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ERROR ESTIMATES FOR THE SECOND ORDER SEMI-DISCRETE STABILIZED GAUGE-UZAWA METHOD FOR THE NAVIER-STOKES EQUATIONS

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Abstract. The Gauge-Uzawa method [GUM], which is a projection type algorithm to solve the time depend Navier-Stokes equations, has been constructed in [14] and enhanced in [15, 17] to apply to more complicated problems. Even though GUM possesses many advantages theoretically and numerically, the studies on GUM have been limited on the first order backward Euler scheme except normal mode error estimate in [16]. The goal of this paper is to research the 2nd order GUM. Because the classical 2nd order GUM which is studied in [16] needs rather strong stability condition, we modify GUM to be unconditionally stable method using BDF2 time marching. The stabilized GUM is equivalent to the rotational form of pressure correction method and the errors are already estimated in [8] for the Stokes equations. In this paper, we will evaluate errors of the stabilized GUM for the Navier-Stokes equations. We also prove that the stabilized GUM is an unconditionally stable method for the Naiver-Stokes equations. So we conclude that the rotational form of pressure correction method in [8] is also unconditionally stable scheme and that the accuracy results in [8] are valid for the Navier-Stokes equations.

Key Words. Projection method, Gauge-Uzawa method, the rotational form of pressure correction method, Navier-Stokes equations, incompressible fluids

1. Introduction

Given an open bounded polyhedral domain Ω in $\mathbb{R}^d$, with $d = 2$ or 3, we consider the time-dependent Navier-Stokes equations of incompressible fluids:

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mu\triangle\mathbf{u} = \mathbf{f}, &\quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0, &\quad \text{in } \Omega, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}^0, &\quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

with vanishing Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$ and pressure mean-value $\int_\Omega p = 0$. The primitive variables are the (vector) velocity $\mathbf{u}$ and the (scalar) pressure p. The viscosity $\mu = Re^{-1}$ is the reciprocal of the Reynolds number Re. Hereafter, vectors are denoted in boldface.

Pressure p can be viewed in (1.1) as a Lagrange multiplier corresponding to the incompressibility condition $\nabla\cdot\mathbf{u} = 0$. This coupling is responsible for compatibility conditions between the spaces for $\mathbf{u}$ and p, characterized by the celebrated inf-sup condition, and associated numerical difficulties [5, 20]. On the other hand, projection methods were

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introduced independently by Chorin [1] and Temam [19] in the late 60's to decouple $\mathbf{u}$ and p and thus reduce the computational cost. And the methods quickly gained popularity in the computational fluid dynamics community, and over the years, an enormous amount of efforts have been devoted to develop more accurate and efficient projection type schemes, we refer to [16, 7] for comprehensive and up-to-date review on this subject.

Because most engineers prefer higher order methods, many projection methods have been built using 2nd order time discrete schemes which are the Crank-Nicolson scheme and 2nd order backward difference formulation [BDF2]. In general, both of them have same accuracy, but BDF2 displays better numerical behaviors on stability than the Crank-Nicolson scheme. So most of new methods, the pressure-correction in [6], the velocity-correction in [10, 3], and the consistent splitting method in [9] have been studied with respect to BDF2 for time. In addition, the rotational form of pressure-correction method has been introduced in [21] with embarking BDF2 and then the errors have been evaluated via energy estimate in [8] and via normal mode analysis in [16] for the Stokes equations. We also study the method in Algorithm 2 below and discuss about the difficulty to control non-linear term in Remark 1 below. One of the goal of this paper is to extend the accuracy results to the Navier-Stokes equations.

On the other direction, the Gauge-Uzawa method [GUM] has been constructed in [14] to solve (1.1) and enhanced to solve more complicated problems which are the Boussinesq equations in [15] and the non-constant density fluid problems in [17]. However, GUM has been studied only for the 1st order backward Euler scheme for time except normal mode error analysis in [16]. The goal of this paper is to research for the BDF2 GUM to solve Navier-Stokes equations. The classical GUM in [16] displays superior numerical behavior on accuracy, but the method requires rather strong stability condition. In [16, 7], it is known that the classical GUM is a equivalent to the consistent splitting scheme in [9]. So both methods request high computational cost due to tiny τ to make hold the stability constraint. In this paper, we newly construct the stabilized BDF2 GUM and prove optimal error estimates for the Navier-Stokes equations. We will also prove that the method is unconditionally stable scheme for any time step τ. In addition, we discover that the stabilized BDF2 GUM is equivalent to the rotational form of pressure correction method in [8]. So we conclude that the rotational form of pressure correction method is also unconditionally stable for any τ and that the error decay results in [8] are extended to the Navier-Stokes equations.

In this paper, we will use standard notations. Let $H^s(\Omega)$ be the Sobolev space with s derivatives in $L^2(\Omega)$, $\mathbf{L}^2(\Omega) = \left(L^2(\Omega)\right)^d$ and $\mathbf{H}^s(\Omega) = \left(H^s(\Omega)\right)^d$, where $d = 2, 3$. Let $\|\cdot\|_0$ denote the $\mathbf{L}^2(\Omega)$ norm, and $\langle \cdot\,, \cdot\rangle$ the corresponding inner product. Let $\|\cdot\|_s$ denote the norm of $H^s(\Omega)$ for $s \in \mathbb{R}$. In addition, we will denote τ as the time marching size. Also we will use δ as difference of 2 consecutive functions, for example, for any sequence function z^{n+1},

$$\delta z^{n+1} = z^{n+1} - z^n, \quad \delta\delta z^{n+1} = \delta(\delta z^{n+1}) = z^{n+1} - 2z^n + z^{n-1}, \quad \cdots .$$

This paper is organized as follows. In §2, we will derive the 2nd order stabilized GUMs and the rotational form of pressure correction method in [8]. And then we state main theorems for stability and accuracy. We introduce some well-known lemma in §3 to use in theoretical proofs. We then prove stability of the stabilized GUM in §4 and estimate errors of the stabilized GUM in §5. We finally conclude in §6 with numerical tests to compare with theoretical results.

2. The stabilized Gauge-Uzawa method

In this section. we will derive the stabilized BDF2 time discrete GUM and the rotational form of projection method in [8, 16]. We will conclude that both of them are equivalent algorithms and state theoretical results for stability in Theorem 1 and for error estimates in Theorem 2 for the Navier-Stokes equations.

Because the standard 2nd order GUM has been precisely derived in [16] from the gauge method in [4, 13] via changing variables, we here introduce GUM briefly and directly from BDF2 time discrete Stokes equations:

$$\frac{3\mathbf{u}^{n+1}-4\mathbf{u}^{n}+\mathbf{u}^{n-1}}{2\tau}+\nabla p^{n+1}-\mu\triangle\mathbf{u}^{n+1}=\mathbf{f}(t^{n+1}). \tag{2.1}$$

GUM hires artificial variables $\widehat{\mathbf{u}}^{n+1}$ and ϕ^{n+1} satisfying

$$\widehat{\mathbf{u}}^{n+1}=\mathbf{u}^{n+1}-\nabla\left(\phi^{n+1}-2\phi^{n}+\phi^{n-1}\right). \tag{2.2}$$

The main strategy of GUM is to compute $\widehat{\mathbf{u}}^{n+1}$ and ϕ^{n+1}, and then calculate $\mathbf{u}^{n+1}$ by addition of the 2 functions. In the view of (2.2), $\widehat{\mathbf{u}}^{n+1}$ and ϕ^{n+1} depend each other, so the role of $\widehat{\mathbf{u}}^{n+1}$ will be decided automatically, provided that of ϕ^{n+1} is given. We will define ϕ^{n+1} soon. If we insert (2.2) into (2.1), then we obtain

$$\begin{aligned}\frac{3\widehat{\mathbf{u}}^{n+1}-4\mathbf{u}^{n}+\mathbf{u}^{n-1}}{2\tau}&+\nabla\left(p^{n+1}+\frac{3\phi^{n+1}-6\phi^{n}+3\phi^{n-1}}{2\tau}\right)\\&-\mu\triangle\left(\widehat{\mathbf{u}}^{n+1}+\nabla\left(\phi^{n+1}-2\phi^{n}+\phi^{n-1}\right)\right)=\mathbf{f}(t^{n+1}).\end{aligned} \tag{2.3}$$

We now contemplate to define ϕ^{n+1} to split (2.3) into uncoupled 2 equations. The classical GUM in [13, 14, 16] impose ϕ^{n+1} as a solution of time discrete heat equation, like

$$\frac{3\phi^{n+1}-4\phi^{n}+\phi^{n-1}}{2\tau}-\mu\triangle\phi^{n+1}=-p^{n+1}. \tag{2.4}$$

Then (2.3) becomes

$$\begin{aligned}&\frac{3\widehat{\mathbf{u}}^{n+1}-4\mathbf{u}^{n}+\mathbf{u}^{n-1}}{2\tau}-\mu\triangle\widehat{\mathbf{u}}^{n+1}\\&-\nabla\left(\frac{\phi^{n}-\phi^{n-1}}{\tau}-\mu\triangle\left(2\phi^{n}-\phi^{n-1}\right)\right)=\mathbf{f}(t^{n+1}).\end{aligned} \tag{2.5}$$

Combining 3 equations (2.2), (2.4) and (2.5) lead the classical GUM. Because this GUM performs superior numerical behavior for accuracy, we have concentrated to study this GUM. But the method requires rather strong stability constraint for τ and the theoretical proof is still open problem. So, in this paper, we modify the role of ϕ^{n+1} as a solution of

$$\frac{3\phi^{n+1}-3\phi^{n}}{2\tau}-\mu\triangle\left(\phi^{n+1}-\phi^{n}\right)=-p^{n+1}. \tag{2.6}$$

Then we can rewrite (2.3) by

$$\begin{aligned}\frac{3\widehat{\mathbf{u}}^{n+1}-4\mathbf{u}^{n}+\mathbf{u}^{n-1}}{2\tau}-\nabla&\left(\frac{3(\phi^{n}-\phi^{n-1})}{2\tau}-\mu\triangle\left(\phi^{n}-\phi^{n-1}\right)\right)\\&-\mu\triangle\widehat{\mathbf{u}}^{n+1}=\mathbf{f}(t^{n+1}).\end{aligned} \tag{2.7}$$

We note that (2.6) is not a time discrete scheme of heat equation or other PDEs. So ϕ has no physical meaning and is only a parameter to construct GUM. Because the functions of ϕ in (2.2), (2.6) and (2.7) are represented by the subtraction of 2 consecutive functions, we use simple notation $\psi^{n+1}:=\phi^{n+1}-\phi^{n}$. Owing to divergence free condition $\nabla\cdot\,\mathbf{u}^{n+1}=0$, (2.2) gives

$$\begin{aligned}-\triangle\psi^{n+1}&=-\triangle\left(\phi^{n+1}-\phi^{n}\right)\\&=-\triangle\left(\phi^{n}-\phi^{n-1}\right)+\nabla\cdot\,\widehat{\mathbf{u}}^{n+1}=-\triangle\psi^{n}+\nabla\cdot\,\widehat{\mathbf{u}}^{n+1}.\end{aligned} \tag{2.8}$$

To deal with the third order term $\nabla\triangle\phi^n$, which is a source of trouble due to lack of commutativity of the differential operators at the discrete level, we denote $q^{n+1} := \triangle\psi^{n+1}$. So (2.8) can be rewritten by

$$q^{n+1} = q^n - \nabla\cdot \widehat{\mathbf{u}}^{n+1},$$

which is connected with the Uzawa iteration. If we added up convection term in (2.7) with a suitable approximation $\mathbf{u}^{n+1} \approx 2\mathbf{u}^n - \mathbf{u}^{n-1}$, then we end up the second order GUM via gathering above equations.

Algorithm 1 (The stabilized Gauge-Uzawa Method)**.** *Compute* $\mathbf{u}^1$ *and* p^1 *via any first order projection method and set* $\psi^1 = \frac{-2\tau}{3}p^1$ *and* $q^1 = 0$. *Repeat for* $1 \le n \le N = [\frac{T}{\tau} - 1]$.

Step 1: *Set* $\mathbf{u}^* = 2\mathbf{u}^n - \mathbf{u}^{n-1}$ *and find* $\widehat{\mathbf{u}}^{n+1}$ *as the solution of*

$$\begin{aligned} &\frac{3\widehat{\mathbf{u}}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\tau} + \nabla p^n + (\mathbf{u}^* \cdot \nabla)\widehat{\mathbf{u}}^{n+1} - \mu\triangle\widehat{\mathbf{u}}^{n+1} = \mathbf{f}(t^{n+1}), \\ &\widehat{\mathbf{u}}^{n+1}|_\Gamma = \mathbf{0}. \end{aligned} \tag{2.9}$$

Step 2: *Find* ψ^{n+1} *as the solution of*

$$\begin{aligned} -\triangle\psi^{n+1} &= -\triangle\psi^n + \nabla\cdot\widehat{\mathbf{u}}^{n+1}, \\ \partial_{\boldsymbol{\nu}}\psi^{n+1}|_\Gamma &= 0. \end{aligned}$$

Step 3: *Update* $\mathbf{u}^{n+1}$ *and* q^{n+1} *by*

$$\begin{aligned} \mathbf{u}^{n+1} &= \widehat{\mathbf{u}}^{n+1} + \nabla\left(\psi^{n+1} - \psi^n\right) \\ q^{n+1} &= q^n - \nabla\cdot\widehat{\mathbf{u}}^{n+1}. \end{aligned} \tag{2.10}$$

Step 4: *Update pressure* p^{n+1} *by*

$$p^{n+1} = -\frac{3\psi^{n+1}}{2\tau} + \mu q^{n+1}. \tag{2.11}$$

We remark that Algorithm 1 consists with (1.1), like the classical GUM. In order to derive the rotational form of pressure correction projection method which is studied in [8, 16], we denote

$$\xi^{n+1} := -\frac{3\left(\psi^{n+1} - \psi^n\right)}{2\tau}$$

and we subtract 2 consecutive equations of (2.11) to get

$$p^{n+1} = p^n + \xi^{n+1} - \mu\nabla\cdot\widehat{\mathbf{u}}^{n+1}.$$

Then we arrive at the rotational form of pressure correction projection method in [8, 16].

Algorithm 2 (The rotational form of pressure correction projection method)**.** *Repeat for* $1 \le n \le N = [\frac{T}{\tau} - 1]$.

Step 1: *Set* $\mathbf{u}^* = 2\mathbf{u}^n - \mathbf{u}^{n-1}$ *and find* $\widehat{\mathbf{u}}^{n+1}$ *as the solution of* (2.9)

Step 2: *Find* ξ^{n+1} *as the solution of*

$$\begin{aligned} \triangle\xi^{n+1} &= \frac{3}{2\tau}\nabla\cdot\widehat{\mathbf{u}}^{n+1}, \\ \partial_{\boldsymbol{\nu}}\xi^{n+1}|_\Gamma &= 0. \end{aligned}$$

Step 3: *Update* $\mathbf{u}^{n+1}$ *and* p^{n+1} *by*

$$\mathbf{u}^{n+1} = \widehat{\mathbf{u}}^{n+1} - \frac{2\tau}{3}\nabla\xi^{n+1},$$

$$p^{n+1} = \xi^{n+1} + p^n - \mu\nabla\cdot\widehat{\mathbf{u}}^{n+1}. \tag{2.12}$$

Remark 2.1 (Difference between Algorithms 1 and 2). *Algorithms 1 and 2 are basically equivalent in semi-discrete level. Only difference is the represent ion of pressure between* (2.11) *and* (2.12). *The pressure p^{n+1} in* (2.11) *is designed by addition of 2 functions of ψ and q. And both of them can be expressed by $\nabla \cdot \widehat{\mathbf{u}}$, so we can replace p^n in momentum equation* (2.9) *to terms of $\nabla \cdot \widehat{\mathbf{u}}$. This is very crucial fact to prove Lemma 5.3 in §5. On the other hand, the pressure in* (2.12) *is formed by $p^{n+1} - p^n$ which is not matched with the pressure term in the momentum equation* (2.9). *Thus, in* [8], *they carried out error estimate with subtracting of 2 consecutive momentum equations to get $p^n - p^{n-1}$ term and to replace to terms of $\nabla \cdot \widehat{\mathbf{u}}$ by using* (2.12) *without result of Lemma 5.3. So they could prove optimal order accuracy only for the Stokes equations. We also use the same technique in Lemma 5.5, but we can handle convection terms by applying the sub-optimal result of Lemma 5.3. So we could prove the extended result of* [8] *to the Navier Stokes equations.*

We will prove that the following stability lemma in §4. Because Algorithms 1 and 2 are equivalent, we conclude that Algorithm 2 is also unconditionally stable.

Theorem 1 (Stability). *The Algorithm 1 is unconditionally stable in the sense that for all $\tau > 0$ the following a priori bound holds:*

$$
\begin{aligned}
&\|\widehat{\mathbf{u}}^{N+1}\|_0^2 + \|\mathbf{u}^{N+1}\|_0^2 + \|2\mathbf{u}^{N+1} - \mathbf{u}^N\|_0^2 + 3\|\nabla\psi^{N+1}\|_0^2 \\
&+ 2\tau\mu\|q^{N+1}\|_0^2 + \sum_{n=1}^{N}\Big(\|\delta\delta\mathbf{u}^{N+1}\|_0^2 + 3\|\nabla\delta\psi^{n+1}\|_0^2 + \tau\mu\|\nabla\widehat{\mathbf{u}}^{n+1}\|_0^2\Big) \\
&\le \|2\mathbf{u}^1 - \mathbf{u}^0\|_0^2 + \|\mathbf{u}^0\|_0^2 + 3\|\nabla\psi^1\|_0^2 + 2\tau\mu\|q^1\|_0^2 + C\frac{\tau}{\mu}\|\mathbf{f}(t^{n+1})\|_{-1}^2.
\end{aligned} \tag{2.13}
$$

In the error estimate, we resort to a duality argument via the following Stokes equations:

$$
\begin{aligned}
-\triangle\mathbf{v} + \nabla r &= \mathbf{w}, && \text{in } \Omega, \\
\nabla\cdot\mathbf{v} &= 0, && \text{in } \Omega, \\
\mathbf{v} &= 0, && \text{on } \partial\Omega.
\end{aligned} \tag{2.14}
$$

We now state a basic assumption about Ω.

Assumption 1 (Regularity of Ω). *The unique solution $\{\mathbf{v}, r\}$ of the steady Stokes equations* (2.14) *satisfies*

$$
\|\mathbf{v}\|_2 + \|r\|_1 \le C\|\mathbf{w}\|_0.
$$

We remark that the validity of Assumption 1 is known if $\partial\Omega$ is of class $\mathbf{C}^2$ [2, 11], or if $\partial\Omega$ is a two-dimensional convex polygon [12], and is generally believed for convex polyhedral [11].

In order to launch Algorithm 1, we need to set $(\mathbf{u}^1, p^1)$ via any first order projection method. The following assumption is used to control initial error.

Assumption 2 (Initial setting). *Let $(\mathbf{u}(t^1), p(t^1))$ be the exact solution of* (1.1) *at $t = t^1$. The initial value $(\mathbf{u}^1, p^1)$ satisfies*

$$
\|\mathbf{u}(t^1) - \mathbf{u}^1\|_0 \le C\tau^2 \quad \textit{and} \quad \|\mathbf{u}(t^1) - \mathbf{u}^1\|_1 + \|p(t^1) - p^1\|_0 \le C\tau.
$$

We will prove the following main theorem through several lemmas in §5.

Theorem 2 (Error estimates). *Suppose the exact solution of* (1.1) *is smooth enough. If Assumptions 1 and 2 hold, then the errors of Algorithms 1 and 2 will be bounded by*

$$\tau \sum_{n=1}^{N} \left(\left\| \mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1} \right\|_0^2 + \left\| \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}^{n+1} \right\|_0^2 \right) \leq C\tau^4,$$
$$\tau \sum_{n=1}^{N} \left(\left\| \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}^{n+1} \right\|_1^2 + \left\| p(t^{n+1}) - p^{n+1} \right\|_0^2 \right) \leq C\tau^2.$$

Furthermore, if assumption 2 hold, then we have

$$\left\| \nabla \cdot \widehat{\mathbf{u}}^{n+1} \right\|_0 \leq C\tau^{\frac{3}{2}}. \tag{2.15}$$

3. Preliminaries

This section is mainly devoted to reviewing some well-known lemmas. The basic mathematical theories summarized in here can be found in [5, 20]. We first define the trilinear form $\mathcal{N}$ associated with the convection term in (1.1)

$$\mathcal{N}(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} d\mathbf{x},$$

for which the following properties are well known [5, 20].

Lemma 3.1 (Properties of $\mathcal{N}$). *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)$ *and* $\nabla \cdot \mathbf{u} = 0$. *If*

$$\mathbf{u} \cdot \boldsymbol{\nu} = 0 \quad \text{or} \quad \mathbf{v} = \mathbf{0} \quad \text{on } \partial\Omega,$$

then

$$\mathcal{N}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -\mathcal{N}(\mathbf{u}, \mathbf{w}, \mathbf{v}) \quad \text{and} \quad \mathcal{N}(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0.$$

The Sobolev imbedding lemma yields the following results, which will be used later in dealing with the convection term in (1.1).

Lemma 3.2 (Bounds on Trilinear Form). *If* $d \leq 4$, *then*

$$\int_\Omega \mathbf{u} \cdot \mathbf{v} \cdot \mathbf{w} d\mathbf{x} \leq \begin{cases} C\|\mathbf{u}\|_0 \|\mathbf{v}\|_1 \|\mathbf{w}\|_1 \\ C\|\mathbf{u}\|_2 \|\mathbf{v}\|_0 \|\mathbf{w}\|_0, \end{cases} \tag{3.1}$$

and if $d \leq 3$, *then*

$$\int_\Omega \mathbf{u} \cdot \mathbf{v} \cdot \mathbf{w} d\mathbf{x} \leq C\|\mathbf{u}\|_1 \|\mathbf{v}\|_0^{1/2} \|\mathbf{v}\|_1^{1/2} \|\mathbf{w}\|_0.$$

The following elementary but crucial relation is derived in [20].

Lemma 3.3 (div-grad relation). *If* $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, *then*

$$\|\nabla \cdot \mathbf{v}\|_0 \leq \|\nabla \mathbf{v}\|_0.$$

We now introduce a well known lemma.

Lemma 3.4 (Orthogonality between divergence free and curl free functions). *Let* $\mathbf{v} \in \mathbf{H}^1(\Omega)$ *and* $q \in \mathbf{L}^2(\Omega)$. *If* $\nabla \cdot \mathbf{u} = 0$ *and* $\mathbf{u} \cdot \boldsymbol{\nu} = 0$ *on* $\partial\Omega$, *then*

$$\langle \mathbf{u}, \nabla q \rangle = 0.$$

We will use the following algebraic identities frequently to treat time derivative terms.

Lemma 3.5 (Inner product of time derivative terms)**.** *For any sequence* $\{z^n\}_{n=0}^N$*, we have*

$$2\left\langle 3z^{n+1} - 4z^n + z^{n-1}\,,\, z^{n+1}\right\rangle = \left\|z^{n+1}\right\|_0^2 + \left\|2z^{n+1} - z^n\right\|_0^2 + \left\|\delta\delta z^{n+1}\right\|_0^2 - \left\|z^n\right\|_0^2 - \left\|2z^n - z^{n-1}\right\|_0^2, \tag{3.2}$$

$$2\left\langle z^{n+1} - z^n\,,\, z^{n+1}\right\rangle = \left\|z^{n+1}\right\|_0^2 - \left\|z^n\right\|_0^2 + \left\|z^{n+1} - z^n\right\|_0^2, \tag{3.3}$$

and

$$2\left\langle z^{n+1} - z^n\,,\, z^n\right\rangle = \left\|z^{n+1}\right\|_0^2 - \left\|z^n\right\|_0^2 - \left\|z^{n+1} - z^n\right\|_0^2. \tag{3.4}$$

4. Proof of stability Theorem 1

The goal of this section is to prove the stability Theorem 1. We first rewrite the momentum equation (2.9) by using (2.10) and (2.11) as follows:

$$\begin{aligned}\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\tau} &+ \left((2\mathbf{u}^n - \mathbf{u}^{n-1})\cdot\nabla\right)\widehat{\mathbf{u}}^{n+1} \\ &-\nabla\left(\frac{3\psi^{n+1}}{2\tau} - \mu q^n\right) - \mu\triangle\widehat{\mathbf{u}}^{n+1} = \mathbf{f}(t^{n+1}).\end{aligned}$$

We now multiply $4\tau\widehat{\mathbf{u}}^{n+1} \in \mathbf{H}_0^1(\Omega)$ and use (3.2) to get

$$\begin{aligned}&\left\|\mathbf{u}^{n+1}\right\|_0^2 + \left\|2\mathbf{u}^{n+1} - \mathbf{u}^n\right\|_0^2 + \left\|\delta\delta\mathbf{u}^{n+1}\right\|_0^2 - \left\|\mathbf{u}^n\right\|_0^2 \\ &-\left\|2\mathbf{u}^n - \mathbf{u}^{n-1}\right\|_0^2 + 4\tau\mu\left\|\nabla\widehat{\mathbf{u}}^{n+1}\right\|_0^2 = \sum_{i=1}^3 A_i,\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}&A_1 = 6\left\langle\nabla\psi^{n+1}\,,\,\widehat{\mathbf{u}}^{n+1}\right\rangle, \quad A_2 = 4\tau\left\langle\mathbf{f}(t^{n+1})\,,\,\widehat{\mathbf{u}}^{n+1}\right\rangle, \\ &A_3 = 4\tau\mu\left\langle q^n\,,\,\nabla\cdot\widehat{\mathbf{u}}^{n+1}\right\rangle.\end{aligned}$$

We note here that convection term is vanished by Lemma 3.1. In conjunction with $\widehat{\mathbf{u}}^{n+1} = \mathbf{u}^{n+1} - \nabla\delta\psi^{n+1}$ and Lemma 3.4, (3.3) yields

$$A_1 = -6\left\langle\nabla\psi^{n+1}\,,\,\nabla\delta\psi^{n+1}\right\rangle = -3\left(\left\|\nabla\psi^{n+1}\right\|_0^2 - \left\|\nabla\psi^n\right\|_0^2 + \left\|\nabla\delta\psi^{n+1}\right\|_0^2\right).$$

Clearly, we have

$$A_2 \le C\frac{\tau}{\mu}\left\|\mathbf{f}(t^{n+1})\right\|_{-1}^2 + \tau\mu\left\|\nabla\widehat{\mathbf{u}}^{n+1}\right\|_0^2.$$

In the view of (2.10) and Lemma 3.3, we have $\left\|\delta q^{n+1}\right\|_0^2 = \left\|\nabla\cdot\widehat{\mathbf{u}}^{n+1}\right\|_0^2 \le \left\|\nabla\widehat{\mathbf{u}}^{n+1}\right\|_0^2$, whence

$$\begin{aligned}A_3 &= -\,4\mu\tau\left\langle q^n\,,\,\delta q^{n+1}\right\rangle = -2\mu\tau\left(\left\|q^{n+1}\right\|_0^2 - \left\|q^n\right\|_0^2 - \left\|\delta q^{n+1}\right\|_0^2\right) \\ &\le -2\mu\tau\left(\left\|q^{n+1}\right\|_0^2 - \left\|q^n\right\|_0^2\right) + 2\mu\tau\left\|\nabla\widehat{\mathbf{u}}^{n+1}\right\|_0^2.\end{aligned}$$

Inserting A_1-A_3 back into (4.1) and summing over n from 1 to N lead (2.13) by help of $\left\|\widehat{\mathbf{u}}^{n+1}\right\|_0^2 = \left\|\mathbf{u}^{n+1}\right\|_0^2 + \left\|\nabla\delta\psi^{n+1}\right\|_0^2$. ■

5. Error estimates

In this section, we will prove Theorem 2 for Algorithm 1. Because both Algorithms 1 and 2 are equivalent, we conclude that Algorithm 2 also hold Theorem 2. We first prove that the convergence rates of velocity and of time-derivative of velocity are order 1 and 2, respectively. And then we improve the rate to order 2 and estimate pressure error.

Let $\big(\mathbf{u}(t^{n+1}), p(t^{n+1})\big)$ be the exact solution of (1.1) at the time step t^{n+1}. If $\big(\widehat{\mathbf{u}}^{n+1}, \mathbf{u}^{n+1}, p^{n+1}, \psi^{n+1}, q^{n+1}\big)$ is the solution of the Algorithms 1, then we denote the corresponding error by

$$\widehat{\mathbf{E}}^{n+1} = \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}^{n+1}, \quad \mathbf{E}^{n+1} = \mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1},$$
$$e^{n+1} = p(t^{n+1}) - p^{n+1}, \qquad \varepsilon^{n+1} = p(t^{n+1}) + \frac{3\psi^{n+1}}{2\tau}.$$

We observe again that $\widehat{\mathbf{u}}^{n+1} = \mathbf{0}$ on $\partial\Omega$ and $\nabla\cdot\, \widehat{\mathbf{u}}^{n+1} \neq 0$ in Ω, whereas $\nabla\cdot\, \mathbf{u}^{n+1} = 0$ in Ω and $\mathbf{u}^{n+1} = \nabla\delta\psi^{n+1} \neq \mathbf{0}$ on $\partial\Omega$. The following lemma results directly from Lemma 3.4.

Lemma 5.1 (Properties of Error Functions). *For all n, m non-negative integers, we have*

$$\nabla\cdot\, \mathbf{E}^{n+1} = 0, \ \textit{in } \Omega, \quad \widehat{\mathbf{E}}^{n+1} = \mathbf{0}, \ \textit{on } \partial\Omega,$$
$$\langle \mathbf{E}^n \,,\, \nabla\psi^m \rangle = 0, \qquad \textit{and} \qquad \left\langle \widehat{\mathbf{E}}^n \,,\, \mathbf{E}^m \right\rangle = \langle \mathbf{E}^n \,,\, \mathbf{E}^m \rangle \,.$$

Lemma 5.2 (Additional Properties of Error Functions). *We have*

$$\widehat{\mathbf{E}}^{n+1} = \mathbf{E}^{n+1} + \nabla\delta\psi^{n+1}, \tag{5.1}$$

$$\big\|\delta q^{n+1}\big\|_0^2 = \big\|\triangle\delta\psi^{n+1}\big\|_0^2 = \Big\|\nabla\cdot\, \widehat{\mathbf{E}}^{n+1}\Big\|_0^2 \le \Big\|\nabla\widehat{\mathbf{E}}^{n+1}\Big\|_0^2, \tag{5.2}$$

$$\Big\|\widehat{\mathbf{E}}^{n+1}\Big\|_0^2 = \big\|\mathbf{E}^{n+1}\big\|_0^2 + \big\|\nabla\delta\psi^{n+1}\big\|_0^2, \tag{5.3}$$

$$\big\|\mathbf{E}^{n+1}\big\|_1^2 \le C\left(\Big\|\widehat{\mathbf{E}}^{n+1}\Big\|_1^2 + \big\|\delta\psi^{n+1}\big\|_2^2\right) \le C\Big\|\nabla\widehat{\mathbf{E}}^{n+1}\Big\|_0^2. \tag{5.4}$$

To examine Algorithm 1, we first show that the semi-discrete solution $\mathbf{u}^{n+1}$ converge to $\mathbf{u}(t^{n+1})$ with order 1 (see Lemma 5.3) and $\delta\mathbf{u}^{n+1}$ converge to $\delta\mathbf{u}(t^{n+1})$ with order 2 in $L^\infty(0,T;\mathbf{L}^2(\Omega))$ (see Lemma 5.5). We then improve the rate of convergence of $\mathbf{u}^{n+1}$ to order 2 in $L^2(0,T;\mathbf{L}^2(\Omega))$ in Lemma 5.7. The results of Lemmas 5.3 and 5.5 are instrumental in deriving Lemma 5.7.

Lemma 5.3 (Reduced rate of convergence for velocity). *Suppose the exact solution of* (1.1) *is smooth enough. If Assumption 2 hold, then the velocity error functions satisfy*

$$\begin{aligned}\Big\|\widehat{\mathbf{E}}^{N+1}\Big\|_0^2 + \big\|\mathbf{E}^{N+1}\big\|_0^2 + \big\|2\mathbf{E}^{N+1} - \mathbf{E}^N\big\|_0^2 + \frac{4\tau^2}{3}\big\|\nabla\varepsilon^{N+1}\big\|_0^2 + 2\mu\tau\big\|q^{N+1}\big\|_0^2 \\ +\mu\tau\sum_{n=1}^N \Big\|\nabla\widehat{\mathbf{E}}^{n+1}\Big\|_0^2 + \sum_{n=1}^N \Big(\big\|\delta\delta\mathbf{E}^{n+1}\big\|_0^2 + \big\|\nabla\delta\psi^{n+1}\big\|_0^2\Big) \le C\tau^2.\end{aligned} \tag{5.5}$$

Proof. By virtue of Taylor expansion for the exact velocity $\mathbf{u}(t)$, we get

$$\begin{aligned}\frac{3\mathbf{u}(t^{n+1}) - 4\mathbf{u}(t^n) + \mathbf{u}(t^{n-1})}{2\tau} &+ \big(\mathbf{u}(t^{n+1})\cdot\nabla\big)\,\mathbf{u}(t^{n+1}) \\ &+ \nabla p(t^{n+1}) - \mu\triangle\mathbf{u}(t^{n+1}) = \mathbf{R}^{n+1} + \mathbf{f}(t^{n+1}),\end{aligned} \tag{5.6}$$

where $\mathbf{R}^{n+1} := \frac{1}{4\tau}\int_{t^{n-1}}^{t^{n+1}} \mathbf{u}_{ttt}(s)(t-s)^2 ds - \frac{1}{\tau}\int_{t^n}^{t^{n+1}} \mathbf{u}_{ttt}(s)(t-s)^2 ds$ is the truncation error. We replace p^n in (2.9) to (2.11) and then subtract from (5.6) to get

$$\begin{aligned}\frac{3\widehat{\mathbf{E}}^{n+1}-4\mathbf{E}^n+\mathbf{E}^{n-1}}{2\tau} &+ \left(\mathbf{u}(t^{n+1})\cdot\nabla\right)\mathbf{u}(t^{n+1}) - \left(\left(2\mathbf{u}^n-\mathbf{u}^{n-1}\right)\cdot\nabla\right)\widehat{\mathbf{u}}^{n+1} \\ &+\nabla\left(\delta p(t^{n+1})+\varepsilon^n-\mu q^n\right) - \mu\triangle\widehat{\mathbf{E}}^{n+1} = \mathbf{R}^{n+1}.\end{aligned} \tag{5.7}$$

Multiplying (5.7) with $4\tau\widehat{\mathbf{E}}^{n+1}\in\mathbf{H}_0^1(\Omega)$ and invoking (3.2) and (5.3), we arrive at

$$\begin{aligned}&\left\|\mathbf{E}^{n+1}\right\|_0^2 + \left\|2\mathbf{E}^{n+1}-\mathbf{E}^n\right\|_0^2 + \left\|\delta\delta\mathbf{E}^{n+1}\right\|_0^2 + 6\left\|\nabla\delta\psi^{n+1}\right\|_0^2 \\ &-\|\mathbf{E}^n\|_0^2 - \left\|2\mathbf{E}^n-\mathbf{E}^{n-1}\right\|_0^2 + 4\mu\tau\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 = \sum_{n=1}^{5} A_i,\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}&A_1 := -4\tau\mathcal{N}\left(\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}), \widehat{\mathbf{E}}^{n+1}\right) + 4\tau\mathcal{N}\left(2\mathbf{u}^n-\mathbf{u}^{n-1}, \widehat{\mathbf{u}}^{n+1}, \widehat{\mathbf{E}}^{n+1}\right),\\ &A_2 := -4\tau\left\langle\nabla\delta p(t^{n+1}), \widehat{\mathbf{E}}^{n+1}\right\rangle, \qquad A_3 := -4\tau\left\langle\nabla\varepsilon^n, \widehat{\mathbf{E}}^{n+1}\right\rangle,\\ &A_4 := 4\mu\tau\left\langle\nabla q^n, \widehat{\mathbf{E}}^{n+1}\right\rangle, \qquad A_5 := 4\tau\left\langle\mathbf{R}^{n+1}, \widehat{\mathbf{E}}^{n+1}\right\rangle.\end{aligned}$$

We now estimate terms A_1 to A_5 separately. To estimate the convection term A_1, we first note $\mathcal{N}\left(2\mathbf{u}^n-\mathbf{u}^{n-1}, \widehat{\mathbf{E}}^{n+1}, \widehat{\mathbf{E}}^{n+1}\right)=0$ by Lemma 3.1. We next invoke $\left\|\mathbf{u}(t^{n+1})\right\|_2\le M$ and infer

$$\begin{aligned}A_1 =& -4\tau\mathcal{N}\left(\delta\delta\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}), \widehat{\mathbf{E}}^{n+1}\right) - 4\tau\mathcal{N}\left(2\mathbf{E}^n-\mathbf{E}^{n-1}, \mathbf{u}(t^{n+1}), \widehat{\mathbf{E}}^{n+1}\right)\\ &-4\tau\mathcal{N}\left(2\mathbf{u}^n-\mathbf{u}^{n-1}, \widehat{\mathbf{E}}^{n+1}, \widehat{\mathbf{E}}^{n+1}\right)\\ \le& C\tau\left(\left\|\delta\delta\mathbf{u}(t^{n+1})\right\|_0 + \left\|2\mathbf{E}^n-\mathbf{E}^{n-1}\right\|_0\right)\left\|\mathbf{u}(t^{n+1})\right\|_2\left\|\widehat{\mathbf{E}}^{n+1}\right\|_1\\ \le& \frac{\mu\tau}{2}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{C\tau^4}{\mu}\int_{t^{n-1}}^{t^{n+1}}\|\mathbf{u}_{tt}(t)\|_0^2dt + \frac{C\tau}{\mu}\left\|2\mathbf{E}^n-\mathbf{E}^{n-1}\right\|_0^2.\end{aligned}$$

We note again (5.1) which is $\widehat{\mathbf{E}}^{n+1}=\mathbf{E}^{n+1}+\nabla\delta\psi^{n+1}$. On employing Lemma 3.4, we obtain

$$A_2 = -4\tau\left\langle\nabla\delta p(t^{n+1}), \nabla\delta\psi^{n+1}\right\rangle \le \left\|\nabla\delta\psi^{n+1}\right\|_0^2 + C\tau^3\int_{t^n}^{t^{n+1}}\|\nabla p_t(t)\|_0^2dt.$$

In conjunction with $\varepsilon^{n+1}=p(t^{n+1})+\frac{3\psi^{n+1}}{2\tau}$ and (3.4), A_3 becomes

$$\begin{aligned}A_3 =& -4\tau\left\langle\nabla\varepsilon^n, \nabla\delta\psi^{n+1}\right\rangle = -\frac{8\tau^2}{3}\left\langle\nabla\varepsilon^n, \nabla\left(\delta\varepsilon^{n+1}-\delta p(t^{n+1})\right)\right\rangle\\ \le& -\frac{4\tau^2}{3}\left(\left\|\nabla\varepsilon^{n+1}\right\|_0^2 - \|\nabla\varepsilon^n\|_0^2\right) + \frac{4\tau^2}{3}\left\|\nabla\delta\varepsilon^{n+1}\right\|_0^2 + C\tau^3\|\nabla\varepsilon^n\|_0^2\\ &+C\tau^2\int_{t^n}^{t^{n+1}}\|\nabla p_t(t)\|_0^2dt.\end{aligned}$$

If we now apply inequality $(a+b)^2\le 4a^2+\frac{4}{3}b^2$, then we can get

$$\begin{aligned}\frac{4\tau^2}{3}\left\|\nabla\delta\varepsilon^{n+1}\right\|_0^2 &= \frac{4\tau^2}{3}\left\|\nabla\delta p(t^{n+1})+\frac{3}{2\tau}\nabla\delta\psi^{n+1}\right\|_0^2\\ &\le C\tau^2\left\|\nabla\delta p(t)^{n+1}\right\|_0^2 + 4\left\|\nabla\delta\psi^{n+1}\right\|_0^2\end{aligned}$$

and arrive at

$$
\begin{aligned}
A_3 \leq & -\frac{4\tau^2}{3}\left(\left\|\nabla\varepsilon^{n+1}\right\|_0^2 - \|\nabla\varepsilon^n\|_0^2\right) + 4\left\|\nabla\delta\psi^{n+1}\right\|_0^2 \\
& + C\tau^3\|\nabla\varepsilon^n\|_0^2 + C\tau^2\int_{t^n}^{t^{n+1}} \|\nabla p_t(t)\|_0^2 dt.
\end{aligned} \tag{5.9}
$$

We now use (5.2) and $\nabla\cdot\widehat{\mathbf{E}}^{n+1} = \delta q^{n+1}$ to lead

$$
\begin{aligned}
A_4 &= -4\mu\tau\left\langle q^n\,,\,\delta q^{n+1}\right\rangle = -2\mu\tau\left(\left\|q^{n+1}\right\|_0^2 - \|q^n\|_0^2 - \left\|\delta q^{n+1}\right\|_0^2\right) \\
&\leq -2\mu\tau\left(\left\|q^{n+1}\right\|_0^2 - \|q^n\|_0^2\right) + 2\mu\tau\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2.
\end{aligned}
$$

Using Cauchy Schwartz inequality, We readily get

$$
\begin{aligned}
A_5 &\leq \frac{\mu\tau}{2}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{C\tau}{\mu}\left\|\mathbf{R}^{n+1}\right\|_{-1}^2 \\
&\leq \frac{\mu\tau}{2}\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{C\tau^4}{\mu}\int_{t^{n-1}}^{t^{n+1}} \|\mathbf{u}_{ttt}(s)\|_{-1}^2 dt.
\end{aligned}
$$

Replacing A_1-A_5 back into (5.8) and summing over n from 1 to N imply

$$
\begin{aligned}
&\left\|\mathbf{E}^{N+1}\right\|_0^2 + \left\|2\mathbf{E}^{N+1} - \mathbf{E}^N\right\|_0^2 + \sum_{n=1}^{N}\left(\left\|\delta\delta\mathbf{E}^{n+1}\right\|_0^2 + \left\|\nabla\delta\psi^{n+1}\right\|_0^2 + \mu\tau\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2\right) \\
&+ \frac{4\tau^2}{3}\left\|\nabla\varepsilon^{N+1}\right\|_0^2 + 2\mu\tau\left\|q^{N+1}\right\|_0^2 \leq C\tau^3\sum_{n=1}^{N}\|\nabla\varepsilon^n\|_0^2 + \frac{C\tau}{\mu}\sum_{n=1}^{N}\left\|2\mathbf{E}^n - \mathbf{E}^{n-1}\right\|_0 \\
&+ \frac{4\tau^2}{3}\left\|\nabla\varepsilon^1\right\|_0^2 + 2\mu\tau\left\|q^1\right\|_0^2 + 5\left\|\mathbf{E}^1\right\|_0^2 + C\tau^2\int_0^{t^{N+1}} \|\nabla p_t(t)\|_0^2 dt \\
&+ \frac{C\tau^4}{\mu}\int_0^{t^{N+1}} \left(\|\mathbf{u}_{tt}(t)\|_0^2 + \|\mathbf{u}_{ttt}(s)\|_{-1}^2\right) dt.
\end{aligned}
$$

We note $\mathbf{E}^0 = \mathbf{0}$ and $q^1 = 0$ (see Algorithm 1) in above estimate. We also note $\varepsilon^1 = p(t^1) + \frac{3}{2\tau}\psi^1 = e^1$. By the discrete Grönwall lemma and Assumption 2, we finally arrive at (5.5) according to (5.3). ■

Remark 5.4 (Suboptimal order)**.** The rate of convergence of order 1 of Lemma 5.3 is due to the presence of $\int_0^{t^{N+1}} \|\nabla p_t(t)\|_0^2 dt$. To improve upon this, We must get rid of the term. However, this suboptimal result is essential to control convection term in proofs of next lemmas to get optimal order.

To derive initial error of the time-derivative for velocity, we fix $n = 1$ in above proof. Then the obstacle term (5.9) to get optimal estimate can be replaced by

$$
\begin{aligned}
A_3 \leq & -2\tau^2\left(\left\|\nabla\varepsilon^2\right\|_0^2 - \left\|\nabla\varepsilon^1\right\|_0^2\right) + 4\left\|\nabla\delta\psi^2\right\|_0^2 \\
& + C\tau^2\left\|\nabla\varepsilon^1\right\|_0^2 + C\tau^3\int_{t^n}^{t^{n+1}} \|\nabla p_t(t)\|_0^2 dt.
\end{aligned}
$$

By help of Assumption 2, (5.5) becomes, for the case $n = 1$,

$$
\begin{aligned}
&\left\|\mathbf{E}^2\right\|_0^2 + \left\|2\mathbf{E}^2 - \mathbf{E}^1\right\|_0^2 + \mu\tau\left\|\nabla\widehat{\mathbf{E}}^2\right\|_0^2 + \left\|\delta\delta\mathbf{E}^2\right\|_0^2 \\
&+ 2\tau^2\left\|\nabla\varepsilon^2\right\|_0^2 + 2\mu\tau\left\|q^2\right\|_0^2 + \left\|\nabla\delta\psi^2\right\|_0^2 \leq C\tau^4.
\end{aligned} \tag{5.10}
$$

Lemma 5.5 (Error estimate for time-derivative of velocity). *Suppose the exact solution of* (1.1) *is smooth enough. If Assumption 2 hold, then the time derivative velocity error functions satisfy*

$$\begin{aligned}&\|\delta\mathbf{E}^{N+1}\|_0^2+\|2\delta\mathbf{E}^{N+1}-\delta\mathbf{E}^{N}\|_0^2+\frac{4\tau^2}{3}\|\nabla\delta\varepsilon^{N+1}\|_0^2+2\mu\tau\|\delta q^{N+1}\|_0^2\\&+\left\|\delta\widehat{\mathbf{E}}^{N+1}\right\|_0^2+\sum_{n=2}^{N}\left(\|\delta\delta\delta\mathbf{E}^{N+1}\|_0^2+\|\nabla\delta\delta\psi^{n+1}\|_0+\mu\tau\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2\right)\le C\tau^4.\end{aligned}\tag{5.11}$$

Proof. Subtracting two consecutive formulas (5.7) and multiplying by $4\tau\delta\widehat{\mathbf{E}}^{n+1}$ yield

$$\begin{aligned}&\|\delta\mathbf{E}^{n+1}\|_0^2+\|2\delta\mathbf{E}^{n+1}-\delta\mathbf{E}^{n}\|_0^2+\|\delta\delta\delta\mathbf{E}^{n+1}\|_0^2+6\|\nabla\delta\delta\psi^{n+1}\|_0^2\\&-\|\delta\mathbf{E}^{n}\|_0^2-\|2\delta\mathbf{E}^{n}-\delta\mathbf{E}^{n-1}\|_0^2+4\mu\tau\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2=\sum_{i=1}^{5}A_i,\end{aligned}\tag{5.12}$$

where

$$\begin{aligned}A_1:=&-4\tau\mathcal{N}\left(\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1}),\delta\widehat{\mathbf{E}}^{n+1}\right)+4\tau\mathcal{N}\left(2\mathbf{u}^n-\mathbf{u}^{n-1},\widehat{\mathbf{u}}^{n+1},\delta\widehat{\mathbf{E}}^{n+1}\right)\\&+4\tau\mathcal{N}\left(\mathbf{u}(t^{n}),\mathbf{u}(t^{n}),\delta\widehat{\mathbf{E}}^{n+1}\right)-4\tau\mathcal{N}\left(2\mathbf{u}^{n-1}-\mathbf{u}^{n-2},\widehat{\mathbf{u}}^{n},\delta\widehat{\mathbf{E}}^{n+1}\right),\\A_2:=&-4\tau\left\langle\nabla\delta\delta p(t^{n+1}),\delta\widehat{\mathbf{E}}^{n+1}\right\rangle,\qquad A_3:=-4\tau\left\langle\nabla\delta\varepsilon^n,\delta\widehat{\mathbf{E}}^{n+1}\right\rangle,\\A_4:=&4\mu\tau\left\langle\nabla\delta q^n,\delta\widehat{\mathbf{E}}^{n+1}\right\rangle,\qquad A_5:=4\mu\tau\left\langle\delta\mathbf{R}^{n+1},\delta\widehat{\mathbf{E}}^{n+1}\right\rangle.\end{aligned}$$

We now estimate each term A_1 to A_5 separately. The convection term A_1 can be rewritten as follows:

$$\begin{aligned}A_1=&-4\tau\mathcal{N}\left(\delta\delta\mathbf{u}(t^{n+1}),\mathbf{u}(t^{n+1}),\delta\widehat{\mathbf{E}}^{n+1}\right)-4\tau\mathcal{N}\left(2\mathbf{u}^n-\mathbf{u}^{n-1},\widehat{\mathbf{E}}^{n+1},\delta\widehat{\mathbf{E}}^{n+1}\right)\\&-4\tau\mathcal{N}\left(2\mathbf{E}^n-\mathbf{E}^{n-1},\mathbf{u}(t^{n+1}),\delta\widehat{\mathbf{E}}^{n+1}\right)+4\tau\mathcal{N}\left(\delta\delta\mathbf{u}(t^{n}),\mathbf{u}(t^{n}),\delta\widehat{\mathbf{E}}^{n+1}\right)\\&+4\tau\mathcal{N}\left(2\mathbf{E}^{n-1}-\mathbf{E}^{n-2},\mathbf{u}(t^{n}),\delta\widehat{\mathbf{E}}^{n+1}\right)+4\tau\mathcal{N}\left(2\mathbf{u}^{n-1}-\mathbf{u}^{n-2},\widehat{\mathbf{E}}^{n},\delta\widehat{\mathbf{E}}^{n+1}\right),\end{aligned}$$

and we denote by $A_{1,i}$, for $i=1,2,\cdots,6$ the six terms in the right hand side. In estimating convection terms, we will use Lemma 3.2 frequently without notice. We recall $\|\mathbf{u}(t)\|_2\le C$ to arrive at

$$\begin{aligned}A_{1,1}+A_{1,4}\le&C\tau\left(\|\delta\delta\mathbf{u}(t^{n+1})\|_0\|\mathbf{u}(t^{n+1})\|_2+\|\delta\delta\mathbf{u}(t^{n})\|_0\|\mathbf{u}(t^{n})\|_2\right)\left\|\delta\widehat{\mathbf{E}}^{n+1}\right\|_1\\\le&\frac{\mu\tau}{4}\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\frac{C\tau^4}{\mu}\int_{t^{n-2}}^{t^{n+1}}\|\mathbf{u}_{tt}(t)\|_0^2dt.\end{aligned}$$

The result in Lemma 5.3, $\|2\mathbf{E}^n-\mathbf{E}^{n-1}\|_0+\left\|\widehat{\mathbf{E}}^n\right\|_0+\sqrt{\tau}\left\|\widehat{\mathbf{E}}^n\right\|_1\le C\tau$, is essential to treat next 2 convection terms. We have

$$\begin{aligned}A_{1,3}+A_{1,5}=&-4\tau\mathcal{N}\left(2\mathbf{E}^n-\mathbf{E}^{n-1},\delta\mathbf{u}(t^{n+1}),\delta\widehat{\mathbf{E}}^{n+1}\right)\\&-4\tau\mathcal{N}\left(2\delta\mathbf{E}^n-\delta\mathbf{E}^{n-1},\mathbf{u}(t^{n}),\delta\widehat{\mathbf{E}}^{n+1}\right)\\\le&C\tau\left(\|2\mathbf{E}^n-\mathbf{E}^{n-1}\|_0\|\delta\mathbf{u}(t^{n+1})\|_2+\|2\delta\mathbf{E}^n-\delta\mathbf{E}^{n-1}\|_0\|\mathbf{u}(t^n)\|_2\right)\left\|\delta\widehat{\mathbf{E}}^{n+1}\right\|_1\\\le&C\tau^2\|\delta\mathbf{u}(t^{n+1})\|_2\left\|\delta\widehat{\mathbf{E}}^{n+1}\right\|_1+C\tau\|2\delta\mathbf{E}^n-\delta\mathbf{E}^{n-1}\|_0\left\|\delta\widehat{\mathbf{E}}^{n+1}\right\|_1\\\le&\frac{\mu\tau}{4}\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\frac{C\tau}{\mu}\|2\delta\mathbf{E}^n-\delta\mathbf{E}^{n-1}\|_0^2+\frac{C\tau^4}{\mu}\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|_2^2dt.\end{aligned}$$

We note $\mathcal{N}\left(2\mathbf{u}^n - \mathbf{u}^{n-1}\,,\, \delta\widehat{\mathbf{E}}^{n+1}\,,\, \delta\widehat{\mathbf{E}}^{n+1}\right) = 0$ which comes from Lemma 3.1 and

$$\left\|2\delta\mathbf{E}^n - \delta\mathbf{E}^{n-1}\right\|_1 \le C\left\|2\nabla\delta\widehat{\mathbf{E}}^n - \nabla\delta\widehat{\mathbf{E}}^{n-1}\right\|_0 \le C\left(\left\|\nabla\delta\widehat{\mathbf{E}}^n\right\|_0 + \left\|\nabla\delta\widehat{\mathbf{E}}^{n-1}\right\|_0\right)$$

which comes from (5.4). Then we obtain

$$\begin{aligned}
A_{1,2} + A_{1,6} = & -4\tau\mathcal{N}\left(2\delta\mathbf{u}^n - \delta\mathbf{u}^{n-1}\,,\, \widehat{\mathbf{E}}^n\,,\, \delta\widehat{\mathbf{E}}^{n+1}\right)\\
=& 4\tau\mathcal{N}\left(2\delta\mathbf{E}^n - \delta\mathbf{E}^{n-1} - 2\delta\mathbf{u}(t^n) + \delta\mathbf{u}(t^{n-1})\,,\, \widehat{\mathbf{E}}^n\,,\, \delta\widehat{\mathbf{E}}^{n+1}\right)\\
\le& C\tau\left(\left\|2\delta\mathbf{E}^n - \delta\mathbf{E}^{n-1}\right\|_1\left\|\widehat{\mathbf{E}}^n\right\|_1 + \left\|2\delta\mathbf{u}(t^n) - \delta\mathbf{u}(t^{n-1})\right\|_2\left\|\widehat{\mathbf{E}}^n\right\|_0\right)\left\|\delta\widehat{\mathbf{E}}^{n+1}\right\|_1\\
\le& C\tau\sqrt{\tau}\left\|2\delta\mathbf{E}^n - \delta\mathbf{E}^{n-1}\right\|_1\left\|\delta\widehat{\mathbf{E}}^{n+1}\right\|_1 + C\tau^2\left\|2\delta\mathbf{u}(t^n) - \delta\mathbf{u}(t^{n-1})\right\|_2\left\|\delta\widehat{\mathbf{E}}^{n+1}\right\|_1\\
\le& \frac{C\tau^2}{\mu}\left(\left\|\nabla\delta\widehat{\mathbf{E}}^n\right\|_0^2 + \left\|\nabla\delta\widehat{\mathbf{E}}^{n-1}\right\|_0^2\right) + \frac{\mu\tau}{4}\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{C\tau^4}{\mu}\int_{t^{n-2}}^{t^n}\|\mathbf{u}_t\|_2^2 dt.
\end{aligned}$$

Marking use of $\delta\widehat{\mathbf{E}}^{n+1} = \delta\mathbf{E}^{n+1} + \nabla\delta\delta\psi^{n+1}$, we arrive at

$$A_2 = -4\tau\left\langle\nabla\delta\delta p(t^{n+1})\,,\, \nabla\delta\delta\psi^{n+1}\right\rangle \le \left\|\nabla\delta\delta\psi^{n+1}\right\|_0^2 + C\tau^5\int_{t^{n-1}}^{t^{n+1}}\|\nabla p_{tt}(t)\|_0^2 dt.$$

By the same technique (5.9), we readily get

$$\begin{aligned}
A_3 =& -4\tau\left\langle\nabla\delta\varepsilon^n\,,\, \nabla\delta\delta\psi^{n+1}\right\rangle = -\frac{8\tau^2}{3}\left\langle\nabla\delta\varepsilon^n\,,\, \nabla\left(\delta\delta\varepsilon^{n+1} - \delta\delta p(t^{n+1})\right)\right\rangle\\
\le& -\frac{4\tau^2}{3}\left(\left\|\nabla\delta\varepsilon^{n+1}\right\|_0^2 - \|\nabla\delta\varepsilon^n\|_0^2 - \left\|\nabla\delta\delta\varepsilon^{n+1}\right\|_0^2\right)\\
&+ C\tau^3\|\nabla\delta\varepsilon^n\|_0^2 + C\tau^4\int_{t^{n-1}}^{t^{n+1}}\|\nabla p_{tt}(t)\|_0^2 dt.
\end{aligned}$$

Since we have

$$\frac{4\tau^2}{3}\left\|\nabla\delta\delta\varepsilon^{n+1}\right\|_0^2 \le 4\left\|\nabla\delta\delta\psi^{n+1}\right\|_0^2 + C\tau^5\int_{t^{n-2}}^{t^{n+1}}\|\nabla p_{tt}(t)\|_0^2 dt,$$

we conclude

$$\begin{aligned}
A_3 \le& -\frac{4\tau^2}{3}\left(\left\|\nabla\delta\varepsilon^{n+1}\right\|_0^2 - \|\nabla\delta\varepsilon^n\|_0^2\right) + 4\left\|\nabla\delta\delta\psi^{n+1}\right\|_0^2\\
&+ C\tau^3\|\nabla\delta\varepsilon^n\|_0^2 + C\tau^4\int_{t^{n-1}}^{t^{n+1}}\|\nabla p_{tt}(t)\|_0^2 dt.
\end{aligned}$$

We note again (5.2) and $\nabla\cdot\widehat{\mathbf{E}}^{n+1} = \delta q^{n+1}$. Then we have

$$\begin{aligned}
A_4 =& -4\mu\tau\left\langle\delta q^n\,,\, \delta\delta q^{n+1}\right\rangle = -2\mu\tau\left(\left\|\delta q^{n+1}\right\|_0^2 - \|\delta q^n\|_0^2 - \left\|\delta\delta q^{n+1}\right\|_0^2\right)\\
\le& -2\mu\tau\left(\left\|\delta q^{n+1}\right\|_0^2 - \|\delta q^n\|_0^2\right) + 2\mu\tau\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2.
\end{aligned}$$

On the other hand, the truncation error term can be bounded by

$$A_5 \le \frac{\mu\tau}{4}\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{C\tau^4}{\mu}\int_{t^{n-2}}^{t^{n+1}}\|\mathbf{u}_{ttt}(t)\|_{-1}^2 dt.$$

Inserting above estimates into (5.12) and summing for n from 2 to N yield

$$\begin{aligned}
&\left\|\delta\mathbf{E}^{N+1}\right\|_0^2 + \left\|2\delta\mathbf{E}^{N+1} - \delta\mathbf{E}^{N}\right\|_0^2 + \sum_{n=2}^{N}\left(\left\|\delta\delta\delta\mathbf{E}^{N+1}\right\|_0^2 + \left\|\nabla\delta\delta\psi^{n+1}\right\|_0\right) \\
&+ \mu\tau\sum_{n=2}^{N}\left\|\nabla\delta\widehat{\mathbf{E}}^{n+1}\right\|_0^2 + \frac{4\tau^2}{3}\left\|\nabla\delta\varepsilon^{N+1}\right\|_0^2 + 2\mu\tau\left\|\delta q^{N+1}\right\|_0^2 \\
&\leq \left\|\delta\mathbf{E}^{2}\right\|_0^2 + \left\|2\delta\mathbf{E}^{2} - \delta\mathbf{E}^{1}\right\|_0^2 + \frac{4\tau^2}{3}\left\|\nabla\delta\varepsilon^{2}\right\|_0^2 + C\tau^2\sum_{n=1}^{N}\left\|\nabla\delta\widehat{\mathbf{E}}^{n}\right\|_0^2 \\
&+ \frac{C\tau}{\mu}\sum_{n=2}^{N}\left\|2\delta\mathbf{E}^{n} - \delta\mathbf{E}^{n-1}\right\|_0^2 + C\tau^3\sum_{n=2}^{N}\left\|\nabla\delta\varepsilon^{n}\right\|_0^2 + 2\mu\tau\left\|\delta q^{2}\right\|_0^2 \\
&+ \frac{C\tau^4}{\mu}\int_0^{t^{N+1}}\left(\left\|\mathbf{u}_{tt}(t)\right\|_0^2 + \left\|\mathbf{u}_t\right\|_2^2 + \left\|\nabla p_{tt}(t)\right\|_0^2 + \left\|\mathbf{u}_{ttt}(t)\right\|_{-1}^2\right)dt.
\end{aligned}$$

We note here that the fourth term $\tau^2\sum_{n=2}^{N}\left\|\nabla\delta\widehat{\mathbf{E}}^{n}\right\|_0^2$ in the right hand side can be removed by cancellation with $\tau\sum_{n=2}^{N}\left\|\nabla\delta\widehat{\mathbf{E}}^{n}\right\|_0^2$ on the left hand side, provided τ is small enough. If we apply Grönwall inequality and then use (5.10), then we arrive at (5.11) and complete the proof. ∎

Remark 5.6 (Estimate of (2.15))**.** *Because* $\nabla\cdot\widehat{\mathbf{E}}^{n+1} = \delta q^{n+1}$*, Lemma 5.5 directly yields* $\left\|\nabla\cdot\widehat{\mathbf{E}}^{n+1}\right\|_0 \leq C\tau^{\frac{3}{2}}$ *which is one of the our assertion.*

We will use duality argument with the Stokes equations

$$\begin{aligned}
-\triangle\mathbf{v}^{n+1} + \nabla r^{n+1} &= \mathbf{E}^{n+1}, \quad \text{in } \Omega, \\
\nabla\cdot\mathbf{v}^{n+1} &= 0, \qquad\quad \text{in } \Omega,
\end{aligned} \tag{5.13}$$

with vanishing Dirichlet boundary condition $\mathbf{v}^{n+1} = 0$ on $\partial\Omega$. According to Assumption 1, $(\mathbf{v}^{n+1}, r^{n+1}) \in \mathbf{H}_0^1(\Omega)\times H^1(\Omega)$ are strong solution of (5.13) and satisfy

$$\left\|\mathbf{v}^{n+1}\right\|_2 + \left\|r^{n+1}\right\|_1 \leq \left\|\mathbf{E}^{n+1}\right\|_0. \tag{5.14}$$

Lemma 5.7 (Full rate of convergence for velocity)**.** *Let the exact solution of* (1.1) *is smooth enough. If Assumptions 1 and 2 hold, then we have*

$$\begin{aligned}
&\left\|\nabla\mathbf{v}^{N+1}\right\|_0^2 + \left\|\nabla\left(2\mathbf{v}^{N+1} - \mathbf{v}^{N}\right)\right\|_0^2 + \sum_{n=1}^{N}\left\|\nabla\delta\delta\mathbf{v}^{n+1}\right\|_0^2 \\
&\qquad + 2\mu\tau\sum_{n=1}^{N}\left\|\mathbf{E}^{n+1}\right\|_0^2 \leq C\tau^4.
\end{aligned} \tag{5.15}$$

Proof. Let $(\mathbf{v}^{n+1}, r^{n+1})$ be the solution of (5.13). Then it satisfies

$$\left\langle\widehat{\mathbf{E}}^{n+1}, \mathbf{v}^{n+1}\right\rangle = \left\langle-\triangle\mathbf{v}^{n+1} + \nabla r^{n+1}, \mathbf{v}^{n+1}\right\rangle = \left\langle\nabla\mathbf{v}^{n+1}, \nabla\mathbf{v}^{n+1}\right\rangle$$

as well as

$$\left\langle-\triangle\widehat{\mathbf{E}}^{n+1}, \mathbf{v}^{n+1}\right\rangle = \left\langle\widehat{\mathbf{E}}^{n+1}, -\triangle\mathbf{v}^{n+1}\right\rangle = \left\|\mathbf{E}^{n+1}\right\|_0^2 - \left\langle\widehat{\mathbf{E}}^{n+1}, \nabla r^{n+1}\right\rangle.$$

So testing (5.7) with $4\tau\mathbf{v}^{n+1}$ yields

$$\begin{aligned}\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2+\left\|\nabla\left(2\mathbf{v}^{n+1}-\mathbf{v}^n\right)\right\|_0^2+\left\|\nabla\delta\delta\mathbf{v}^{n+1}\right\|_0^2-\left\|\nabla\mathbf{v}^n\right\|_0^2\\ -\left\|\nabla\left(2\mathbf{v}^n-\mathbf{v}^{n-1}\right)\right\|_0^2+4\mu\tau\left\|\mathbf{E}^{n+1}\right\|_0^2=\sum_{i=1}^{3}A_i,\end{aligned}\tag{5.16}$$

where

$$\begin{aligned}A_1 :&= 4\tau\mathcal{N}\left(2\mathbf{u}^n-\mathbf{u}^{n-1}\,,\,\widehat{\mathbf{u}}^{n+1}\,,\,\mathbf{v}^{n+1}\right)-4\tau\mathcal{N}\left(\mathbf{u}(t^{n+1})\,,\,\mathbf{u}(t^{n+1})\,,\,\mathbf{v}^{n+1}\right),\\ A_2 :&= 4\mu\tau\left\langle\widehat{\mathbf{E}}^{n+1}\,,\,\nabla r^{n+1}\right\rangle, \qquad\qquad A_3 := 4\tau\left\langle\mathbf{R}^{n+1}\,,\,\mathbf{v}^{n+1}\right\rangle.\end{aligned}$$

We now estimate A_1 to A_3 separately. The convection term A_1 can be rewritten as follows:

$$\begin{aligned}A_1 =&-4\tau\mathcal{N}\left(\delta\delta\mathbf{u}(t^{n+1})\,,\,\mathbf{u}(t^{n+1})\,,\,\mathbf{v}^{n+1}\right)-4\tau\mathcal{N}\left(2\mathbf{E}^n-\mathbf{E}^{n-1}\,,\,\mathbf{u}(t^{n+1})\,,\,\mathbf{v}^{n+1}\right)\\ &-4\tau\mathcal{N}\left(2\mathbf{u}^n-\mathbf{u}^{n-1}\,,\,\widehat{\mathbf{E}}^{n+1}\,,\,\mathbf{v}^{n+1}\right)=\sum_{i=1}^{3}A_{1,i}.\end{aligned}$$

To estimate convection terms, we will use frequently Lemma 3.2 without notice. Using $\left\|\mathbf{u}(t^{n+1})\right\|_2\le M$, we can readily get

$$\begin{aligned}A_{1,1}&\le C\tau\left\|\delta\delta\mathbf{u}(t^{n+1})\right\|_0\left\|\mathbf{u}(t^{n+1})\right\|_2\left\|\mathbf{v}^{n+1}\right\|_1\\ &\le C\tau\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2+C\mu\tau^4\int_{t^{n-1}}^{t^{n+1}}\|\mathbf{u}_{tt}(t)\|_0^2dt.\end{aligned}$$

We use $2\mathbf{E}^n-\mathbf{E}^{n-1}=\mathbf{E}^{n+1}-\delta\delta\mathbf{E}^{n+1}$ to obtain

$$\begin{aligned}A_{1,2}\le&C\tau\left(\left\|\delta\delta\mathbf{E}^{n+1}\right\|_0+\left\|\mathbf{E}^{n+1}\right\|_0\right)\left\|\mathbf{u}(t^{n+1})\right\|_2\left\|\mathbf{v}^{n+1}\right\|_1\\ \le&\mu\tau\left\|\mathbf{E}^{n+1}\right\|_0^2+\frac{C\tau}{\mu}\left\|\delta\delta\mathbf{E}^{n+1}\right\|_0^2+\frac{C\tau}{\mu}\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2.\end{aligned}$$

If we apply $\left\|2\mathbf{E}^n-\mathbf{E}^{n-1}\right\|_1\le C\sqrt{\tau}$ which derives from (5.4) and Lemma 5.3, then we can derive, by using Lemma 3.2,

$$\begin{aligned}A_{1,3}=&4\tau\mathcal{N}\left(2\mathbf{E}^n-\mathbf{E}^{n-1}-2\mathbf{u}(t^n)+\mathbf{u}(t^{n-1})\,,\,\widehat{\mathbf{E}}^{n+1}\,,\,\mathbf{v}^{n+1}\right)\\ \le&C\tau\left\|2\mathbf{u}(t^n)-\mathbf{u}(t^{n-1})\right\|_2\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0\left\|\mathbf{v}^{n+1}\right\|_1\\ &+C\tau\left\|2\mathbf{E}^n-\mathbf{E}^{n-1}\right\|_1\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^{\frac12}\left\|\widehat{\mathbf{E}}^{n+1}\right\|_1^{\frac12}\left\|\mathbf{v}^{n+1}\right\|_1\\ \le&\frac{\mu\tau}{2}\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0^2+\frac{C\tau}{\mu}\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2+C\tau^2\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0\\ \le&\mu\tau\left(\left\|\mathbf{E}^{n+1}\right\|_0^2+\left\|\nabla\delta\psi^{n+1}\right\|_0^2\right)+\frac{C\tau}{\mu}\left\|\nabla\mathbf{v}^{n+1}\right\|_0^2+C\tau^3\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0^2.\end{aligned}$$

Because we have

$$\begin{aligned}\left\|\nabla\delta\psi^{n+1}\right\|_0^2&=\frac{4\tau^2}{9}\left\|\nabla\left(\delta\varepsilon^{n+1}-\delta p(t^{n+1})\right)\right\|_0^2\\ &\le C\tau^2\left\|\nabla\delta\varepsilon^{n+1}\right\|_0^2+C\tau^3\int_{t^n}^{t^{n+1}}\|\nabla p_t(t)\|_0^2dt,\end{aligned}\tag{5.17}$$

$A_{1,3}$ term can be rewritten by

$$\begin{aligned} A_{1,3} \leq & \mu\tau \big\|\mathbf{E}^{n+1}\big\|_0^2 + \frac{C\tau}{\mu}\big\|\nabla \mathbf{v}^{n+1}\big\|_0^2 + C\tau^3 \Big\|\nabla \widehat{\mathbf{E}}^{n+1}\Big\|_0^2 \\ & + C\tau^3 \big\|\nabla \delta\varepsilon^{n+1}\big\|_0^2 + C\tau^4 \int_{t^n}^{t^{n+1}} \|\nabla p_t(t)\|_0^2 dt. \end{aligned}$$

In conjunction with $\big\|\nabla r^{n+1}\big\|_0 \leq C\big\|\mathbf{E}^{n+1}\big\|_0$ from (5.14), Lemma 3.4 yields

$$A_2 = 4\mu\tau \left\langle \nabla\delta\psi^{n+1} \,,\, \nabla r^{n+1}\right\rangle \leq \mu\tau\big\|\mathbf{E}^{n+1}\big\|_0^2 + \frac{C\tau}{\mu}\big\|\nabla\delta\psi^{n+1}\big\|_0^2.$$

If we apply (5.17) again, then we arrive at

$$A_2 \leq \mu\tau\big\|\mathbf{E}^{n+1}\big\|_0^2 + \frac{C\tau^3}{\mu}\big\|\nabla\delta\varepsilon^{n+1}\big\|_0^2 + C\tau^4 \int_{t^n}^{t^{n+1}} \|\nabla p_t(t)\|_0^2 dt.$$

On the other hand, the truncation error term becomes

$$A_3 = 4\tau \left\langle \mathbf{R}^{n+1} \,,\, \mathbf{v}^{n+1}\right\rangle \leq C\tau\big\|\nabla \mathbf{v}^{n+1}\big\|_0^2 + C\tau^4 \int_{t^{n-1}}^{t^{n+1}} \|\mathbf{u}_{ttt}(s)\|_0^2 dt.$$

Invoking $\mathbf{v}^0 = \mathbf{0}$, inserting above estimates from A_1 and A_3 into (5.16) and summing over n from 1 to N give us

$$\begin{aligned} & \big\|\nabla \mathbf{v}^{N+1}\big\|_0^2 + \big\|\nabla\left(2\mathbf{v}^{N+1} - \mathbf{v}^N\right)\big\|_0^2 + \sum_{n=1}^{N}\left(\big\|\nabla\delta\delta\mathbf{v}^{n+1}\big\|_0^2 + \mu\tau\big\|\mathbf{E}^{n+1}\big\|_0^2\right) \\ & \leq C\tau \sum_{n=1}^{N} \big\|\nabla \mathbf{v}^{n+1}\big\|_0^2 + 5\big\|\nabla \mathbf{v}^1\big\|_0^2 + C\tau^3 \sum_{n=1}^{N}\left(\Big\|\nabla\widehat{\mathbf{E}}^{n+1}\Big\|_0^2 + \big\|\nabla\delta\varepsilon^{n+1}\big\|_0^2\right) \\ & \quad + \frac{C\tau}{\mu}\sum_{n=1}^{N}\big\|\delta\delta\mathbf{E}^{n+1}\big\|_0^2 + C\mu\tau^4 \int_0^{t^{N+1}} \left(\|\mathbf{u}_{tt}(t)\|_0^2 + \|\mathbf{u}_{ttt}(t)\|_0^2 + \|\nabla p_t(t)\|_0^2\right) dt. \end{aligned}$$

We note $\big\|\delta\delta\mathbf{E}^{n+1}\big\|_0^2 \leq 2\big\|2\delta\mathbf{E}^{n+1} - \delta\mathbf{E}^n\big\|_0^2 + 2\big\|\delta\mathbf{E}^{n+1}\big\|_0^2 \leq C\tau^4$ which is result of Lemma 5.5. Applying the discrete Grönwall inequality allows us to remove the first term on the right hand side. With the aid of Lemmas 5.3 and 5.5, we finally obtain (5.15). ∎

We now estimate the pressure error in $L^2(0,T;L^2(\Omega))$. This hinges on the error estimate for the time derivative of velocity of Lemma 5.5.

Lemma 5.8 (Pressure Error estimate)**.** *Let the exact solution of* (1.1) *is smooth enough. If Assumptions 1 and 2 hold, then we have*

$$\tau \sum_{n=1}^{N} \big\|e^{n+1}\big\|_0^2 \leq C\tau^2. \tag{5.18}$$

Proof. In conjunction with (2.11), we can rewrite (5.7) as

$$\begin{aligned} \frac{3\mathbf{E}^{n+1} - 4\mathbf{E}^n + \mathbf{E}^{n-1}}{2\tau} + \left(\mathbf{u}(t^{n+1})\cdot\nabla\right)\mathbf{u}(t^{n+1}) - \left(\left(2\mathbf{u}^n - \mathbf{u}^{n-1}\right)\cdot\nabla\right)\widehat{\mathbf{u}}^{n+1} & \\ + \nabla e^{n+1} - \mu\triangle\mathbf{E}^{n+1} = \mathbf{R}^{n+1}. & \end{aligned} \tag{5.19}$$

We recall in [5, 20] the existence of $\beta > 0$ such that (inf-sup condition)

$$\beta\|q\|_0 \leq \sup_{\mathbf{w}\in\mathbf{H}_0^1(\Omega)} \frac{\langle q \,,\, \nabla\cdot\mathbf{w}\rangle}{\|\nabla\mathbf{w}\|_0}, \qquad \forall q \in L_0^2(\Omega). \tag{5.20}$$

Consequently, it suffices to estimate $\left\langle e^{n+1}, \nabla\cdot \mathbf{w}\right\rangle$ in terms of $\|\nabla \mathbf{w}\|_0$. Multiplying (5.19) by $\mathbf{w} \in \mathbf{H}^1_0(\Omega)$, and utilizing Lemma 3.4, we end up with

$$\begin{aligned}\left\langle e^{n+1}, \nabla\cdot \mathbf{w}\right\rangle &= \frac{1}{2\tau}\left\langle 3\mathbf{E}^{n+1} - 4\mathbf{E}^n + \mathbf{E}^{n-1}, \mathbf{w}\right\rangle + \mu\left\langle \nabla\widehat{\mathbf{E}}^{n+1}, \nabla \mathbf{w}\right\rangle \\ &+ \mathcal{N}\left(\delta\delta\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}), \mathbf{w}\right) + \mathcal{N}\left(2\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n), \widehat{\mathbf{E}}^{n+1}, \mathbf{w}\right) \\ &+ \mathcal{N}\left(2\mathbf{E}^n - \mathbf{E}^{n-1}, \widehat{\mathbf{u}}^{n+1}, \mathbf{w}\right) - \left\langle \mathbf{R}^{n+1}, \mathbf{w}\right\rangle = \sum_{i=1}^{6} A_i.\end{aligned} \tag{5.21}$$

We now proceed to estimate each term A_1 to A_6 separately. We first note that

$$A_1 \le \frac{3}{2\tau}\left(\left\|\delta\mathbf{E}^{n+1}\right\|_0 + \|\delta\mathbf{E}^n\|_0\right)\|\mathbf{w}\|_0 \le \frac{C}{\tau}\left(\left\|\delta\mathbf{E}^{n+1}\right\|_0 + \|\delta\mathbf{E}^n\|_0\right)\|\nabla\mathbf{w}\|_0$$

and

$$A_2 \le C\left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0\|\nabla\mathbf{w}\|_0.$$

Term A_3 and A_4 can be dealt with the aid of (3.1) and $\left\|\mathbf{u}(t^{n+1})\right\|_2 \le M$ as follows:

$$A_3 \le C\left\|\delta\delta\mathbf{u}(t^{n+1})\right\|_0\left\|\mathbf{u}(t^{n+1})\right\|_2\|\mathbf{w}\|_1 \le C\left\|\delta\delta\mathbf{u}(t^{n+1})\right\|_1\|\nabla\mathbf{w}\|_1$$

and

$$A_4 \le C\left\|2\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\right\|_2\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0\|\mathbf{w}\|_1 \le C\left\|\widehat{\mathbf{E}}^{n+1}\right\|_0\|\nabla\mathbf{w}\|_0.$$

In light of $\left\|\widehat{\mathbf{u}}^{n+1}\right\|_1 = \left\|\widehat{\mathbf{E}}^{n+1} - \mathbf{u}(t^{n+1})\right\|_1 \le C$ from Lemma 5.3, we readily get

$$A_5 \le C\left\|2\mathbf{E}^n - \mathbf{E}^{n-1}\right\|_1\left\|\widehat{\mathbf{u}}^{n+1}\right\|_1\|\mathbf{w}\|_1 \le C\left(\left\|\nabla\mathbf{E}^{n+1}\right\|_0 + \|\nabla\mathbf{E}^n\|_0\right)\|\nabla\mathbf{w}\|_0.$$

On the other hand, we have

$$A_6 \le \left\|\mathbf{R}^{n+1}\right\|_{-1}\|\nabla\mathbf{w}\|_0.$$

Inserting the estimates for A_1 to A_6 back into (5.21), and employing (5.20), we obtain

$$\begin{aligned}C\left\|e^{n+1}\right\|_0 \le& \frac{1}{\tau}\left(\left\|\delta\mathbf{E}^{n+1}\right\|_0 + \|\delta\mathbf{E}^n\|_0\right) + \left(\left\|\nabla\mathbf{E}^{n+1}\right\|_0 + \|\nabla\mathbf{E}^n\|_0\right) \\ &+ \left\|\nabla\widehat{\mathbf{E}}^{n+1}\right\|_0 + \left\|\delta\delta\mathbf{u}(t^{n+1})\right\|_1 + \left\|\widehat{\mathbf{E}}^{n+1}\right\|_0 + \left\|\mathbf{R}^{n+1}\right\|_{-1}.\end{aligned}$$

If we now square, multiply by τ, and sum over n from 1 to N, then Lemmas 5.5 and 5.3 derives (5.18). ∎

6. Numerical experiments

In this section, we carried out numerical experiments to compare to theoretical results. We first perform with an known solution to test error decay order in Theorem 2 and then compute driven cavity flows under very unstable condition to check stability result in Theorem 1. They were both computed within the finite element toolbox ALBERTA [18]. We note that the ALBERTA hires triangular mesh of the Union Jack shape.

In the first experiments, we choose square domain $[0, 1] \times [0, 1]$ and impose forcing term the exact solution to become

$$\begin{aligned}u &= \cos(t)\left(x^2 - 2x^3 + x^4\right)\left(2y - 6y^2 + 4y^3\right), \\ v &= -\cos(t)\left(y^2 - 2y^3 + y^4\right)\left(2x - 6x^2 + 4x^3\right), \\ p &= \cos(t)\left(x^2 + y^2 - \frac{2}{3}\right).\end{aligned}$$

Table 1 is the error decay for Algorithm 1. In this computation, we use Taylor-Hood (P2-P1) finite element on the uniform mesh. We impose $\tau = h$ and $\mu = 1$. The velocity error in $L^\infty(0, 1, \mathbf{L}^2(\Omega))$ converges order 2 which is the assertion of this paper. But error decays for others, except pressure in $L^\infty(0,1,L^\infty(\Omega))$ are around order 1.8 (less than 2), in contrast the classical GUM in [16]. So we conclude that the classical GUM performs more accurate than Algorithm 1. However, the main advantage of Algorithm 1 is stability without any condition for τ and we will check the stability in the next experiment.

TABLE 1. Error decay for Algorithm 1

$\tau = h$	1/16	1/32	1/64	1/128	1/256
$\|E\|_0$	0.000766413	0.000231018	6.42159e-05	1.70043e-05	4.38265e-06
	Order	1.730117	1.847003	1.917031	1.956024
$\|E\|_{L^\infty}$	0.00197285	0.000618625	0.000180551	5.0349e-05	1.36219e-05
	Order	1.673144	1.776659	1.842371	1.886035
$\|E\|_1$	0.0123447	0.00406798	0.00126886	0.000382078	0.000112936
	Order	1.601507	1.680780	1.731594	1.758362
$\|e\|_0$	0.0145993	0.0047309	0.00146086	0.000430436	0.000123065
	Order	1.625713	1.695297	1.762947	1.806378
$\|e\|_{L^\infty}$	0.207157	0.0865286	0.0379158	0.0156967	0.00626727
	Order	1.259476	1.190378	1.272338	1.324552

In order to examine the stability result in Theorem 1 via numerical test, we challenge to compute the driven cavity flow with extremely weak stability conditions, Reynolds numbers is 10,000 ($\mu = 1/10,000$), $h = 1/256$, and $\tau = 0.5$. Most of projection type methods have upper bound for τ to make hold the stability constraint, like $\tau \le Ch$, where C depends on the Reynolds numbers. So, the smaller τ has to be imposed for the bigger Reynolds numbers problem. In this case $\mu = 1/10,000$ and $h = 1/256$, very small τ at least $\tau \le h$ is essential in general. But Algorithm 1 becomes released from the limitation of the time marching size τ by Theorem 1. To check this numerically, we hire $\tau = 0.5$ as big as possible.

Figure 1 is numerical result of Algorithm 1 and displays still stable even for high viscosity flow with $\mu = 1/10,000$ under unstable conditions $h = 1/256$ and $\tau = 0.5$. We thus conclude that Algorithm 1 is unconditionally stable and consists to Theorem 1. We note here that this experiment is to verify only stability for any τ, not to check accuracy. So we impose $\tau = 0.5$ and the big τ is the main reason of the oscillations in Figure 1. We need to use a reasonable τ to obtain more accurate results, because stability and accuracy do not depend each other.

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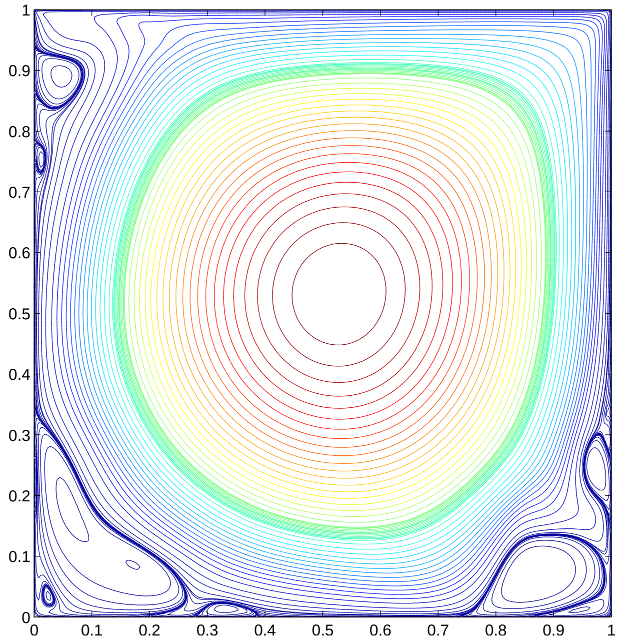

FIGURE 1. Driven cavity for Algorithm 1 with $\mu = 1/10,000$, $h = 1/256$, $\tau = 0.5$

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