$$
$$+ c\sum_{j=n-2}^{n-1}(\|A_1^{\frac{1}{2}}e^j\|_{L^2}^2\|A_1e^j\|_{L^2}^2+\|A_2^{\frac{1}{2}}\varepsilon^j\|_{L^2}^2) + c\|E_1^n\|_{L^2}^2. \tag{4.33}$$

$$\|d_t\varepsilon^n\|_{L^2}^2 \le c\|A_2\hat{\varepsilon}^n\|_{L^2}^2 + c\sum_{k=n-2}^{n-1}[\|u(t_k)\|_{H^3}^2\|A_2^{\frac{1}{2}}\varepsilon^k\|_{L^2}^2 + c(1+\|\theta(t_k)\|_{H^3}^2)\|A_1^{\frac{1}{2}}e^k\|_{L^2}^2]$$
$$+ c\sum_{k=n-2}^{n-1}(\|A_1^{\frac{1}{2}}e^k\|_{L^2}^2\|A_1e^k\|_{L^2}^2+\|A_2^{\frac{1}{2}}\varepsilon^k\|_{L^2}^2\|A_2\varepsilon^k\|_{L^2}^2) + c\|E_2^n\|_{L^2}^2. \tag{4.34}$$

Moreover, taking $v = \bar{v}$ in (4.14) and using (2.1)-(2.2), Lemma 3.2 and the inf-sup inequality:

$$\|q\|_{L^2(\omega)} \le c\sup_{\bar{v}\in X_0}\frac{(\nabla\cdot\bar{v},q)_\omega}{\|\nabla\bar{v}\|_{L^2(\omega)}},$$

we obtain

$$\|r^n\|_{L^2(\omega)} \le c\|d_te^n\|_{L^2} + c\|A_1\hat{e}^n\|_{L^2} + c\sum_{j=n-2}^{n-1}(1+\|u(t_j)\|_{H^3})\|A_1^{\frac{1}{2}}e^j\|_{L^2}$$
$$+ c\sum_{j=n-2}^{n-1}(\|A_1^{\frac{1}{2}}e^j\|_{L^2}\|A_1e^j\|_{L^2}+\|A_2^{\frac{1}{2}}\varepsilon^j\|_{L^2}) + c\|E_1^n\|_{L^2}.$$

Combining the above inequality with (4.33) yields

$$\|d_t e^n\|_{L^2}^2 + \|r^n\|_{L^2(\omega)}^2 \le c\|A_1\hat{e}^n\|_{L^2}^2 + c\sum_{j=n-2}^{n-1}(1+\|u(t_j)\|_{H^3}^2)\|A_1^{\frac{1}{2}}e^j\|_{L^2}^2$$

$$+ c\sum_{j=n-2}^{n-1}(\|A_1^{\frac{1}{2}}e^j\|_{L^2}^2\|A_1e^j\|_{L^2}^2 + \|A_2^{\frac{1}{2}}\varepsilon^j\|_{L^2}^2) + c\|E_1^n\|_{L^2}^2. \tag{4.35}$$

Multiplying (4.35) and (4.34) by $\tau\sigma^i(t_n)$ and summing from $n=2$ to $n=m$, respectively, and using the inequalities: $\sigma(t_n)\le\sigma(t_{n-1})+\tau,\ \sigma(t_n)\le\sigma(t_{n-2})+2\tau$, we deduce (4.31) and (4.32). The proof ends.

5. Proof of Theorem 1.2

In this section, we shall provide the second order error estimates of the numerical solution (u^n,p^n,θ^n) by the negative norm technique and the Gronwall lemma. Moreover, based on the error estimates of (u^n,p^n,θ^n) and Theorem 2.1, we deduce the stability results of (u^n,p^n,θ^n). Finally, Theorem 1.2 is proven.

Lemma 5.1. Under the assumptions of Theorem 4.1, (u^n,p^n,θ^n) satisfies the following error estimate:

$$\sum_{n=1}^{m}[\|\hat{e}^n\|_{L^2}^2\tau + \|\hat{\varepsilon}^n\|_{L^2}^2\tau + \frac{1}{2}\|A_1^{-\frac{1}{2}}(e^n-e^{n-1})\|_{L^2}^2 + \frac{1}{2}\|A_2^{-\frac{1}{2}}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2]$$

$$+\|A_1^{-\frac{1}{2}}e^m\|_{L^2}^2 + \|A_2^{-\frac{1}{2}}\varepsilon^m\|_{L^2}^2 + \frac{\tau}{2}\|e^m\|_{L^2}^2 + \frac{\tau}{2}\|\varepsilon^m\|_{L^2}^2 \le \kappa\tau^3,\ 1\le m\le N. \tag{5.1}$$

Proof. Taking $(v,q)=2(A_1^{-1}e^n,0)\tau$ in (4.17) and $\phi=2A_2^{-1}\varepsilon^n\tau$ in (4.18), adding these two relations, we find

$$\begin{aligned}
&[\|A_1^{-\frac{1}{2}}e^n\|_{L^2}^2 + \|A_2^{-\frac{1}{2}}\varepsilon^n\|_{L^2}^2 + \frac{\tau}{2}\|e^n\|_{L^2}^2 + \frac{\tau}{2}\|\varepsilon^n\|_{L^2}^2]\\
&-[\|A_1^{-\frac{1}{2}}e^{n-1}\|_{L^2}^2 + \|A_2^{-\frac{1}{2}}\varepsilon^{n-1}\|_{L^2}^2 + \frac{\tau}{2}\|e^{n-1}\|_{L^2}^2 + \frac{\tau}{2}\|\varepsilon^{n-1}\|_{L^2}^2]\\
&+\|A_1^{-\frac{1}{2}}(e^n-e^{n-1})\|_{L^2}^2 + \|A_2^{-\frac{1}{2}}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2 + 2\tau\|\hat{e}^n\|_{L^2}^2 + 2\tau\|\hat{\varepsilon}^n\|_{L^2}^2\\
&+3\tau[b(e^{n-1},u(t_{n-1}),A_1^{-1}e^n) + b(u(t_{n-1}),e^{n-1},A_1^{-1}e^n) - b(e^{n-1},e^{n-1},A_1^{-1}e^n)]\\
&-\tau[b(e^{n-2},u(t_{n-2}),A_1^{-1}e^n) + b(u(t_{n-2}),e^{n-2},A_1^{-1}e^n) - b(e^{n-2},e^{n-2},A_1^{-1}e^n)]\\
&+3\tau[b(e^{n-1},\theta(t_{n-1}),A_2^{-1}\varepsilon^n) + b(u(t_{n-1}),\varepsilon^{n-1},A_2^{-1}\varepsilon^n) - b(e^{n-1},\varepsilon^{n-1},A_2^{-1}\varepsilon^n)]\\
&-\tau[b(e^{n-2},\theta(t_{n-2}),A_2^{-1}\varepsilon^n) + b(u(t_{n-2}),\varepsilon^{n-2},A_2^{-1}\varepsilon^n) - b(e^{n-2},\varepsilon^{n-2},A_2^{-1}\varepsilon^n)]\\
&+2\tau(f\vec{k}\times Z(e^n),A_1^{-1}e^n)_\Omega - 2\gamma\tau(\int_{-d}^{z}\nabla Z(\varepsilon^n)d\xi,A_1^{-1}e^n)_\Omega\\
&\qquad + 2\sigma\tau(\int_{-d}^{z}\nabla\cdot Z(e^n)d\xi,A_2^{-1}\varepsilon^n)_\Omega\\
&\qquad = 2\tau(E_1^n,A_1^{-1}e^n)_\Omega + 2\tau(E_2^n,A_2^{-1}\varepsilon^n)_\Omega.
\end{aligned}\tag{5.2}$$

Using (2.1)-(2.2), (2.6) and Lemma 3.2, integration by parts and some simple calculations and noting $e^n=\hat{e}^n+\frac{1}{2}(e^n-e^{n-1}),\ \varepsilon^n=\hat{\varepsilon}^n+\frac{1}{2}(\varepsilon^n-\varepsilon^{n-1})$ and

$e^n = e^n - e^{n-1} + e^{n-1},\ \varepsilon^n = \varepsilon^n - \varepsilon^{n-1} + \varepsilon^{n-1}$, there hold the following estimates:

$$
\begin{aligned}
&|b(e^j, u(t_j), A_1^{-1}e^n)|\tau + |b(u(t_j), e^j, A_1^{-1}e^n)|\tau \\
&\le \frac{1}{96}\|\hat{e}^{n-1}\|_{L^2}^2\tau + \frac{1}{96}\|e^{n-1} - e^{n-2}\|_{L^2}^2\tau + \frac{1}{96}\|A_1^{-\frac{1}{2}}(e^n - e^{n-1})\|_{L^2}^2 \\
&+ \frac{1}{96}\|A_1^{-\frac{1}{2}}(e^{n-1} - e^{n-2})\|_{L^2}^2 + c\sum_{j=n-2}^{n-1}\|u(t_j)\|_{H^3}^2(\|A_1^{-\frac{1}{2}}e^j\|_{L^2}^2 + \frac{\tau}{2}\|e^j\|_{L^2}^2), \\
&3|b(e^{n-1}, \theta(t_{n-1}), A_2^{-1}\varepsilon^n)|\tau + |b(e^{n-2}, \theta(t_{n-2}), A_2^{-1}\varepsilon^n)|\tau \\
&\le \frac{1}{32}\|\hat{e}^{n-1}\|_{L^2}^2\tau + \frac{1}{32}\|e^{n-1} - e^{n-2}\|_{L^2}^2\tau + \frac{1}{32}\|A_2^{-\frac{1}{2}}(\varepsilon^n - \varepsilon^{n-1})\|_{L^2}^2 \\
&+ \frac{1}{32}\|A_2^{-\frac{1}{2}}(\varepsilon^{n-1} - \varepsilon^{n-2})\|_{L^2}^2 + \tau c\sum_{j=n-2}^{n-1}\|\theta(t_j)\|_{H^3}^2(\|A_2^{-\frac{1}{2}}\varepsilon^j\|_{L^2}^2 + \frac{\tau}{2}\|e^j\|_{L^2}^2), \\
&3|b(u(t_{n-1}), \varepsilon^{n-1}, A_2^{-1}\varepsilon^n)|\tau + |b(u(t_{n-2}), \varepsilon^{n-2}, A_2^{-1}\varepsilon^n)|\tau \\
&\le \frac{1}{32}\|\hat{\varepsilon}^{n-1}\|_{L^2}^2\tau + \frac{1}{32}\|\varepsilon^{n-1} - \varepsilon^{n-2}\|_{L^2}^2\tau + \frac{1}{32}\|A_2^{-\frac{1}{2}}(\varepsilon^n - \varepsilon^{n-1})\|_{L^2}^2 \\
&+ \frac{1}{32}\|A_2^{-\frac{1}{2}}(\varepsilon^{n-1} - \varepsilon^{n-2})\|_{L^2}^2 + \tau c\sum_{j=n-2}^{n-1}\|u(t_j)\|_{H^3}^2(\|A_2^{-\frac{1}{2}}\varepsilon^j\|_{L^2}^2 + \frac{\tau}{2}\|\varepsilon^j\|_{L^2}^2), \\
&3|b(e^{n-1}, e^{n-1}, A_1^{-1}e^n)|\tau + |b(e^{n-2}, e^{n-2}, A_1^{-1}e^n)|\tau \\
&\le \frac{1}{32}\|\hat{e}^n\|_{L^2}^2\tau + \frac{1}{32}\|e^n - e^{n-1}\|_{L^2}^2\tau + c\tau\sum_{j=n-2}^{n-1}\|A_1^{\frac{1}{2}}e^j\|_{L^2}^4, \\
&3|b(e^{n-1}, \varepsilon^{n-1}, A_2^{-1}\varepsilon^n)|\tau + |b(e^{n-2}, \varepsilon^{n-2}, A_2^{-1}\varepsilon^n)| \le \frac{1}{32}\|\hat{\varepsilon}^n\|_{L^2}^2\tau \\
&+ \frac{1}{32}\|\varepsilon^n - \varepsilon^{n-1}\|_{L^2}^2\tau + c\tau\sum_{j=n-2}^{n-1}\|A_1^{\frac{1}{2}}e^j\|_{L^2}^2\|A_2^{\frac{1}{2}}\varepsilon^j\|_{L^2}^2, \\
&2\gamma|(\int_{-d}^z \nabla Z(\varepsilon^n)d\xi, A_1^{-1}e^n)_\Omega|\tau \le c\|Z(\varepsilon^n)\|_{L^2}\|A_1^{-\frac{1}{2}}e^n\|_{L^2}\tau \\
&\le \frac{1}{32}\|A_1^{-\frac{1}{2}}(e^n - e^{n-1})\|_{L^2}^2 + \frac{1}{32}\|\hat{\varepsilon}^{n-1}\|_{L^2}^2\tau + \frac{1}{32}\|\varepsilon^{n-1} - \varepsilon^{n-2}\|_{L^2}^2\tau \\
&+ c\tau(\|A_1^{-\frac{1}{2}}e^{n-1}\|_{L^2}^2 + \frac{\tau}{2}\|\varepsilon^{n-1}\|_{L^2}^2 + \frac{\tau}{2}\|\varepsilon^{n-2}\|_{L^2}^2), \\
&2\sigma|(\int_{-d}^z \nabla\cdot Z(e^n)d\xi, A_2^{-1}\varepsilon^n)_\Omega|\tau \le \frac{1}{32}\|\hat{e}^{n-1}\|_{L^2}^2\tau + \frac{1}{32}\|e^{n-1} - e^{n-2}\|_{L^2}^2\tau \\
&+ \frac{1}{32}\|A_2^{-\frac{1}{2}}(\varepsilon^n - \varepsilon^{n-1})\|_{L^2}^2 + c\tau(\|A_2^{-\frac{1}{2}}\varepsilon^{n-1}\|_{L^2}^2 + \frac{\tau}{2}\|e^{n-1}\|_{L^2}^2 + \frac{\tau}{2}\|e^{n-2}\|_{L^2}^2), \\
&2\tau|(f\vec{k}\times Z(e^n), A_1^{-1}e^n)_\Omega| \le \frac{1}{32}\|\hat{e}^{n-1}\|_{L^2}^2\tau + \frac{1}{32}\|e^{n-1} - e^{n-2}\|_{L^2}^2\tau \\
&+ \frac{1}{32}\|A_1^{-\frac{1}{2}}(e^n - e^{n-1})\|_{L^2}^2 + c\tau(\|A_1^{-\frac{1}{2}}e^{n-1}\|_{L^2}^2 + \frac{\tau}{2}\|e^{n-1}\|_{L^2}^2 + \frac{\tau}{2}\|e^{n-2}\|_{L^2}^2), \\
&2|(E_1^n, A_1^{-1}e^n)_\Omega|\tau \le \frac{1}{32}\|\hat{e}^n\|_{L^2}^2\tau + \frac{1}{32}\|e^n - e^{n-1}\|_{L^2}^2\tau + c\|A_1^{-1}PE_1^n\|_{L^2}^2\tau, \\
&2|(E_2^n, A_2^{-1}\varepsilon^n)_\Omega|\tau \le \frac{1}{32}\|\hat{\varepsilon}^n\|_{L^2}^2\tau + \frac{1}{32}\|\varepsilon^n - \varepsilon^{n-1}\|_{L^2}^2\tau + c\|A_2^{-1}E_2^n\|_{L^2}^2\tau.
\end{aligned}
$$

Combining (5.2) with the above inequalities yields

$$
\begin{aligned}
&[\|A_1^{-\frac{1}{2}}e^n\|_{L^2}^2+\|A_2^{-\frac{1}{2}}\varepsilon^n\|_{L^2}^2+\frac{\tau}{2}\|e^n\|_{L^2}^2+\frac{\tau}{2}\|\varepsilon^n\|_{L^2}^2]\\
&\qquad-[\|A_1^{-\frac{1}{2}}e^{n-1}\|_{L^2}^2+\|A_2^{-\frac{1}{2}}\varepsilon^{n-1}\|_{L^2}^2+\frac{\tau}{2}\|e^{n-1}\|_{L^2}^2+\frac{\tau}{2}\|\varepsilon^{n-1}\|_{L^2}^2]\\
&\qquad+\frac{3}{2}[\|\hat{e}^n\|_{L^2}^2\tau+\|\hat{\varepsilon}^n\|_{L^2}^2]\tau-\frac{1}{2}[\|\hat{e}^{n-1}\|_{L^2}^2\tau+\|\hat{\varepsilon}^{n-1}\|_{L^2}^2]\tau\\
&\qquad+\frac{3}{4}[\|A_1^{-\frac{1}{2}}(e^n-e^{n-1})\|_{L^2}^2+\|A_2^{-\frac{1}{2}}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2]\\
&\qquad-\frac{1}{4}[\|A_1^{-\frac{1}{2}}(e^{n-1}-e^{n-2})\|_{L^2}^2+\|A_2^{-\frac{1}{2}}(\varepsilon^{n-1}-\varepsilon^{n-2})\|_{L^2}^2]\\
&\qquad\le\frac{\tau}{2}\sum_{j=n-2}^{n-1}d_j[\|A_1^{-\frac{1}{2}}e^j\|_{L^2}^2+\|A_2^{-\frac{1}{2}}\varepsilon^j\|_{L^2}^2+\frac{\tau}{2}\|e^j\|_{L^2}^2+\frac{\tau}{2}\|\varepsilon^j\|_{L^2}^2]\\
&\qquad+c\tau\sum_{j=n-1}^{n}\|e^j-e^{j-1}\|_{L^2}^2+c\tau\sum_{j=n-2}^{n-1}(\|A_1^{\frac{1}{2}}e^j\|_{L^2}^2+\|A_2^{\frac{1}{2}}\varepsilon^j\|_{L^2}^2)\|A_1^{\frac{1}{2}}e^j\|_{L^2}^2\\
&\qquad+c\tau\sum_{j=n-1}^{n}\|\varepsilon^j-\varepsilon^{j-1}\|_{L^2}^2+c\|A_1^{-1}PE_1^n\|_{L^2}^2\tau+c\|A_2^{-1}E_2^n\|_{L^2}^2\tau,
\end{aligned}
\tag{5.3}
$$

for $2\le n\le m$.

Summing (5.3) from $n=2$ to m and using Lemma 3.1 and Lemma 4.1 and Theorem 4.1, we deduce

$$
\begin{aligned}
&\|A_1^{-\frac{1}{2}}e^m\|_{L^2}^2+\|A_2^{-\frac{1}{2}}\varepsilon^m\|_{L^2}^2+\frac{\tau}{2}\|e^m\|_{L^2}^2+\frac{\tau}{2}\|\varepsilon^m\|_{L^2}^2\\
&+\sum_{n=1}^{m}[\|\hat{e}^n\|_{L^2}^2\tau+\|\hat{\varepsilon}^n\|_{L^2}^2\tau+\frac{1}{2}\|A_1^{-\frac{1}{2}}(e^n-e^{n-1})\|_{L^2}^2+\|A_2^{-\frac{1}{2}}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2]\\
&\le C\tau^3+\tau\sum_{n=1}^{m-1}d_n(\|A_1^{-\frac{1}{2}}e^n\|_{L^2}^2+\|A_1^{-\frac{1}{2}}\varepsilon^n\|_{L^2}^2+\frac{\tau}{2}\|e^n\|_{L^2}^2+\frac{\tau}{2}\|\varepsilon^n\|_{L^2}^2),
\end{aligned}
\tag{5.4}
$$

for $1\le m\le N$. Applying Lemma 2.1 to (5.4) and using (4.26), we deduce (5.1).

Lemma 5.2. Under the assumptions of Theorem 4.1, (u^n,p^n,θ^n) satisfies the following error estimate:

$$
\begin{aligned}
&\sum_{n=1}^{m}[\|A_1^{-\frac{1}{2}}\hat{e}^n\|_{L^2}^2\tau+\|A_2^{-\frac{1}{2}}\hat{\varepsilon}^n\|_{L^2}^2\tau+\frac{1}{2}\|A_1^{-1}(e^n-e^{n-1})\|_{L^2}^2+\frac{1}{2}\|A_2^{-1}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2]\\
&\qquad+\|A_1^{-1}e^m\|_{L^2}^2+\|A_2^{-1}\varepsilon^m\|_{L^2}^2+\frac{\tau}{2}\|A_1^{-\frac{1}{2}}e^m\|_{L^2}^2+\frac{\tau}{2}\|A_2^{-\frac{1}{2}}\varepsilon^m\|_{L^2}^2\le\kappa\tau^4,
\end{aligned}
\tag{5.5}
$$

for $1\le m\le N$.

Proof. Taking $(v,q) = 2(A_1^{-2}e^n, 0)\tau$ in (4.17) and $\phi = 2A_2^{-2}\varepsilon^n\tau$ in (4.18), adding these two relations, we find

$$
\begin{aligned}
&[\|A_1^{-1}e^n\|_{L^2}^2 + \|A_2^{-1}\varepsilon^n\|_{L^2}^2 + \frac{\tau}{2}\|A_1^{-\frac{1}{2}}e^n\|_{L^2}^2 + \frac{\tau}{2}\|A_2^{-\frac{1}{2}}\varepsilon^n\|_{L^2}^2]\\
&-[\|A_1^{-1}e^{n-1}\|_{L^2}^2 + \|A_2^{-1}\varepsilon^{n-1}\|_{L^2}^2 + \frac{\tau}{2}\|A_1^{-\frac{1}{2}}e^{n-1}\|_{L^2}^2 + \frac{\tau}{2}\|A_2^{-\frac{1}{2}}\varepsilon^{n-1}\|_{L^2}^2]\\
&+\|A_1^{-1}(e^n - e^{n-1})\|_{L^2}^2 + \|A_2^{-1}(\varepsilon^n - \varepsilon^{n-1})\|_{L^2}^2 + 2\tau\|A_1^{-\frac{1}{2}}\hat{e}^n\|_{L^2}^2 + 2\tau\|A_2^{-\frac{1}{2}}\hat{\varepsilon}^n\|_{L^2}^2\\
&+3\tau[b(e^{n-1}, u(t_{n-1}), A_1^{-2}e^n) + b(u(t_{n-1}), e^{n-1}, A_1^{-2}e^n) - b(e^{n-1}, e^{n-1}, A_1^{-2}e^n)]\\
&-\tau[b(e^{n-2}, u(t_{n-2}), A_1^{-2}e^n) + b(u(t_{n-2}), e^{n-2}, A_1^{-2}e^n) - b(e^{n-2}, e^{n-2}, A_1^{-2}e^n)]\\
&+3\tau[b(e^{n-1}, \theta(t_{n-1}), A_2^{-2}\varepsilon^n) + b(u(t_{n-1}), \varepsilon^{n-1}, A_2^{-2}\varepsilon^n) - b(e^{n-1}, \varepsilon^{n-1}, A_2^{-2}\varepsilon^n)]\\
&-\tau[b(e^{n-2}, \theta(t_{n-2}), A_2^{-2}\varepsilon^n) + b(u(t_{n-2}), \varepsilon^{n-2}, A_2^{-2}\varepsilon^n) - b(e^{n-2}, \varepsilon^{n-2}, A_2^{-2}\varepsilon^n)]\\
&+2\tau(f\vec{k}\times Z(e^n), A_1^{-2}e^n)_\Omega - 2\gamma\tau(\int_{-d}^{z}\nabla Z(\varepsilon^n)d\xi, A_1^{-2}e^n)_\Omega\\
&+2\sigma\tau(\int_{-d}^{z}\nabla\cdot Z(e^n)d\xi, A_2^{-2}\varepsilon^n)_\Omega = 2\tau(E_1^n, A_1^{-2}e^n)_\Omega + 2\tau(E_2^n, A_2^{-2}\varepsilon^n)_\Omega.
\end{aligned}\tag{5.6}
$$

Using (2.1)-(2.2), (2.6) and Lemma 3.2, integration by parts and some simple calculations and noting $e^n = \hat{e}^n + \frac{1}{2}(e^n - e^{n-1})$, $\varepsilon^n = \hat{\varepsilon}^n + \frac{1}{2}(\varepsilon^n - \varepsilon^{n-1})$ and $e^n = e^n - e^{n-1} + e^{n-1}$, $\varepsilon^n = \varepsilon^n - \varepsilon^{n-1} + \varepsilon^{n-1}$, there hold the following estimates:

$$
\begin{aligned}
&|b(e^j, u(t_j), A_1^{-2}e^n)|\tau + |b(u(t_j), e^j, A_1^{-1}e^n)|\tau\\
&\le \frac{1}{96}\|A_1^{-\frac{1}{2}}\hat{e}^{n-1}\|_{L^2}^2\tau + \frac{1}{96}\|A_1^{-\frac{1}{2}}(e^{n-1} - e^{n-2})\|_{L^2}^2\tau + \frac{1}{96}\sum_{j=n-1}^{n}\|A_1^{-1}(e^j - e^{j-1})\|_{L^2}^2\\
&+ c(\tau^2\|u(t_j)\|_{H^2}^2\|e^j\|_{L^2}^2 + \tau\|u(t_j)\|_{H^3}^2\|A_1^{-1}e^j\|_{L^2}^2),\\
&3|b(e^{n-1}, \theta(t_{n-1}), A_2^{-2}\varepsilon^n)|\tau + |b(e^{n-2}, \theta(t_{n-2}), A_2^{-2}\varepsilon^n)|\tau\\
&\le \frac{1}{32}\|A_1^{-\frac{1}{2}}\hat{e}^{n-1}\|_{L^2}^2\tau + \frac{1}{32}\|A_1^{-\frac{1}{2}}(e^{n-1} - e^{n-2})\|_{L^2}^2\tau + \frac{1}{32}\sum_{j=n-1}^{n}\|A_2^{-1}(\varepsilon^j - \varepsilon^{j-1})\|_{L^2}^2\\
&+ c\sum_{j=n-2}^{n-1}(\|\theta(t_j)\|_{H^2}^2\|e^j\|_{L^2}^2\tau^2 + c\|\theta(t_j)\|_{H^3}^2\|A_2^{-1}\varepsilon^j\|_{L^2}^2\tau),\\
&3|b(u(t_{n-1}), \varepsilon^{n-1}, A_2^{-2}\varepsilon^n)|\tau + |b(u(t_{n-2}), \varepsilon^{n-2}, A_2^{-2}\varepsilon^n)|\tau\\
&\le \frac{1}{32}\|A_2^{-\frac{1}{2}}\hat{\varepsilon}^{n-1}\|_{L^2}^2\tau + \frac{1}{32}\|A_2^{-\frac{1}{2}}(\varepsilon^{n-1} - \varepsilon^{n-2})\|_{L^2}^2\tau + \frac{1}{32}\sum_{j=n-1}^{n}\|A_2^{-1}(\varepsilon^j - \varepsilon^{j-1})\|_{L^2}^2\\
&+ c\sum_{j=n-2}^{n-1}(\|u(t_j)\|_{H^2}^2\|\varepsilon^j\|_{L^2}^2\tau^2 + \|u(t_j)\|_{H^3}^2\|A_2^{-1}\varepsilon^j\|_{L^2}^2\tau),
\end{aligned}
$$

and

$$
\begin{aligned}
&|b(e^j, e^j, A_1^{-2}e^n)|\tau \le \frac{1}{96}\|A_1^{-\frac{1}{2}}\hat{e}^n\|_{L^2}^2\tau + \frac{1}{96}\|A_1^{-1}(e^n - e^{n-1})\|_{L^2}^2 \\
&+ c\tau(\|e^{n-2}\|_{L^2}^2 + \tau\|A_1^{\frac{1}{2}}e^{n-2}\|_{L^2}^2)\|A_1^{\frac{1}{2}}e^{n-2}\|_{L^2}^2, \\
&3|b(e^j, \varepsilon^j, A_2^{-2}\varepsilon^n)|\tau \le \frac{1}{32}\|A_2^{-\frac{1}{2}}\hat{\varepsilon}^n\|_{L^2}^2\tau + \frac{1}{32}\|A_2^{-1}(\varepsilon^n - \varepsilon^{n-1})\|_{L^2}^2 \\
&+ c\tau(\|\varepsilon^j\|_{L^2}^2 + \tau\|A_1^{\frac{1}{2}}\varepsilon^j\|_{L^2}^2)\|A_1^{\frac{1}{2}}e^j\|_{L^2}^2, \\
&2\gamma|(\int_{-d}^{z}\nabla Z(\varepsilon^n)d\xi, A_1^{-2}e^n)_\Omega|\tau \le \frac{1}{32}\|A_1^{-\frac{1}{2}}\hat{e}^n\|_{L^2}^2\tau + \frac{1}{32}\|A_1^{-1}(e^n - e^{n-1})\|_{L^2}^2 \\
&+ c[\|A_1^{-1}\varepsilon^{n-1}\|_{L^2}^2 + \|A_2^{-1}\varepsilon^{n-2}\|_{L^2}^2 + \frac{\tau}{2}(\|A_2^{-\frac{1}{2}}\varepsilon^{n-1}\|_{L^2}^2 + \|A_2^{-\frac{1}{2}}\varepsilon^{n-2}\|_{L^2}^2)]\tau, \\
&2\sigma|(\int_{-d}^{z}\nabla\cdot Z(e^n)d\xi, A_2^{-2}\varepsilon^n)_\Omega|\tau \le \frac{1}{32}\|A_2^{-\frac{1}{2}}\hat{\varepsilon}^n\|_{L^2}^2\tau + \frac{1}{32}\|A_2^{-1}(\varepsilon^n - \varepsilon^{n-1})\|_{L^2}^2 \\
&+ c[\|A_1^{-1}e^{n-1}\|_{L^2}^2 + \|A_1^{-1}e^{n-2}\|_{L^2}^2 + \frac{\tau}{2}(\|A_1^{-\frac{1}{2}}e^{n-1}\|_{L^2}^2 + \|A_1^{-\frac{1}{2}}e^{n-2}\|_{L^2}^2)]\tau, \\
&2\tau|(f\vec{k}\times Z(e^n), A_1^{-2}e^n)_\Omega| = 2\tau|(f\vec{k}\times A_1^{-2}e^n, Z(e^n))_\Omega| \\
&\le \frac{1}{32}\|A_1^{-\frac{1}{2}}\hat{e}^n\|_{L^2}^2\tau + \frac{1}{32}\|A_1^{-1}(e^n - e^{n-1})\|_{L^2}^2 + c(\|A_1^{-1}e^{n-1}\|_{L^2}^2 + \|A_1^{-1}e^{n-2}\|_{L^2}^2)\tau, \\
&2|(E_1^n, A_1^{-2}e^n)_\Omega|\tau \le \frac{1}{32}\|A_1^{-\frac{1}{2}}\hat{e}^n\|_{L^2}^2\tau + \frac{1}{32}\|A_1^{-\frac{1}{2}}(e^n - e^{n-1})\|_{L^2}^2\tau + c\|A_1^{-\frac{3}{2}}PE_1^n\|_{L^2}^2\tau, \\
&2|(E_2^n, A_2^{-2}\varepsilon^n)_\Omega|\tau \le \frac{1}{32}\|A_2^{-\frac{1}{2}}\hat{\varepsilon}^n\|_{L^2}^2\tau + \frac{1}{32}\|A_2^{-\frac{1}{2}}(\varepsilon^n - \varepsilon^{n-1})\|_{L^2}^2\tau + c\|A_2^{-\frac{3}{2}}E_2^n\|_{L^2}^2\tau,
\end{aligned}
$$

for $j = n-1,\ n-2$. Combining (5.6) with the above inequalities yields

$$
\begin{aligned}
&[\|A_1^{-1}e^n\|_{L^2}^2 + \|A_2^{-1}\varepsilon^n\|_{L^2}^2 + \frac{\tau}{2}\|A_1^{-\frac{1}{2}}e^n\|_{L^2}^2 + \frac{\tau}{2}\|A_2^{-\frac{1}{2}}\varepsilon^n\|_{L^2}^2] \\
&\qquad - [\|A_1^{-1}e^{n-1}\|_{L^2}^2 + \|A_2^{-1}\varepsilon^{n-1}\|_{L^2}^2 + \frac{\tau}{2}\|A_1^{-\frac{1}{2}}e^{n-1}\|_{L^2}^2 + \frac{\tau}{2}\|A_2^{-\frac{1}{2}}\varepsilon^{n-1}\|_{L^2}^2] \\
&\qquad + \frac{3}{2}[\|A_1^{-\frac{1}{2}}\hat{e}^n\|_{L^2}^2\tau + \|A_2^{-\frac{1}{2}}\hat{\varepsilon}^n\|_{L^2}^2]\tau - \frac{1}{2}[\|A_1^{-\frac{1}{2}}\hat{e}^{n-1}\|_{L^2}^2\tau + \|A_2^{-\frac{1}{2}}\hat{\varepsilon}^{n-1}\|_{L^2}^2]\tau \\
&\qquad + \frac{3}{4}[\|A_1^{-1}(e^n - e^{n-1})\|_{L^2}^2 + \|A_2^{-1}(\varepsilon^n - \varepsilon^{n-1})\|_{L^2}^2] \\
&\qquad - \frac{1}{4}[\|A_1^{-1}(e^{n-1} - e^{n-2})\|_{L^2}^2 + \|A_2^{-1}(\varepsilon^{n-1} - \varepsilon^{n-2})\|_{L^2}^2] \\
&\qquad \le \frac{\tau}{2}\sum_{j=n-2}^{n-1} d_j[\|A_1^{-1}e^j\|_{L^2}^2 + \|A_2^{-1}\varepsilon^j\|_{L^2}^2 + \frac{\tau}{2}\|A_1^{-\frac{1}{2}}e^j\|_{L^2}^2 + \frac{\tau}{2}\|A_2^{-\frac{1}{2}}\varepsilon^j\|_{L^2}^2] \\
&\qquad + c\tau\sum_{j=n-1}^{n}(\|A_1^{-\frac{1}{2}}(e^j - e^{j-1})\|_{L^2}^2 + \|A_2^{-\frac{1}{2}}(\varepsilon^j - \varepsilon^{j-1})\|_{L^2}^2) \\
&\qquad + c\tau^2\sum_{j=n-2}^{n-1}(\|u(t_j)\|_{H^2}^2 + \|\theta(t_j)\|_{H^2}^2)(\|e^j\|_{L^2}^2 + \|\varepsilon^j\|_{L^2}^2) \\
&\qquad + c\|A_1^{-\frac{3}{2}}PE_1^n\|_{L^2}^2\tau + c\|A_2^{-\frac{3}{2}}E_2^n\|_{L^2}^2\tau,
\end{aligned}
\tag{5.7}
$$

for $2 \le n \le N$.

Summing (5.7) from $n=2$ to m and using Lemma 3.1, Lemma 4.1, Lemma 5.1, Theorem 2.1 and Theorem 4.1, we deduce

$$\begin{aligned}
&\|A_1^{-1}e^m\|_{L^2}^2+\|A_2^{-1}\varepsilon^m\|_{L^2}^2+\frac{\tau}{2}\|A_1^{-\frac{1}{2}}e^m\|_{L^2}^2+\frac{\tau}{2}\|A_2^{-\frac{1}{2}}\varepsilon^m\|_{L^2}^2+\tau\sum_{n=1}^{m}\|A_1^{-\frac{1}{2}}\hat{e}^n\|_{L^2}^2\\
&+\tau\sum_{n=1}^{m}[\|A_2^{-\frac{1}{2}}\hat{\varepsilon}^n\|_{L^2}^2+\|A_1^{-1}(e^n-e^{n-1})\|_{L^2}^2+\|A_2^{-1}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2]\le C\tau^4\\
&+\tau\sum_{n=1}^{m-1}d_n(\|A_1^{-1}e^n\|_{L^2}^2+\|A_1^{-1}\varepsilon^n\|_{L^2}^2+\frac{\tau}{2}\|A_1^{-\frac{1}{2}}e^n\|_{L^2}^2+\frac{\tau}{2}\|A_2^{-\frac{1}{2}}\varepsilon^n\|_{L^2}^2),
\end{aligned}\tag{5.8}$$

for $1\le m\le N$. Applying Lemma 2.1 to (5.8) and using (4.26), we deduce (5.5).

Lemma 5.3. Under the assumption of Theorem 4.1, (u^m,p^m,θ^m) satisfies the following error estimate:

$$\begin{aligned}
&\sum_{n=1}^{m}\sigma(t_n)[\|\hat{e}^n\|_{L^2}^2\tau+\|\hat{\varepsilon}^n\|_{L^2}^2\tau+\|A_1^{-\frac{1}{2}}(e^n-e^{n-1})\|_{L^2}^2+\|A_2^{-\frac{1}{2}}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2]\\
&+\sigma(t_m)[\|A_1^{-\frac{1}{2}}e^m\|_{L^2}^2+\|A_2^{-\frac{1}{2}}\varepsilon^m\|_{L^2}^2+\frac{\tau}{2}\|e^m\|_{L^2}^2\tau+\frac{\tau}{2}\|\varepsilon^m\|_{L^2}^2]\le\kappa\tau^4,
\end{aligned}\tag{5.9}$$

$$\begin{aligned}
&\sum_{n=1}^{m}\sigma(t_n)[\|A_1^{\frac{1}{2}}\hat{e}^n\|_{L^2}^2\tau+\|A_2^{\frac{1}{2}}\hat{\varepsilon}^n\|_{L^2}^2\tau+\|e^n-e^{n-1}\|_{L^2}^2+\|\varepsilon^n-\varepsilon^{n-1}\|_{L^2}^2]\\
&+\sigma(t_m)[\|e^m\|_{L^2}^2+\|\varepsilon^m\|_{L^2}^2+\frac{\tau}{2}\|A_1^{\frac{1}{2}}e^m\|_{L^2}^2\tau+\frac{\tau}{2}\|A_2^{\frac{1}{2}}\varepsilon^m\|_{L^2}^2]\le\kappa\tau^3,
\end{aligned}\tag{5.10}$$

$$\begin{aligned}
&\sum_{n=1}^{m}\sigma(t_n)[\|A_1\hat{e}^n\|_{L^2}^2\tau+\|A_2\hat{\varepsilon}^n\|_{L^2}^2\tau+\|A_1^{\frac{1}{2}}(e^n-e^{n-1})\|_{L^2}^2+\|A_2^{\frac{1}{2}}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2]\\
&+\sigma(t_m)[\|A_1^{\frac{1}{2}}e^m\|_{L^2}^2+\|A_2^{\frac{1}{2}}\varepsilon^m\|_{L^2}^2+\frac{\tau}{2}\|A_1e^m\|_{L^2}^2+\frac{\tau}{2}\|A_2\varepsilon^m\|_{L^2}^2]\le\kappa\tau^2,
\end{aligned}\tag{5.11}$$

for all $1\le m\le N$.

Proof. Multiplying (4.24) with $k=1$ by $\sigma(t_n)$ and noting $\sigma(t_n)\le\sigma(t_{n-1})+\tau,\ \sigma(t_n)\le\sigma(t_{n-2})+2\tau$, and using Lemma 2.1 and (4.26), we deduce (5.11).

Multiplying (4.24) with $k=0$ by $\sigma(t_n)$, using Lemma 2.1 and (4.26), we deduce (5.10).

Now, multiplying (5.3) by $\sigma(t_n)$ and summing from $n=2$ to $n=m$, using (5.10), Lemma 3.1, Lemma 4.1, Lemma 5.1, Lemma 5.2 and Theorem 4.1, we deduce

$$\begin{aligned}
&\sigma(t_m)[\|A_1^{-\frac{1}{2}}e^m\|_{L^2}^2+\|A_2^{-\frac{1}{2}}\varepsilon^m\|_{L^2}^2+\frac{\tau}{2}\|e^m\|_{L^2}^2+\frac{\tau}{2}\|\varepsilon^m\|_{L^2}^2]\\
&+\frac{\tau}{2}\sum_{n=1}^{m}\sigma(t_n)\|\hat{e}^n\|_{L^2}^2+\frac{\tau}{2}\sum_{n=1}^{m}\sigma(t_n)\|\hat{\varepsilon}^n\|_{L^2}^2\\
&+\frac{1}{8}\sum_{n=1}^{m}(\|A_1^{-\frac{1}{2}}(e^n-e^{n-1})\|_{L^2}^2+\|A_2^{-\frac{1}{2}}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2)\le\kappa\tau^4\\
&+3\tau\sum_{n=1}^{m-1}d_n\sigma(t_n)[\|A_1^{-\frac{1}{2}}e^n\|_{L^2}^2+\|A_2^{-\frac{1}{2}}\varepsilon^n\|_{L^2}^2+\frac{\tau}{2}\|e^n\|_{L^2}^2+\frac{\tau}{2}\|\varepsilon^n\|_{L^2}^2],
\end{aligned}\tag{5.12}$$

Applying Lemma 2.1 to (5.12) and using (4.26), we deduce (5.9).

Lemma 5.4. Under the assumption of Theorem 4.1, (u^n, p^n, θ^n) satisfies the following error estimate:

$$\sum_{n=1}^{m}\sigma^2(t_n)[\|A_1^{\frac{1}{2}}\hat{e}^n\|_{L^2}^2\tau+\|A_2^{\frac{1}{2}}\hat{\varepsilon}^n\|_{L^2}^2\tau+\|e^n-e^{n-1}\|_{L^2}^2+\|\varepsilon^n-\varepsilon^{n-1}\|_{L^2}^2]$$
$$+\sigma^2(t_m)[\|e^m\|_{L^2}^2+\|\varepsilon^m\|_{L^2}^2+\frac{\tau}{2}\|A_1^{\frac{1}{2}}e^m\|_{L^2}^2\tau+\frac{\tau}{2}\|A_2^{\frac{1}{2}}\varepsilon^m\|_{L^2}^2]\le\kappa\tau^4, \tag{5.13}$$

$$\sum_{n=1}^{m}\sigma^2(t_n)[\|A_1\hat{e}^n\|_{L^2}^2\tau+\|A_2\hat{\varepsilon}^n\|_{L^2}^2\tau+\|A_1^{\frac{1}{2}}(e^n-e^{n-1})\|_{L^2}^2+\|A_2^{\frac{1}{2}}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2]$$
$$+\sigma^2(t_m)[\|A_1^{\frac{1}{2}}e^m\|_{L^2}^2+\|A_2^{\frac{1}{2}}\varepsilon^m\|_{L^2}^2+\frac{\tau}{2}\|A_1e^m\|_{L^2}^2+\frac{\tau}{2}\|A_2\varepsilon^m\|_{L^2}^2]\le\kappa\tau^3, \tag{5.14}$$

for all $1\le m\le N$.

Proof. Multiplying (4.24) with $k=1$ by $\sigma^2(t_n)$, and summing from $n=2$ to $n=m$, using Lemma 3.1, Lemma 4.1, Lemma 5.1 and Theorem 4.1, we deduce

$$\sum_{n=1}^{m}\sigma^2(t_n)[\|A_1\hat{e}^n\|_{L^2}^2\tau+\|A_2\hat{\varepsilon}^n\|_{L^2}^2+\|A_1^{\frac{1}{2}}(e^n-e^{n-1})\|_{L^2}^2+\|A_2^{\frac{1}{2}}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2]$$
$$+\sigma^2(t_m)[\|A_1^{\frac{1}{2}}e^m\|_{L^2}^2+\|A_2^{\frac{1}{2}}\varepsilon^m\|_{L^2}^2+\frac{\tau}{2}\|A_1e^m\|_{L^2}^2+\frac{\tau}{2}\|A_2\varepsilon^m\|_{L^2}^2]\le\kappa\tau^3$$
$$+9\tau\sum_{n=1}^{m-1}d_n\sigma^2(t_n)[\|A_1^{\frac{1}{2}}e^n\|_{L^2}^2+\|A_2^{\frac{1}{2}}\varepsilon^n\|_{L^2}^2+\frac{\tau}{2}\|A_1e^n\|_{L^2}^2+\frac{\tau}{2}\|A_2\varepsilon^n\|_{L^2}^2], \tag{5.15}$$

for all $1\le m\le N$. Applying Lemma 2.1 to (5.15) and using (4.26), we deduce (5.14).

Multiplying (4.24) with $k=0$ by $\sigma^2(t_n)$, and summing from $n=2$ to $n=m$, using Lemma 3.1, Lemma 4.1, Lemma 5.1 and Theorem 4.1, we deduce

$$\tau\sum_{n=1}^{m}\sigma^2(t_n)[\|A_1^{\frac{1}{2}}\hat{e}^n\|_{L^2}^2+\|A_2^{\frac{1}{2}}\hat{\varepsilon}^n\|_{L^2}^2+\frac{1}{4}\|e^n-e^{n-1}\|_{L^2}^2+\frac{1}{4}\|\varepsilon^n-\varepsilon^{n-1}\|_{L^2}^2]$$
$$+\sigma^2(t_m)[\|e^m\|_{L^2}^2+\|\varepsilon^m\|_{L^2}^2+\frac{1}{2}\|A_1^{\frac{1}{2}}e^m\|_{L^2}^2\tau+\frac{1}{2}\|A_2^{\frac{1}{2}}\varepsilon^m\|_{L^2}^2\tau]\le\kappa\tau^4$$
$$+9\tau\sum_{n=1}^{m-1}d_n\sigma^2(t_n)[\|e^n\|_{L^2}^2+\|\varepsilon^n\|_{L^2}^2+\frac{1}{2}\|A_1^{\frac{1}{2}}e^n\|_{L^2}^2\tau+\frac{1}{2}\|A_2^{\frac{1}{2}}\varepsilon^n\|_{L^2}^2\tau], \tag{5.16}$$

for all $1\le m\le N$. Applying Lemma 2.1 to (5.16) and using (4.26), we deduce (5.14).

Lemma 5.5. Under the assumption of Theorem 4.1, (u^m, p^m, θ^m) satisfies the following error estimate:

$$\sum_{n=1}^{m}\sigma^3(t_n)[\|A_1\hat{e}^n\|_{L^2}^2\tau+\|A_2\hat{\varepsilon}^n\|_{L^2}^2\tau+\|A_1^{\frac{1}{2}}(e^n-e^{n-1})\|_{L^2}^2+\|A_2^{\frac{1}{2}}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2]$$
$$+\sigma^3(t_m)[\|A_1^{\frac{1}{2}}e^m\|_{L^2}^2+\|A_2^{\frac{1}{2}}\varepsilon^m\|_{L^2}^2+\frac{\tau}{2}\|A_1e^m\|_{L^2}^2+\frac{\tau}{2}\|A_2\varepsilon^m\|_{L^2}^2]\le\kappa\tau^4, \tag{5.17}$$

for all $1\le m\le N$.

Proof. Multiplying (4.24) with $k=1$ by $\sigma^3(t_n)$ and noting $\sigma(t_n)\le\sigma(t_{n-1})+\tau,\ \sigma(t_n)\le\sigma(t_{n-2})+2\tau$, we obtain

$$\begin{aligned}
&\sigma^3(t_n)[\|A_1^{\frac12}e^n\|_{L^2}^2+\|A_2^{\frac12}\varepsilon^n\|_{L^2}^2+\frac{\tau}{2}\|A_1e^n\|_{L^2}^2+\frac{\tau}{2}\|A_2\varepsilon^n\|_{L^2}^2]\\
&-\sigma^3(t_{n-1})[\|A_1^{\frac12}e^{n-1}\|_{L^2}^2+\|A_2^{\frac12}\varepsilon^{n-1}\|_{L^2}^2+\frac{\tau}{2}\|A_1e^{n-1}\|_{L^2}^2+\frac{\tau}{2}\|A_2\varepsilon^{n-1}\|_{L^2}^2]\\
&+\tau\sigma^3(t_n)(\|A_1\hat e^n\|_{L^2}^2\tau+\|A_2\hat\varepsilon^n\|_{L^2}^2)\\
&+\frac34\sigma^3(t_n)(\|A_1^{\frac12}(e^n-e^{n-1})\|_{L^2}^2+\|A_2^{\frac12}(\varepsilon^n-\varepsilon^{n-1})\|_{L^2}^2)\\
&-\frac12\sigma^3(t_{n-1})(\|A_1^{\frac12}(e^{n-1}-e^{n-2})\|_{L^2}^2+\|A_2^{\frac12}(\varepsilon^{n-1}-\varepsilon^{n-2})\|_{L^2}^2)\\
&\le 3\tau\sigma^2(t_{n-1})[\|A_1^{\frac12}e^{n-1}\|_{L^2}^2+\|A_2^{\frac12}\varepsilon^{n-1}\|_{L^2}^2\\
&+\frac{\tau}{2}\|A_1e^{n-1}\|_{L^2}^2+\frac{\tau}{2}\|A_2\varepsilon^{n-1}\|_{L^2}^2]\\
&+\tau\frac{27}{2}\sum_{j=n-2}^{n-1}d_j\sigma^3(t_j)(\|A_1^{\frac12}e^j\|_{L^2}^2+\|A_2^{\frac12}\varepsilon^j\|_{L^2}^2+\frac{\tau}{2}\|A_1e^j\|_{L^2}^2+\frac{\tau}{2}\|A_2\varepsilon^j\|_{L^2}^2)\\
&+c\sigma^3(t_n)[\|E_1^n\|_{L^2}^2\tau+c\|A_1^{\frac12}PE_1^n\|_{L^2}^2\tau^2+c\|E_2^n\|_{L^2}^2\tau+c\|A_2^{\frac12}E_2^n\|_{L^2}^2\tau^2]\\
&+\tau c(\|A_1^{\frac12}\hat e^{n-1}\|_{L^2}^2+\|A_1^{\frac12}(e^{n-1}-e^{n-2})\|_{L^2}^2+\|A_1(e^{n-1}-e^{n-2})\|_{L^2}^2\tau)\\
&\times\sum_{j=n-2}^{n-1}\sigma^2(t_j)(\|A_1e^j\|_{L^2}^2+\|A_2\varepsilon^j\|_{L^2}^2)\\
&+\tau c(\|A_2^{\frac12}\hat\varepsilon^{n-1}\|_{L^2}^2+\|A_2^{\frac12}(\varepsilon^{n-1}-\varepsilon^{n-2})\|_{L^2}^2+\|A_2(\varepsilon^{n-1}-\varepsilon^{n-2})\|_{L^2}^2\tau)\\
&\times\sum_{j=n-2}^{n-1}\sigma^2(t_j)(\|A_1e^j\|_{L^2}^2+\|A_2\varepsilon^j\|_{L^2}^2).
\end{aligned}\tag{5.18}$$

Summing (5.18) from $n=2$ to $n=m$, using Lemma 3.1, Lemma 4.1, Lemma 5.1-Lemma 5.4 and Theorem 4.1, we deduce

$$\begin{aligned}
&\sum_{n=1}^{m}\sigma^3(t_n)[\|A_1\hat e^n\|_{L^2}^2\tau+\|A_2\hat\varepsilon^n\|_{L^2}^2\tau+\frac14(\|A_1^{\frac12}d_te^n\|_{L^2}^2+\|A_2^{\frac12}d_t\varepsilon^n\|_{L^2}^2)\tau^2]\\
&+\sigma^3(t_m)[\|A_1^{\frac12}e^m\|_{L^2}^2+\|A_2^{\frac12}\varepsilon^m\|_{L^2}^2+\frac{\tau}{2}\|A_1e^m\|_{L^2}^2+\frac{\tau}{2}\|A_2\varepsilon^m\|_{L^2}^2]\le\kappa\tau^4\\
&+\tau\sum_{n=1}^{m-1}27d_n\sigma^3(t_n)[\|A_1^{\frac12}e^n\|_{L^2}^2+\|A_2^{\frac12}\varepsilon^n\|_{L^2}^2+\frac{\tau}{2}\|A_1e^n\|_{L^2}^2+\frac{\tau}{2}\|A_2\varepsilon^n\|_{L^2}^2],
\end{aligned}\tag{5.19}$$

for all $1\le m\le N$. Applying Lemma 2.1 to (5.19) and using (4.26), we deduce (5.17).

Finally, combining Theorem 4.1 and (4.26) with Theorem 2.1 and using Assumption **A2** has completed the proof of the stability results in Theorem 1.2 and combining Lemma 5.4 and Lemma 5.5 with Theorem 4.2 has completed the proof of the convergence results in Theorem 1.2.

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