INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 1, Pages 74–98

NUMERICAL ANALYSIS OF A FINITE ELEMENT, CRANK-NICOLSON DISCRETIZATION FOR MHD FLOWS AT SMALL MAGNETIC REYNOLDS NUMBERS

GAMZE YUKSEL AND ROSS INGRAM

Abstract. We consider the finite element method for time dependent MHD flow at small magnetic Reynolds number. We make a second (and common) simplification in the model by assuming the time scales of the electrical and magnetic components are such that the electrical field responds instantaneously to changes in the fluid motion. This report gives a comprehensive error analysis for both the semi-discrete and a fully-discrete approximation. Finally, the effectiveness of the method is illustrated in several numeral experiments.

Key words. Navier-Stokes, MHD, finite element, Crank-Nicolson

1. Introduction

Magnetohydrodynamics (MHD) is the theory of macroscopic interaction of electrically conducting fluid and electromagnetic fields. Many interesting MHD-flows involve a viscous, incompressible, electrically conducting fluid that interacts with an electromagnetic field. The governing equations for these MHD flows are the Navier-Stokes (NS) equations (NSE) coupled with the pre-Maxwell equations (via the Lorentz force and Ohm's Law). The resulting system of equations (see e.g. Chapter 2 in [21]) often requires an unrealistic amount of computing power and storage to properly resolve the flow details. A simplification of the usual MHD equations can be made by noting that most terrestrial applications involve small R_m; e.g. most industrial flows involving liquid metal have $R_m < 10^{-2}$. Moreover, it is customary to solve a quasi-static approximation when an external magnetic field is present R_m is small since the time scale of the fluid velocity is much shorter than that of the electromagnetic field [3]. We provide herein a stability and convergence analysis of a fully discrete finite element (FE) discretization for time-dependent MHD flow at a small Re_m and under a quasi-static approximation. Magnetic damping of jets, vortices, and turbulence are several applications, [3, 18, 20, 22].

Let Ω be an open, regular domain in $\mathbb{R}^d$ ($d = 2$ or 3). Let $R_m = UL/\eta > 0$ where U, L are the characteristic speed and length of the problem, $\eta > 0$ is the magnetic diffusivity. The dimensionless quasi-static MHD model is given by: Given time $T > 0$, body force $\mathbf{f}$, interaction parameter $N > 0$, Hartmann number $M > 0$, and domain $\Omega_T := (0, T] \times \Omega$, find velocity $\mathbf{u} : \Omega_T \to \mathbb{R}^d$, pressure $p : \Omega_T \to \mathbb{R}$, electric current density $\mathbf{j} : \Omega_T \to \mathbb{R}^d$, magnetic field $\mathbf{B} : \Omega_T \to \mathbb{R}^d$, and electric potential $\phi : \Omega_T \to \mathbb{R}$ satisfying:

$$
\begin{array}{rclcrcl}
N^{-1}(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) & = & \mathbf{f} + M^{-2}\Delta\mathbf{u} - \nabla p + \mathbf{j}\times\mathbf{B}, & \quad & \nabla\cdot\mathbf{u} & = & 0\\
-\nabla\phi + \mathbf{u}\times\mathbf{B} & = & \mathbf{j}, & & \nabla\cdot\mathbf{j} & = & 0\\
\nabla\times\mathbf{B} & = & R_m\mathbf{j}, & & \nabla\cdot\mathbf{B} & = & 0
\end{array}
\tag{1}
$$

Received by the editors October 8, 2010 and, in revised form, December 6, 2011.
This work was partially supported by National Science Foundation Grant Division of Mathematical Sciences 080385.

subject to boundary and initial conditions

$$\begin{array}{ll} \mathbf{u}(\mathbf{x},t) = 0, & \forall(\mathbf{x},t) \in \partial\Omega \times (0,T] \\ \phi(\mathbf{x},t) = 0, & \forall(\mathbf{x},t) \in \partial\Omega \times (0,T] \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), & \forall \mathbf{x} \in \Omega \end{array} \tag{2}$$

where $\mathbf{u}_0 \in V$ and $\nabla \cdot \mathbf{u}_0 = 0$. When $R_m << 1$, then $\mathbf{j}$ and $\nabla \times \mathbf{B}$ in (1)(3a) decouple. Suppose further that $\mathbf{B}$ is an applied (and known) magnetic field. Then (1) reduces to the simplified MHD (SMHD) system studied herein: Find $\mathbf{u}$, p, ϕ satisfying

$$\begin{array}{rcl} N^{-1}(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) & = & \mathbf{f} + M^{-2}\Delta \mathbf{u} - \nabla p + \mathbf{B} \times \nabla \phi + (\mathbf{u} \times \mathbf{B}) \times \mathbf{B} \\ \nabla \cdot \mathbf{u} & = & 0 \\ -\Delta \phi + \nabla \cdot (\mathbf{u} \times \mathbf{B}) & = & 0. \end{array} \tag{3}$$

subject to (2). This is the time dependent version of the model first proposed by Peterson [19].

We provide a brief overview of previous applications and analyses of MHD flows (high and low R_m) in Section 1.1. In Section 2, we present notation and a weak formulation of (3) required in our stability and convergence analysis. In this report we prove stability estimates for any solution $\mathbf{u}$, p, ϕ to a semi-discrete and fully discrete approximation of (3) in Propositions 3.2, 4.2 respectively. We use these estimates to prove optimal error estimates in two steps:

- Semi-discrete (FE in space), Section 3
- Fully-discrete (FE in space, Crank-Nicolson time-stepping), Section 4

Let $h > 0$ and $\Delta t > 0$ be a representative measure of the spatial and time discretization. We investigate the interplay between spatial and time-stepping errors. We prove that the method is unconditionally stable and, for small enough Δt, the errors satisfy

$$\text{error}(\mathbf{u}, p, \phi) < \mathcal{O}(h^r + \Delta t^2) \to 0, \quad \text{as } h, \Delta t \to 0$$

where r is the order of the FE approximation. See Theorems 3.3, 4.3 and Corollaries 3.4, 4.5.

1.1. Overview of MHD models. Applications of the MHD equations arise in astronomy and geophysics as well as numerous engineering problems including liquid metal cooling of nuclear reactors [2, 7], electromagnetic casting of metals [16], controlled thermonuclear fusion and plasma confinement [8, 23], climate change forecasting and sea water propulsion [15]. Theoretical analysis and mathematical modeling of the MHD equations can be found in [3, 10]. Existence of solutions to the continuous and a discrete MHD problem without conditions on the boundary data of $\mathbf{u}$ is derived in [24]. Existence and uniqueness of weak solutions to the equilibrium MHD equations is proven by Gunzburger, Meir, and Peterson in [6]. Meir and Schmidt provide an optimal convergence estimate of a FE discretization of the equilibrium MHD equations in [17]. To the best of our knowledge, the first rigorous numerical analysis of MHD problems was conducted by Peterson [19] by considering a small R_m, steady-state, incompressible, electrically conducting fluid flow subjected to an undisturbed external magnetic field. Further developments can be found in [1, 12, 13].

2. Problem formulation

We use standard notation for Lebesgue and Sobolev spaces and their norms. Fix $p \geq 1$. Let $L^p(\Omega)$ denote the linear space of all real Lebesgue-measurable functions bounded in the usual norm denoted by $||\cdot||_{L^p(\Omega)}$. Let $(\cdot,\cdot)_\Omega$ and $||\cdot||_\Omega$ be the standard $L^2(\Omega)$-inner product and norm. Fix $k \in \mathbb{R}$. The Sobolev space $W_p^k(\Omega)$ is equipped with the usual norm denoted by $||\cdot||_{W_p^k(\Omega)}$. Identify $||\cdot||_{k,p,\Omega} := ||\cdot||_{W_p^k(\Omega)}$, $H^k(\Omega) := W_2^k(\Omega)$, $||\cdot||_{k,\Omega} := ||\cdot||_{W_2^k(\Omega)}$ with $|\cdot|_{k,\Omega}$ the corresponding semi-norm. Let the context determine whether $W_p^k(\Omega)$ denotes a scalar, vector, or tensor function space. For example let $\mathbf{v} : \Omega \to \mathbb{R}^d$. Then, $\mathbf{v} \in H^1(\Omega)$ implies that $\mathbf{v} \in H^1(\Omega)^d$ and $\nabla \mathbf{v} \in H^1(\Omega)$ implies that $\nabla \mathbf{v} \in H^1(\Omega)^{d\times d}$. Define $H_0^1(\Omega) := \{\mathbf{v} \in H^1(\Omega) \ : \ \mathbf{v}|_{\partial\Omega} = 0\}$. The dual space of $H_0^1(\Omega)$ is denoted $W_2^{-1}(\Omega) := (H_0^1(\Omega))'$ and equipped with the norm

$$||\mathbf{f}||_{-1,\Omega} := \sup_{0\neq \mathbf{v}\in H_0^1(\Omega)} \frac{< \mathbf{f}, \mathbf{v} >_{W_2^{-1}(\Omega)\times H_0^1(\Omega)}}{|\mathbf{v}|_{1,\Omega}}.$$

For brevity, omit Ω in the definitions above. For example, $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$, $H^1 = H^1(\Omega)$, and $V = V(\Omega)$. Fix time $T > 0$ and $m \geq 1$. Let $W_q^m(0,T; W_p^k(\Omega))$ denote the linear space of all Lebesgue measurable functions from $(0,T)$ onto W_p^k equipped with and bounded in the norm

$$||\mathbf{u}||_{W_q^m(0,T;W_p^k)} := (\int_0^T \sum_{i=0}^m ||\partial_t^{(i)}\mathbf{u}(\cdot,t)||_{W_p^k}^q dt)^{1/q}.$$

Write $W_q^m(W_p^k) = W_q^m(0,T;W_p^k(\Omega))$ and $C^m(W_p^k) = C^m([0,T];W_p^k(\Omega))$. Define

$$Q := \{q \in L^2 : (q,\ 1) = 0\}, \quad X := H_0^1(\Omega)^d, \quad S := H_0^1(\Omega).$$

Let X', S' denote the dual space of X, S respectively. Then a weak formulation of (3), (2) is: find $\mathbf{u} : (0,T] \to X$, $p : (0,T] \to Q$, and $\phi : (0,T] \to S$ for $t \in (0,T]$ satisfying

$$\begin{aligned} &N^{-1}\frac{d}{dt}(\mathbf{u},\mathbf{v}) + N^{-1}(\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{v}) + M^{-2}(\nabla\mathbf{u},\nabla\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) \\ &\qquad + (-\nabla\phi + \mathbf{u}\times\mathbf{B}, \mathbf{v}\times\mathbf{B}) = (\mathbf{f},\mathbf{v}), \quad \forall \mathbf{v} \in X \end{aligned} \tag{4}$$

$$(\nabla\cdot\mathbf{u}, q) = 0, \quad \forall q \in Q \tag{5}$$

$$(\nabla\phi - \mathbf{u}\times\mathbf{B}, \nabla\psi) = 0, \quad \forall \psi \in S \tag{6}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \text{a.e. } \mathbf{x} \in \Omega. \tag{7}$$

We obtain (4) from (3)(a) by applying the following identities.

Lemma 2.1. *For all* $\mathbf{u}$, $\mathbf{v} \in L^2$, $\mathbf{B} \in L^\infty$, $\phi \in H^1$,

$$((\mathbf{u}\times\mathbf{B})\times\mathbf{B},\mathbf{v}) = -(\mathbf{u}\times\mathbf{B},\mathbf{v}\times\mathbf{B}), \quad (\mathbf{B}\times\nabla\phi,\mathbf{u}) = (\mathbf{u}\times\mathbf{B},\nabla\phi). \tag{8}$$

Proof. Follows from scalar triple product identities, see e.g. [11]. □

Herein we write $\overline{B} := ||\mathbf{B}||_{L^\infty(L^\infty)}$. We assume that $(\mathbf{u},\phi)$ is a *strong solution* of the SMHD model satisfying (4), (5), (6), (7) and $\mathbf{u} \in L^4(0,T;V) \cap L^\infty(0,T;L^2)$, $\phi \in L^\infty(0,T;S)$, $\mathbf{u}_t \in L^2(0,T;X')$, and $\mathbf{u}(x,t) \to \mathbf{u}_0(x) \in V$ as $t \to 0^+$. Restrict

$\mathbf{v} \in V$ in (4), (5), (6), (7): find $\mathbf{u} : (0,T] \to V$ and $\phi : (0,T] \to S$ a.e. $t \in (0,T]$ satisfying (6), (7), and

$$\begin{aligned} &N^{-1}\frac{d}{dt}(\mathbf{u},\mathbf{v}) + N^{-1}(\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{v}) + M^{-2}(\nabla\mathbf{u},\nabla\mathbf{v}) \\ &\quad + (-\nabla\phi + \mathbf{u}\times\mathbf{B}, \mathbf{v}\times\mathbf{B}) = (\mathbf{f},\mathbf{v}), \quad \forall \mathbf{v}\in V. \end{aligned} \tag{9}$$

Solving the problem associated with (9), (6), (7) is equivalent to (4), (5), (6), (7).

Fix $h > 0$. Let $\mathcal{T}^h$ be a family of subdivisions (e.g. triangulation) of $\overline{\Omega} \subset \mathbb{R}^d$ satisfying $\overline{\Omega} = \bigcup_{E\in\mathcal{T}^h} E$ so that $diameter(E) \leq h$ and any two (closed) elements in $\mathcal{T}^h$ are either disjoint or share exactly one face, side, or vertex. For example, $\mathcal{T}^h$ consists of triangles for $d = 2$ or tetrahedra for $d = 3$ that are nondegenerate as $h \to 0$. Let $X^h \subset X$, $Q^h \subset Q$, and $S^h \subset S$ be a conforming velocity-pressure-potential mixed FE space. We assume that $X^h \times Q^h \times S^h$ satisfy the following:

Assumption 2.2. *The FE spaces $X^h \times Q^h$ satisfy:*

Uniform inf-sup (LBB) condition:

$$\inf_{q^h\in Q^h} \sup_{\mathbf{v}^h\in X^h} \frac{(q^h, \nabla\cdot\mathbf{v}^h)}{|\mathbf{v}^h|_1 \; ||q^h||} \geq C > 0 \tag{10}$$

FE-approximation:

$$\begin{aligned} \inf_{\mathbf{v}^h\in X^h} ||\mathbf{u}-\mathbf{v}^h||_1 &\leq Ch^k||\mathbf{u}||_{k+1} \\ \inf_{\psi^h\in X^h} ||\phi-\psi^h||_1 &\leq Ch^r||\phi||_{r+1} \\ \inf_{q^h\in Q^h} ||p-q^h|| &\leq Ch^{s+1}||p||_{s+1}. \end{aligned} \tag{11}$$

for some fixed $k, r \geq 0$, $s \geq -1$ when $\mathbf{u} \in H^{k+1} \cap X$, $p \in H^{s+1} \cap Q$, $\phi \in H^{r+1} \cap S$.

Error estimates for the elliptic projection (17) in L^2 and W_2^{-1} require regularity of solutions to the following auxiliary problem.

Assumption 2.3. *Given $\theta \in W_2^{-1}$, find $(\mathbf{w}_\theta, \omega_\theta) \in X \times Q$ satisfying*

$$(\nabla\mathbf{w}_\theta, \nabla\mathbf{v}) - (\omega_\theta, \nabla\cdot\mathbf{v}) - (\nabla\cdot\mathbf{w}_\theta, q) = (\theta, \mathbf{v}), \quad \forall(\mathbf{v}, q) \in X \times Q.$$

This problem is well-known to be well-posed. Suppose further that $(\mathbf{w}_\theta, r_\theta) \in H^{m+2} \times H^{m+1}$ satisfy

$$||\mathbf{w}_\theta||_{m+2} + ||\omega_\theta||_{m+1} \leq C||\theta||_m \tag{12}$$

when $\theta \in H_0^m$ (with $H_0^0 = L^2$).

Indeed, (12) is true if Ω is smooth enough.

2.1. Fundamentals Inequalities. Denote by $C > 0$ a generic constant independent of h, Δt, and ν. We use the fact that $||\nabla\cdot\mathbf{v}|| \leq \sqrt{d}|\mathbf{v}|_1$ throughout without further reference. The following estimates are used frequently in the analysis herein (for proofs see e.g. [4], Chapter II, and references therein). Fix q, $q' \geq 1$ so that $1/q + (1/q') = 1$.

$$\begin{aligned} \text{Young}: &\quad ab \leq \frac{1}{q\delta^{q/q'}}a^q + \frac{\delta}{q'}b^{q'} \;\forall a,\; b,\; \delta > 0 \\ \text{Hölder}: &\quad |(\mathbf{v},\mathbf{w})| \leq ||\mathbf{v}||_{0,q}||\mathbf{w}||_{0,q'} \;\forall \mathbf{v}\in L^q,\; \mathbf{w}\in L^{q'}. \end{aligned}$$

Furthermore, we have

$$\begin{array}{ll}\text{Poincaré :} & ||\mathbf{v}|| \le C|\mathbf{v}|_1 \ \forall \mathbf{v} \in X \\ \text{Sobolev :} & ||\mathbf{v}||_{0,\infty} + ||\mathbf{v}||_{1,3} \le C||\mathbf{v}||_2 \ \forall \mathbf{v} \in H^2.\end{array}$$

Let $V^h := \left\{\mathbf{v} \in X^h : \int_\Omega q\nabla \cdot \mathbf{v} = 0 \quad \forall q \in Q^h\right\}$. Note that in general $V^h \not\subset V$. We use the explicitly skew-symmetric convective term:

$$b^*(\mathbf{u},\mathbf{v},\mathbf{w}) := \frac{1}{2}\left((\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - (\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v})\right) \tag{13}$$

so that

$$b^*(\mathbf{u},\mathbf{v},\mathbf{v}) = 0, \quad \forall \mathbf{u} \in H^1,\ \mathbf{v} \in H^1.$$

Note that $(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{v}) \neq 0$ in general. The following estimates of the convective term are derived using the previous inequalities. See [14] for a compilation of associated estimates. For $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in H^1$,

$$b^*(\mathbf{u},\mathbf{v},\mathbf{w}) = (\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}), \quad \forall \mathbf{u} \in V.$$

If, on the other hand, $\mathbf{u} \in X$,

$$\begin{array}{rcl} b^*(\mathbf{u},\mathbf{v},\mathbf{w}) & \le & C\sqrt{||\mathbf{u}||\ |\mathbf{u}|_1}|\mathbf{v}|_1|\mathbf{w}|_1 \\ b^*(\mathbf{u},\mathbf{v},\mathbf{w}) & \le & C|\mathbf{u}|_1|\mathbf{v}|_1\sqrt{||\mathbf{w}||\ |\mathbf{w}|_1}. \end{array} \tag{14}$$

Moreover,

$$\begin{array}{rcll} b^*(\mathbf{u},\mathbf{v},\mathbf{w}) & \le & C||\mathbf{u}||\ ||\mathbf{v}||_2|\mathbf{w}|_1, & \forall \mathbf{v} \in H^2 \\ b^*(\mathbf{u},\mathbf{v},\mathbf{w}) & \le & C|\mathbf{u}|_1||\mathbf{v}||_2||\mathbf{w}||, & \forall \mathbf{v} \in H^2 \end{array}. \tag{15}$$

We define the elliptic for approximating H^1-functions in X^h and S^h. Estimate (16) is necessary since the discrete pressure is eliminated from the error analysis for velocity by testing with functions in the discretely divergence free space V^h (proved e.g. in [5], see intermediate estimate (1.16) in Theorem II.1.1).

Lemma 2.4. *Suppose that the FE space satisfies Assumption 2.2. Then, for any $\mathbf{u} \in V$, there exists a constant $0 < C < \infty$ depending on (10) so that*

$$\inf_{\mathbf{v}^h \in V^h} |\mathbf{u} - \mathbf{v}^h|_1 \le C \inf_{\mathbf{w}^h \in X^h} |\mathbf{u} - \mathbf{w}^h|_1. \tag{16}$$

The elliptic projection is given by $P_1 : V \to V^h$ so that $\tilde{\mathbf{u}}^h := P_1(\mathbf{u})$ satisfies

$$\int_\Omega \nabla(\mathbf{u} - \tilde{\mathbf{u}}^h) : \nabla\mathbf{v} = 0, \quad \forall \mathbf{v} \in V^h. \tag{17}$$

We similarly define the scalar elliptic projection $P_2 : S \to S^h$ so that $\tilde{\phi}^h := P_2(\phi)$ satisfies

$$\int_\Omega \nabla(\phi - \tilde{\phi}^h) : \nabla\psi = 0, \quad \forall \psi \in S^h. \tag{18}$$

We present an error estimate for P_1 and P_2.

Lemma 2.5. *Fix $\mathbf{u} \in X$ and $\phi \in S$. Suppose that FE space satisfies Assumption 2.2. Then P_1, P_2 given by (17) and (18) are well-defined and satisfy*

$$||\mathbf{u} - P_1(\mathbf{u})||_{-m} \le Ch^{m+1} \inf_{\mathbf{v}^h \in X^h} |\mathbf{u} - \mathbf{v}^h|_1 \tag{19}$$

$$||\phi - P_2(\phi)||_1 \le C \inf_{\phi^h \in X^h} |\phi - \phi^h|_1 \tag{20}$$

for $m = -1$. Suppose further that Assumption 2.3 is satisfied. Then (19) also holds for $m = 0$, 1.

Proof. For $m = -1$, apply Céa's Lemma to get $|\mathbf{u} - \tilde{\mathbf{v}}^h|_1 \le 2 \inf_{\mathbf{v}^h \in V^h} |\mathbf{u} - \mathbf{v}^h|_1$. To recover infimum over all $\mathbf{v}^h \in X^h$ in (19), apply estimate (16). Similarly, Céa's Lemma directly gives (20). To recover estimates for $m = 0$ and 1, follow the procedure in [5] (e.g. Theorem II.1.9). □

3. Semi-discrete approximation

We first state the semi-discrete formulation of (4), (5), (6), (7). Suppose that $f \in X'$, $B \in C^0(L^\infty)$.

Problem 3.1 (Semi-discrete FE-approximation)**.** *Find* $\mathbf{u}^h : ([0,T] \to X^h$*,* $p^h : (0,T] \to Q^h$*,* $\psi^h : (0,T] \to S^h$ *satisfying*

$$N^{-1}(\mathbf{u}_t^h, \mathbf{v}) + N^{-1}b^*(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}) + M^{-2}(\nabla \mathbf{u}^h, \nabla \mathbf{v}) - (p^h, \nabla \cdot \mathbf{v})$$

$$+ (-\nabla \phi^h + \mathbf{u}^h \times \mathbf{B}, \mathbf{v} \times \mathbf{B}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in X^h \tag{21}$$

$$(\nabla \cdot \mathbf{u}^h, q) = 0, \quad \forall q \in Q^h \tag{22}$$

$$(\nabla \phi^h - \mathbf{u}^h \times \mathbf{B}, \nabla \psi) = 0, \quad \forall \psi \in S^h \tag{23}$$

$$\mathbf{u}^h(\mathbf{x}, 0) = \mathbf{u}_0^h(\mathbf{x}) \tag{24}$$

for some $\mathbf{u}_0^h \in V^h$*.*

Restrict $\mathbf{v} \in V^h$ in (21), (22), (23), (24): find $\mathbf{u}^h : (0,T] \to V^h$ and $\phi^h : (0,T] \to S^h$ satisfying (23), (24), and

$$N^{-1}(\mathbf{u}_t^h, \mathbf{v}) + N^{-1}b^*(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v})$$

$$+ M^{-2}(\nabla \mathbf{u}^h, \nabla \mathbf{v}) + (-\nabla \phi^h + \mathbf{u}^h \times \mathbf{B}, \mathbf{v} \times \mathbf{B}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in V^h. \tag{25}$$

Solving the problem associated with (25), (23), (24) is equivalent to (21), (22), (23), (24). Define

$$\mathbf{j}^h := -\nabla \phi^h + \mathbf{u}^h \times \mathbf{B}.$$

This definition makes sense in L^2 as we show in Proposition 3.2. We provide proofs of the *a priori* estimate (Proposition 3.2) and of the convergence estimate (Theorem 3.3).

Proposition 3.2. *Any solution* $(\mathbf{u}^h, p^h, \phi^h)$ *of (21), (22), (23), (24) satisfies* $\mathbf{u}^h \in L^2(0,T;X) \cap L^\infty(L^2)$*,* $\phi^h \in L^\infty(0,T;H^1(\Omega))$ *so that*

$$N^{-1}||\mathbf{u}^h||^2_{L^\infty(L^2)} + M^{-2}||\nabla \mathbf{u}^h||^2_{L^2(L^2)}$$

$$+ \overline{B}^{-2} N^{-1}||\nabla \phi^h||^2_{L^\infty(L^2)} + 2||\mathbf{j}^h||^2_{L^2(L^2)} \le N^{-1}||\mathbf{u}_0^h||^2 + M^2||\mathbf{f}||^2_{L^2(W_2^{-1})} \tag{26}$$

Proof. See Proposition 3.1. □

Fix k, $r \ge 0$, $s \ge -1$, and $k' = \max\{2, k\}$. For the convergence estimate, introduce

$$F^h := G_T h^{k+1}||\mathbf{u}_0^h||_{k+1} + G_T M N h^{s+1}||p||_{L^2(H^{s+1})} + G_T M N h^r ||\phi||_{L^2(H^{r+1})}$$

$$+ G_T N^{-1/2} h^{k'} ||\mathbf{u}_t||_{L^2(H^{k'-1})} + G_T M N^{-1/2} h^k ||\mathbf{u}||_{L^\infty(H^1)} ||\mathbf{u}||_{L^2(H^{k+1})} \tag{27}$$

where $G_T := C \exp(C N^{-3} M^6 \int_0^T |\mathbf{u}(\cdot,t)|_1^4 dt)$. Note that $F^h \to 0$ as $h \to 0$ for smooth enough $\mathbf{u}$, p, ϕ. This is made precise in Corollary 3.4.

Theorem 3.3. *Let* $(\mathbf{u}, p, \phi)$ *solve (4), (5), (6), (7) and* $(\mathbf{u}^h, p^h, \phi^h)$ *solve (21), (22), (23), (24). Suppose that Assumption 2.3 is satisfied and the FE-space satisfies Assumption 2.2. Fix* k, $r \geq 0$, $s \geq -1$, *and* $k' = \max\{2, k\}$. *Suppose that* $\mathbf{u} \in L^2(H^{k+1}) \cap L^\infty(H^k \cap V)$, $\mathbf{u}_t \in L^2(H^{k'-1})$, $p \in L^2(H^{s+1} \cap Q)$, *and* $\phi \in L^2(H^{r+1} \cap S)$. *If* $||\mathbf{u}_0^h - \mathbf{u}_0|| \leq \alpha F^h$ *for some* $\alpha > 0$, *then*

$$||\mathbf{u} - \mathbf{u}^h||_{L^\infty(L^2)} \leq F^h + Ch^k ||\mathbf{u}||_{L^\infty(H^k)} \tag{28}$$

$$||\nabla(\mathbf{u} - \mathbf{u}^h)||_{L^2(L^2)} \leq MN^{-1/2} F^h + Ch^k ||\mathbf{u}||_{L^2(H^{k+1})} \tag{29}$$

$$||\mathbf{j} - \mathbf{j}^h||_{L^2(L^2)} \leq N^{-1/2} F^h + C\overline{B} h^k ||\mathbf{u}||_{L^2(H^{k+1})} + Ch^r ||\phi||_{L^2(H^{r+1})} \tag{30}$$

$$||\nabla(\phi - \phi^h)||_{L^2(L^2)} \leq \overline{B} F^h + C\overline{B} h^k ||\mathbf{u}||_{L^2(H^k)} + Ch^r ||\phi||_{L^2(H^{r+1})} \tag{31}$$

and, if $\phi \in L^\infty(H^{k+1})$,

$$||\nabla(\phi - \phi^h)||_{L^\infty(L^2)} \leq \overline{B} F^h + C\overline{B} h^k ||\mathbf{u}||_{L^\infty(H^k)} + Ch^r ||\phi||_{L^\infty(H^{r+1})} \tag{32}$$

where F^h *is given in (27).*

Proof. See Section 3.2. □

Corollary 3.4. *Under the assumptions of Theorem 3.3, suppose further that* $k = r = s + 1 \geq 2$ *and* $\phi \in L^\infty(H^{k+1})$. *Then*

$$\begin{aligned} &||\mathbf{u} - \mathbf{u}^h||_{L^\infty(L^2)} + ||\nabla(\mathbf{u} - \mathbf{u}^h)||_{L^2(L^2)} \\ &\quad + ||\mathbf{j} - \mathbf{j}^h||_{L^2(L^2)} + ||\nabla(\phi - \phi^h)||_{L^\infty(L^2)} \leq Fh^k \end{aligned} \tag{33}$$

where $F > 0$ *is independent of* $h \to 0$.

Proof. See Section 3.2. □

3.1. Proof of Proposition 3.2. Set $\mathbf{v} = \mathbf{u}^h$, $\psi = \phi^h$ in (25), (23) to get

$$\frac{1}{2N}\frac{d}{dt}||\mathbf{u}^h||^2 + M^{-2}|\mathbf{u}^h|_1^2 + (-\nabla\phi^h + \mathbf{u}^h \times \mathbf{B}, \mathbf{u}^h \times \mathbf{B}) = (\mathbf{f}, \mathbf{u}^h) \tag{34}$$

$$(-\nabla\phi^h + \mathbf{u}^h \times \mathbf{B}, \nabla\phi^h) = 0. \tag{35}$$

Add (34) and (35) to get

$$\frac{1}{2N}\frac{d}{dt}||\mathbf{u}^h||^2 + M^{-2}|\mathbf{u}^h|_1^2 + ||\mathbf{j}^h||^2 = (\mathbf{f}, \mathbf{u}^h).$$

Duality estimate on $H_0^1 \times W_2^{-1}$ and Young give

$$N^{-1}\frac{d}{dt}||\mathbf{u}^h||^2 + M^{-2}|\mathbf{u}^h|_1^2 + 2||\mathbf{j}^h||^2 \leq M^2 ||\mathbf{f}||_{-1}^2.$$

Integrate over $[0, T]$ to get

$$\begin{aligned} &N^{-1}||\mathbf{u}^h||_{L^\infty(L^2)}^2 + M^{-2}||\nabla\mathbf{u}^h||_{L^2(L^2)}^2 \\ &\quad + 2||\mathbf{j}^h||_{L^2(L^2)}^2 \leq N^{-1}||\mathbf{u}_0^h||^2 + M^2||\mathbf{f}||_{L^2(W_2^{-1})}^2 \end{aligned} \tag{36}$$

Apply Cauchy-Schwarz to (35) and simplify to get

$$|\phi^h|_1 \leq \overline{B}||\mathbf{u}^h||.$$

Together with (36), we prove (26).

3.2. Proof of Theorem 3.3. Decompose the errors

$$\begin{array}{llllllll} \mathbf{E}_u &=& \mathbf{U}^h - \eta, & \mathbf{U}^h &=& \mathbf{u}^h - \tilde{\mathbf{u}}^h, & \eta &=& \mathbf{u} - \tilde{\mathbf{u}}^h \\ E_\phi &=& \Phi^h - \zeta, & \Phi^h &=& \phi^h - \tilde{\phi}^h, & \zeta &=& \phi - \tilde{\phi}^h \\ \mathbf{E}_j &=& \mathbf{J}^h - \chi, & \mathbf{J}^h &=& -\nabla\Phi^h + \mathbf{U}^h \times \mathbf{B}, & \chi &=& -\nabla\zeta + \eta \times \mathbf{B}. \end{array}$$

Let $\tilde{\mathbf{u}}^h$, $\tilde{\phi}^h$ be the elliptic projections defined in (17), (18) respectively. Fix $\tilde{q}^h \in Q^h$. Note that $(p^h, \nabla \cdot \mathbf{v}) = 0$ for any $\mathbf{v} \in V^h$. Write

$$R(\mathbf{v}) := N^{-1} b^*(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}) - N^{-1} b^*(\mathbf{u}, \mathbf{u}, \mathbf{v}).$$

Subtract (9) from (25), test with $\mathbf{v} = \mathbf{U}^h$, and apply (17) to get

$$\begin{aligned} &\frac{1}{2N}\frac{d}{dt}||\mathbf{U}^h||^2 + M^{-2}|\mathbf{U}^h|_1^2 + (-\nabla\Phi^h + \mathbf{U}^h \times \mathbf{B}, \mathbf{U}^h \times \mathbf{B}) \\ &\quad = -(p - \tilde{q}^h, \nabla \cdot \mathbf{U}^h) + N^{-1}(\eta_t, \mathbf{U}^h) \\ &\quad + (-\nabla\zeta + \eta \times \mathbf{B}, \mathbf{U}^h \times \mathbf{B}) - R(\mathbf{U}^h). \end{aligned} \tag{37}$$

Subtract (6) from (23) and test with $\psi = \Phi^h$ to get

$$(-\nabla\Phi^h + \mathbf{U}^h \times \mathbf{B}, \nabla\Phi^h) = (-\nabla\zeta + \eta \times \mathbf{B}, \nabla\Phi^h). \tag{38}$$

Subtract (37) and (38) to get

$$\begin{aligned} &\frac{1}{2N}\frac{d}{dt}||\mathbf{U}^h||^2 + M^{-2}|\mathbf{U}^h|_1^2 + ||\mathbf{J}^h||^2 \\ &\quad = N^{-1}(\eta_t, \mathbf{U}^h) - (p - \tilde{q}^h, \nabla \cdot \mathbf{U}^h) + (\chi, \mathbf{J}^h) - R(\mathbf{U}^h). \end{aligned} \tag{39}$$

Fix $\varepsilon' > 0$ to be the Young's inequality constant to be prescribed later. Note that the generic constant C depends on ε'. Apply the duality estimate on $H_0^1 \times W_2^{-1}$, Cauchy-Schwarz, and Young to get

$$\begin{aligned} &|N^{-1}(\eta_t, \mathbf{U}^h) - (p - \tilde{q}^h, \nabla \cdot \mathbf{U}^h) + (\chi, \mathbf{J}^h)| \\ &\leq CM^2||p - \tilde{q}^h||^2 + CM^2N^{-2}||\eta_t||_{-1}^2 + \frac{1}{2}||\chi||^2 + \frac{1}{\varepsilon' M^2}|\mathbf{U}^h|_1^2 + \frac{1}{2}||\mathbf{J}^h||^2. \end{aligned} \tag{40}$$

It remains to bound the convective terms.

Lemma 3.5. *Let* $\mathbf{u}$ *satisfy the regularity assumptions of Theorem 3.3. For any* $\varepsilon' > 0$ *there exists* $C > 0$ *such that*

$$\begin{aligned} |R(\mathbf{U}^h)| &\leq \frac{1}{\varepsilon' M^2}|\mathbf{U}^h|_1^2 \\ &\quad + CN^{-4}M^6(|\eta|_1^4 + |\mathbf{u}|_1^4)||\mathbf{U}^h||^2 + CN^{-2}M^2||\mathbf{u}||_{L^\infty(H^1)}^2|\eta|_1^2. \end{aligned} \tag{41}$$

Proof. First note that

$$R(\mathbf{U}^h) = N^{-1}b^*(\mathbf{U}^h, \mathbf{u}, \mathbf{U}^h) - N^{-1}b^*(\mathbf{u}^h, \eta, \mathbf{U}^h) - N^{-1}b^*(\eta, \mathbf{u}, \mathbf{U}^h)$$

Then (14) and Young give

$$\begin{aligned} &N^{-1}|b^*(\mathbf{U}^h, \mathbf{u}, \mathbf{U}^h) + b^*(\eta, \mathbf{u}, \mathbf{U}^h)| \\ &\leq CN^{-2}M^2(|\mathbf{u}|_1^2|\eta|_1^2 + N^{-2}M^4|\mathbf{u}|_1^4||\mathbf{U}^h||^2) + \frac{1}{2\varepsilon' M^2}|\mathbf{U}^h|_1^2. \end{aligned} \tag{42}$$

Rewrite the remaining convective term to get

$$b^*(\mathbf{u}^h, \eta, \mathbf{U}^h) = b^*(\mathbf{u}, \eta, \mathbf{U}^h) + b^*(\mathbf{U}^h, \eta, \mathbf{U}^h) - b^*(\eta, \eta, \mathbf{U}^h).$$

Then (14) and Young give

$$N^{-1}|b^*(\mathbf{u},\eta,\mathbf{U}^h)+b^*(\eta,\eta,\mathbf{U}^h)+b^*(\mathbf{U}^h,\eta,\mathbf{U}^h)|$$

$$\le CN^{-2}M^2(|\mathbf{u}|_1^2|\eta|_1^2+|\eta|_1^2|\eta|_1^2+N^{-2}M^4|\eta|_1^4||\mathbf{U}^h||^2)+\frac{1}{2\varepsilon' M^2}|\mathbf{U}^h|_1^2. \tag{43}$$

Estimates (42), (43) prove (41). □

Note that

$$|\chi|_1\le|\zeta|_1+\overline{B}|\eta|_1. \tag{44}$$

Apply (19), (20), (11) along with (40), (41), (44) to (39). Pick $\varepsilon'=4$. Fix k, $r\ge 0$, $s\ge -1$, and $k'=\max\{2,k\}$. Then

$$\begin{aligned}&\frac{d}{dt}||\mathbf{U}^h||^2+M^{-2}N|\mathbf{U}^h|_1^2+N||\mathbf{J}^h||^2\\ &\quad\le CN^{-1}h^{2k'}||\mathbf{u}_t||_{k'-1}^2+CM^2Nh^{2k}(N^{-2}|\mathbf{u}|_1^2+\overline{B}^2)|\mathbf{u}|_{k+1}^2\\ &\quad+CM^2Nh^{2s+2}||p||_{s+1}^2+CM^2Nh^{2r}||\phi||_{r+1}^2+CN^{-3}M^6|\mathbf{u}|_1^4||\mathbf{U}^h||^2.\end{aligned} \tag{45}$$

Write

$$G(t):=\exp(\int_0^t A(t')dt'),\quad A(t):=CN^{-3}M^6\int_0^t|\mathbf{u}(\cdot,t')|_1^4dt'. \tag{46}$$

Multiply (45) by $G(-t)$, integrate over $[0,t]$, and multiply the result by $G(t)$ to get

$$\begin{aligned}&||\mathbf{U}^h(\cdot,t)||^2+M^{-2}N\int_0^t|\mathbf{U}^h(\cdot,t')|_1^2dt'+N\int_0^t||\mathbf{J}^h(\cdot,t')||^2dt'\\ &\le G(t)||\mathbf{U}^h(\cdot,0)||^2+G(t)N^{-1}h^{2k'}||\mathbf{u}_t||_{L^2(H^{k'-1})}^2\\ &\quad+G(t)M^2N^{-1}h^{2k}||\mathbf{u}||_{L^\infty(H^1)}^2||\mathbf{u}||_{L^2(H^{k+1})}^2\\ &\quad+G(t)M^2Nh^{2s+2}||p||_{L^2(H^{s+1})}^2+G(t)M^2Nh^{2r}||\phi||_{L^2(H^{r+1})}^2\end{aligned} \tag{47}$$

Apply the triangle inequality $||\mathbf{E}_u||\le||\mathbf{U}^h||+||\eta||$ and $||\mathbf{E}_j||\le||\mathbf{J}^h||+||\chi||$ and elliptic projection estimates (19), (20) to (47) to get (28), (29), (30). Start now with (38). Apply (18) to get

$$|\Phi^h|_1^2=-(\mathbf{E}_u\times\mathbf{B},\nabla\Phi^h)\le\overline{B}||\mathbf{E}_u||\ |\Phi^h|_1. \tag{48}$$

The triangle inequality $||E_\phi||\le||\Phi^h||+||\zeta||$ along with estimates (19), (20), (28) applied to (48) proves (32). Lastly, estimates (11), (19) applied to the results of Theorem 3.3 prove Corollary 3.4.

4. Fully-discrete approximation

The semi-discrete model (Problem 3.1) reduces the SMHD equations to a stiff system of ordinary differential equations. A time-discretization must be chosen carefully to ensure stability and accuracy of the approximation scheme. We investigate a Crank-Nicolson (CN) time-stepping scheme of the SMHD equations by following a similar analysis performed for the semi-discrete formulation (Section 3). Let $0=:t_0<t_1<\ldots<t_K:=T<\infty$ be a discretization of the time interval $[0,T]$ for a constant time step $\Delta t=t_n-t_{n-1}$. Write $z_n=z(t_n)$ and $z_{n+1/2}=\frac{1}{2}(z_n+z_{n+1})$ for any function z on $[0,T]$. Define

$$||\mathbf{u}||_{l^q(m_1,m_2;W_p^k)}:=\begin{cases}(\Delta t\sum_{n=m_1}^{m_2}||\mathbf{u}_n||_{k,p}^q)^{1/q}, & q\in[1,\infty)\\ \max_{m_1\le n\le m_2}||\mathbf{u}_n||_{k,p}, & q=\infty\end{cases}$$

for any $0 \le n = m_1, m_1+1, \ldots, m_2 \le K$. Write $||\mathbf{u}||_{l^q(W_p^k)} = ||\mathbf{u}||_{l^q(0,K;W_p^k)}$. We say that $\mathbf{u} \in l^q(m_1, m_2; W_p^k)$ if the associated norm defined above stays finite as $\Delta t \to 0$.

Algorithm 4.1 (CN-FE)**.** *Given* $\mathbf{u}_0 \in V$*, find* $(\mathbf{u}^h_{n+1}, p^h_{n+1/2}, \phi^h_{n+1/2}) \in X^h \times Q^h \times S^h$ *for each* $n = 0, 1, \ldots, K-1$ *satisfying*

$$\begin{aligned} & N^{-1}(\frac{\mathbf{u}^h_{n+1} - \mathbf{u}^h_n}{\Delta t}, \mathbf{v}) + N^{-1}b^*(\mathbf{u}^h_{n+1/2}, \mathbf{u}^h_{n+1/2}, \mathbf{v}) + M^{-2}(\nabla \mathbf{u}^h_{n+1/2}, \nabla \mathbf{v}) \\ & \quad + (-\nabla\phi^h_{n+1/2} + \mathbf{u}^h_{n+1/2} \times \mathbf{B}_{n+1/2}, \mathbf{v} \times \mathbf{B}_{n+1/2}) \\ & \quad - (p^h_{n+1/2}, \nabla \cdot \mathbf{v}) = (\mathbf{f}_{n+1/2}, \mathbf{v}), \quad \forall \mathbf{v} \in X^h \end{aligned} \tag{49}$$

$$(\nabla \cdot \mathbf{u}^h_{n+1}, q) = 0, \quad \forall q \in Q^h \tag{50}$$

$$(\nabla\phi^h_{n+1/2} - \mathbf{u}^h_{n+1/2} \times \mathbf{B}_{n+1/2}, \nabla\psi) = 0, \quad \forall \psi \in S^h \tag{51}$$

$$\mathbf{u}^h(\mathbf{x}, 0) = \mathbf{u}^h_0(\mathbf{x}). \tag{52}$$

Restrict $\mathbf{v} \in V^h$ in (49), (50), (51), (52): find $(\mathbf{u}^h_{n+1}, p^h_{n+1/2}, \phi^h_{n+1/2}) \in V^h \times Q^h \times S^h$ for each $n = 0, 1, \ldots, K-1$ satisfying (51), (52), and

$$\begin{aligned} & N^{-1}(\frac{\mathbf{u}^h_{n+1} - \mathbf{u}^h_n}{\Delta t}, \mathbf{v}) + N^{-1}b^*(\mathbf{u}^h_{n+1/2}, \mathbf{u}^h_{n+1/2}, \mathbf{v}) + M^{-2}(\nabla \mathbf{u}^h_{n+1/2}, \nabla \mathbf{v}) \\ & \quad + (-\nabla\phi^h_{n+1/2} + \mathbf{u}^h_{n+1/2} \times \mathbf{B}_{n+1/2}, \mathbf{v} \times \mathbf{B}_{n+1/2}) = (\mathbf{f}_{n+1/2}, \mathbf{v}), \quad \forall \mathbf{v} \in V^h. \end{aligned} \tag{53}$$

Solving the problem associated with (53), (51), (52) is equivalent to (49), (50), (51), (52). We prove that $\mathbf{u}^h_n$, ϕ^h_n solving (49), (50), (51), (52) are stable and converge to $\mathbf{u}$, ϕ solving (4), (5), (6), (7).

Proposition 4.2 (Stability)**.** *Suppose that* $(\mathbf{u}^h_{n+1}, p^h_{n+1/2}, \phi^h_{n+1/2})$ *solve (49), (50), (51), (52) for each* $n = 0, 1, \ldots, K-1$*. Then,*

$$\begin{aligned} & ||\mathbf{u}^h||^2_{l^\infty(1,K;L^2)} + M^{-2}N\Delta t \sum_{n=0}^{K-1} |\mathbf{u}^h_{n+1/2}|^2_1 \\ & \qquad + 2N\Delta t \sum_{n=0}^{K-1} ||\mathbf{j}^h_{n+1/2}||^2 \le ||\mathbf{u}^h_0||^2 + M^2 N\Delta t \sum_{n=0}^{K-1} ||\mathbf{f}_{n+1/2}||^2_{-1} \end{aligned} \tag{54}$$

and

$$\max_{0 \le n \le K-1} |\phi^h_{n+1/2}|^2_1 \le \overline{B}^2(||\mathbf{u}^h_0||^2 + M^2 N\Delta t \sum_{n=0}^{K-1} ||\mathbf{f}_{n+1/2}||^2_{-1}). \tag{55}$$

Proof. See Section 4.1. □

Fix $s \geq -1$, k, $r \geq 0$, and $k' = \max\{2, k\}$. For the convergence estimate, introduce

$$\begin{aligned} F^h_{\Delta t} &:= G_K h^k ||\mathbf{u}_0||_k + G_K M N^{-1/2} \Delta t^2 (\sum_{n=0}^{K-1} E^{(1)}_{n+1})^{1/2} \\ &+ G_K N^{1/2} \Delta t^2 (\sum_{n=0}^{K-1} E^{(2)}_{n+1})^{1/2} + G_K M N^{-1/2} h^k ||\mathbf{u}||_{l^\infty(H^1)} ||\mathbf{u}||_{l^2(H^{k+1})} \\ &+ G_K M N^{-1/2} h^{k'} ||\mathbf{u}_t||_{L^2(H^{k'-1})} + G_K N^{1/2} h^k \overline{B} ||\mathbf{u}||_{l^2(H^k)} \\ &+ G_K M N^{1/2} h^{s+1} ||p||_{l^2(H^{s+1})} + G_K N^{1/2} h^r ||\phi||_{l^2(H^{r+1})} \end{aligned} \tag{56}$$

where $E^{(1)}_{n+1}$, $E^{(2)}_{n+1} > 0$ are given in (80), (88) respectively and $G_K :=$ $\exp(\sum_{n=0}^{K} \frac{\kappa_n}{1-\Delta t \kappa_n})$, $\kappa_n := M^6 N^{-3} |\mathbf{u}_{n+1/2}|_1^4 + ||\mathbf{B}_{n+1/2}||^2_{0,\infty}$. Note that $F^h_{\Delta t} \to 0$ as h, $\Delta t \to 0$ for smooth enough $\mathbf{u}$, p, ϕ. This is made precise in Corollary 4.5.

Theorem 4.3. *Let $(\mathbf{u}, p, \phi)$ solve (4), (5), (6), (7) and $(\mathbf{u}^h_{n+1}, p^h_{n+1/2}, \phi^h_{n+1/2})$ solve (49), (50), (51), (52) for each $n = 0, 1, \ldots, K-1$. Suppose that Assumption 2.3 is satisfied and the FE-space satisfies Assumption 2.2. Fix $s \geq -1$ and k, $r \geq 0$, and $k' = \max\{2, k\}$. Suppose further that $\mathbf{u} \in l^\infty(H^2) \cap l^2(H^{k+1})$, $\mathbf{u}_t \in l^\infty(L^2) \cap L^2(H^2) \cap L^2(H^{k'-1})$, $\mathbf{u}_{tt} \in L^2(L^2)$, $\mathbf{u}_{ttt} \in L^2(W_2^{-1})$, $p \in l^2(H^{s+1})$, and $\phi \in l^2(H^{r+1})$. If $\mathbf{B}$ is not constant, require that $\mathbf{B} \in L^\infty(L^\infty)$, $\mathbf{B}_t \in l^\infty(H^1)$, $\mathbf{B}_{tt} \in L^2(L^2)$, $\phi \in l^\infty(H^2)$, $\phi_t \in L^2(H^1)$, and $\phi_{tt} \in L^2(L^2)$. If $\Delta t > 0$ is small enough (e.g. $\Delta t < M^6 N^{-3} |\mathbf{u}_{n+1/2}|_1^4 + ||\mathbf{B}_{n+1/2}||^2_{0,\infty}$ for each n) and $||\mathbf{u}_0 - \mathbf{u}^h_0|| \leq \alpha F^h_{\Delta t}$ for some $\alpha \geq 0$, then*

$$||\mathbf{u} - \mathbf{u}^h||_{l^\infty(1,K;L^2)} \leq C h^k ||\mathbf{u}||_{l^\infty(H^k)} + F^h_{\Delta t} \tag{57}$$

$$(\Delta t \sum_{n=0}^{K-1} |\mathbf{u}_{n+1/2} - \mathbf{u}^h_{n+1/2}|_1^2)^{1/2} \leq C h^k ||\mathbf{u}||_{l^2(H^{k+1})} + F^h_{\Delta t} \tag{58}$$

$$\begin{aligned} (\Delta t \sum_{n=0}^{K-1} ||\mathbf{j}_{n+1/2} - \mathbf{j}^h_{n+1/2}||^2)^{1/2} &\leq C h^r ||\phi||_{l^2(H^{r+1})} \\ &+ C \overline{B} h^k ||\mathbf{u}||_{l^2(H^k)} + F^h_{\Delta t} \end{aligned} \tag{59}$$

and

$$\begin{aligned} \max_{0 \leq n \leq K-1} |\phi_{n+1/2} - \phi^h_{n+1/2}|_1 &\leq C h^r ||\phi||_{l^\infty(H^{r+1})} \\ &+ C \overline{B} h^k ||\mathbf{u}||_{l^\infty(H^k)} + \Delta t^2 (\sum_{n=0}^{K-1} E^{(2)}_{n+1})^{1/2} + \overline{B} F^h_{\Delta t}. \end{aligned} \tag{60}$$

where $F^h_{\Delta t}$, $E^{(2)}_{n+1}$ are given in (56), (88) respectively.

Proof. See Section 4.2. □

Remark 4.4. *The regularity assumptions in Theorem 4.3 can be relaxed by requiring instead, e.g., $t\ \mathbf{u}_{tt} \in L^2(L^2)$, $t^2\ \mathbf{u}_{ttt} \in L^2(W_2^{-1})$, and when $k > 2$, $s > 1$, $t^{k-2}\ \mathbf{u} \in l^2(H^{k+1}) \cap L^2(H^k)$, $t^{s-1}\ p \in l^2(H^{s+1})$. Relaxed regularity assumptions introduce a factor $\min(1, t^{-1})$ into the estimates of Theorem 4.3. Note that the requirement $\mathbf{u} \in L^2(H^k)$ is a consequence of using the elliptic projection in the error analysis to prove Theorem 4.3. This requirement is eliminated by proceeding instead* *with the L^2-projection. The L^2-projection requires more technical machinery and does not provide an improved estimate in the case $k > 2$ (since here we pick $k' = k$ to preserve the optimal convergence rate) relative to the elliptic projection.*

Corollary 4.5. *Under the assumptions of Theorem 4.3, pick $s = k - 1$, $k' = \max\{2, k\}$, and $r = k$. Then*

$$||\mathbf{u} - \mathbf{u}^h||_{l^\infty(1,K;L^2)} + \max_{0\le n\le K-1} |\phi_{n+1/2} - \phi^h_{n+1/2}|_1^2 + \sum_{n=0}^{K-1} (|\mathbf{u}_{n+1/2} - \mathbf{u}^h_{n+1/2}|_1^2 + ||\mathbf{j}_{n+1/2} - \mathbf{j}^h_{n+1/2}||^2) \le C(h^k + \Delta t^2) \tag{61}$$

for some $C > 0$ independent of h, $\Delta t \to 0$.

Proof. See Section 4.2. □

The discrete Gronwall inequality is essential in proving Theorem 4.3, Corollary 4.5.

Lemma 4.6. *Let $D \ge 0$ and κ_n, A_n, B_n, $C_n \ge 0$ for any integer $n \ge 0$ and satisfy*

$$A_K + \Delta t \sum_{n=0}^{K} B_n \le \Delta t \sum_{n=0}^{K} \kappa^n A_n + \Delta t \sum_{n=0}^{K} C_n + D, \quad \forall K \ge 0.$$

Suppose that for all n

$$\Delta t \kappa_n < 1$$

and set $\lambda_n = (1 - \Delta t \kappa_n)^{-1}$. Then,

$$A_K + \Delta t \sum_{n=0}^{K} B_n \le \exp(\Delta t \sum_{n=0}^{K} \lambda_n \kappa_n)(\Delta t \sum_{n=0}^{K} C_n + D), \quad \forall K \ge 0.$$

Lemma 4.6 is proved in Lemma 5.1 of [9]. The estimates in (62), (63), (64) are used in Corollary 4.5: For any $n = 0, 1, \ldots, K-1$, $k \ge -1$

$$||\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t}||_k^2 \le \Delta t^{-1} \int_{t_n}^{t_{n+1}} ||\mathbf{u}_t(\cdot, t)||_k^2 dt \tag{62}$$

$$||\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})||_k^2 \le C\Delta t^3 \int_{t_n}^{t_{n+1}} ||\mathbf{u}_{tt}(\cdot, t)||_k^2 dt \tag{63}$$

$$||\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} - (\mathbf{u}_t)^{n+1/2}||_k^2 \le C\Delta t^3 \int_{t_n}^{t_{n+1}} ||\mathbf{u}_{ttt}(\cdot, t)||_k^2 dt \tag{64}$$

when $\mathbf{u}_t \in L^2(H^k)$, $\mathbf{u}_{tt}(\cdot, t) \in L^2(H^k)$, $\mathbf{u}_{ttt}(\cdot, t) \in L^2(H^k)$. Each estimate (62), (63), (64) is a result of a Taylor expansion with integral remainder, see Appendix A. Write $\sigma(t) := \min\{1, t\}$. In the case that the prescribed regularity is not attainable, (63), (64) are replaced by

$$||\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})||_k^2 \le C\Delta t^3 \sigma(t_{n+1/2})^{-2} \int_{t_n}^{t_{n+1}} \sigma(t)^2 ||\mathbf{u}_{tt}(\cdot, t)||_k^2 dt$$

$$||\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} - (\mathbf{u}_t)^{n+1/2}||_k^2 \le C\Delta t^3 \sigma(t_{n+1/2})^{-2} \int_{t_n}^{t_{n+1}} \sigma(t)^2 ||\mathbf{u}_{ttt}(\cdot, t)||_k^2 dt$$

for $t^2\ \mathbf{u}_{tt}(\cdot, t) \in L^2(H^k)$, $t^2\ \mathbf{u}_{ttt}(\cdot, t) \in L^2(H^k)$. See Appendix A.

4.1. Proof of Proposition 4.2. Recall $\mathbf{j}^h_{n+1/2} = -\nabla\phi^h_{n+1/2} + \mathbf{u}^h_{n+1/2} \times \mathbf{B}_{n+1/2}$. Set $\mathbf{v} = \mathbf{u}^h_{n+1/2}$ in (53) to get

$$\begin{aligned}&\frac{1}{2N\Delta t}\left(||\mathbf{u}^h_{n+1}||^2 - ||\mathbf{u}^h_n||^2\right) + M^{-2}|\mathbf{u}^h_{n+1/2}|^2_1\\ &\quad + (\mathbf{j}^h_{n+1/2}, \mathbf{u}^h_{n+1/2} \times \mathbf{B}_{n+1/2}) = (\mathbf{f}_{n+1/2}, \mathbf{u}^h_{n+1/2}).\end{aligned} \tag{65}$$

Set $\psi = \phi^h_{n+1}$ in (51) to get

$$-(\mathbf{j}^h_{n+1/2}, \nabla\phi^h_{n+1/2}) = 0. \tag{66}$$

Add (65) and (66), apply duality estimate $H^1_0 \times W^{-1}_2$, and Young's inequality. Absorb like-terms from right to left-hand-side. Multiply by Δt and sum from $n = 0$ to $n = K-1$ to get

$$\begin{aligned}&\frac{1}{2N}||\mathbf{u}^h_K||^2 + \frac{1}{2M^2}\Delta t\sum_{n=0}^{K-1}|\mathbf{u}^h_{n+1/2}|^2_1 + \Delta t\sum_{n=0}^{K-1}||\mathbf{j}^h_{n+1/2}||^2\\ &\leq \frac{1}{2N}||\mathbf{u}^h_0||^2 + \frac{M^2}{2}\Delta t\sum_{n=0}^{K-1}||\mathbf{f}_{n+1/2}||^2_{-1}.\end{aligned}$$

Multiply by $2N$ to get (54). Set $\psi = \phi^h_{n+1/2}$ in (51), apply Cauchy-Schwarz, and simplify to get

$$|\phi^h_{n+1/2}|_1 \leq ||\mathbf{u}^h_{n+1/2} \times \mathbf{B}_{n+1/2}|| \leq \overline{B}||\mathbf{u}^h_{n+1/2}|| \leq \overline{B}||\mathbf{u}^h||_{l^\infty(L^2)}. \tag{67}$$

Apply (54) to (67) to prove (55).

4.2. Proof of Theorem 4.3, Corollary 4.5. Define $\mathbf{j}_{n+1/2} := -\nabla\mathbf{u}_{n+1/2} + \mathbf{u}_{n+1/2} \times \overline{B}_{n+1/2}$ and $\mathbf{j}^h_{n+1/2} := -\nabla\mathbf{u}^h_{n+1/2} + \mathbf{u}^h_{n+1/2} \times \overline{B}_{n+1/2}$. For any $\mathbf{v} \in X$, $\psi \in S$, write

$$\begin{aligned}\tau^{(1)}_n(\mathbf{v}) &:= N^{-1}((\mathbf{u}_t)_{n+1/2} - \frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t}, \mathbf{v}) + \hat{\tau}^{(1)}_n(\mathbf{v}) + \hat{\hat{\tau}}^{(1)}_n(\mathbf{v})\\ \hat{\tau}^{(1)}_n(\mathbf{v}) &:= \frac{1}{2N}\sum_{i=0}^{1}b^*(\mathbf{u}_{n+i}, \mathbf{u}_{n+i}, \mathbf{v}) - \frac{1}{N}b^*(\mathbf{u}_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{v})\\ \hat{\hat{\tau}}^{(1)}_n(\mathbf{v}) &:= \frac{1}{2}\sum_{i=0}^{1}(\mathbf{j}_{n+i}, \mathbf{v} \times \mathbf{B}_{n+i}) - (\mathbf{j}_{n+1/2}, \mathbf{v} \times \mathbf{B}_{n+1/2})\end{aligned} \tag{68}$$

$$\tau^{(2)}_n(\psi) := -\frac{1}{2}\sum_{i=0}^{1}(\mathbf{u}_{n+i} \times \mathbf{B}_{n+i}, \nabla\psi) + (\mathbf{u}_{n+1/2} \times \mathbf{B}_{n+1/2}, \nabla\psi). \tag{69}$$

Then sum (4), (6) at $t = t_n$ and t_{n+1} and divide by 2 to get

$$\begin{aligned}&N^{-1}(\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t}, \mathbf{v}) + N^{-1}b^*(\mathbf{u}_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{v}) + M^{-2}(\nabla\mathbf{u}_{n+1/2}, \nabla\mathbf{v})\\ &- (p_{n+1/2}, \nabla\cdot\mathbf{v}) + (\mathbf{j}_{n+1/2}, \mathbf{v} \times \mathbf{B}_{n+1/2}) = (\mathbf{f}_{n+1/2}, \mathbf{v}) - \tau^{(1)}_n(\mathbf{v}), \quad \forall\mathbf{v} \in V\end{aligned} \tag{70}$$

$$-(\mathbf{j}_{n+1/2}, \nabla\psi) = -\tau^{(2)}_n(\psi), \quad \forall\psi \in S. \tag{71}$$

Decompose the velocity, potential, and current density errors:

$$\begin{array}{lclllcllcl}\mathbf{E}_{u,n} &=& \mathbf{U}^h_n - \eta_n, & \mathbf{U}^h_n &=& \mathbf{u}^h_n - \tilde{\mathbf{u}}^h_n, & \eta_n &=& \mathbf{u}_n - \tilde{\mathbf{u}}^h_n\\ E_{\phi,n} &=& \Phi^h_n - \zeta_n, & \Phi^h_n &=& \phi^h_n - \tilde{\phi}^h_n, & \zeta_n &=& \phi_n - \tilde{\phi}^h_n\\ \mathbf{E}_{j,n} &=& \mathbf{J}^h_n - \chi_n, & \mathbf{J}^h_n &=& -\nabla\Phi^h_n + \mathbf{U}^h_n \times \mathbf{B}_n, & \chi_n &=& -\nabla\zeta_n + \eta_n \times \mathbf{B}_n\end{array}$$

Let $\tilde{u}_n^h$, $\tilde{\phi}_n^h$ be the elliptic projection defined in (17), (18). Fix $\tilde{q}_n^h \in Q^h$. Note that $(p^h, \nabla \cdot \mathbf{v}) = 0$ for any $\mathbf{v} \in V^h$. Write

$$R_n^h(\mathbf{v}) := N^{-1} b^*(\mathbf{u}_{n+1/2}^h, \mathbf{u}_{n+1/2}^h, \mathbf{v}) - N^{-1} b^*(\mathbf{u}_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{v})$$

Subtract (70) and (71) from (53) and (51) respectively. Set $\mathbf{v} = \mathbf{U}_{n+1/2}^h$, $\psi = \Phi_{n+1/2}^h$ to get the error equations

$$\begin{aligned} &\frac{1}{2N\Delta t}\left(||\mathbf{U}_{n+1}^h||^2 - ||\mathbf{U}_n^h||^2\right) + M^{-2}|\mathbf{U}_{n+1/2}^h|_1^2 + (\mathbf{J}_{n+1/2}^h, \mathbf{U}_{n+1/2}^h \times \mathbf{B}_{n+1/2}) \\ &\quad = N^{-1}(\frac{\eta_{n+1} - \eta_n}{\Delta t}, \mathbf{U}_{n+1/2}^h) + (\chi_{n+1/2}, \mathbf{U}_{n+1/2}^h \times \mathbf{B}_{n+1/2}) \\ &\quad + (\tilde{q}_{n+1/2}^h - p_{n+1/2}, \nabla \cdot \mathbf{U}_{n+1/2}^h) - R_n^h(\mathbf{U}_{n+1/2}^h) - \tau_n^{(1)}(\mathbf{U}_{n+1/2}^h), \end{aligned} \tag{72}$$

$$-(\mathbf{J}_{n+1/2}^h, \nabla\Phi_{n+1/2}^h) = -(\chi_{n+1/2}, \nabla\Phi_{n+1/2}^h) + \tau_n^{(2)}(\Phi_{n+1/2}^h). \tag{73}$$

Add (73) to (72) and combine terms to get

$$\begin{aligned} &\frac{1}{2N\Delta t}\left(||\mathbf{U}_{n+1}^h||^2 - ||\mathbf{U}_n^h||^2\right) + M^{-2}|\mathbf{U}_{n+1/2}^h|_1^2 + ||\mathbf{J}_{n+1/2}^h||^2 \\ &\quad = N^{-1}(\frac{\eta_{n+1} - \eta_n}{\Delta t}, \mathbf{U}_{n+1/2}^h) + (\tilde{q}_{n+1/2}^h - p_{n+1/2}, \nabla \cdot \mathbf{U}_{n+1/2}^h) \\ &\quad + (\chi_{n+1/2}, \mathbf{J}_{n+1/2}^h) - R_n^h(\mathbf{U}_{n+1/2}^h) - \tau_n^{(1)}(\mathbf{U}_{n+1/2}^h) + \tau_n^{(2)}(\Phi_{n+1/2}^h). \end{aligned} \tag{74}$$

Note that (73) can be written

$$|\Phi_{n+1/2}^h|_1^2 = (\mathbf{U}_{n+1/2}^h \times \mathbf{B}_{n+1/2} - \chi_{n+1/2}, \nabla\Phi_{n+1/2}^h) + \tau_n^{(2)}(\Phi_{n+1/2}^h). \tag{75}$$

Add (75) to (74) to get

$$\begin{aligned} &\frac{1}{2N\Delta t}\left(||\mathbf{U}_{n+1}^h||^2 - ||\mathbf{U}_n^h||^2\right) + M^{-2}|\mathbf{U}_{n+1/2}^h|_1^2 + ||\mathbf{J}_{n+1/2}^h||^2 + |\Phi_{n+1/2}^h|_1^2 \\ &\quad = N^{-1}(\frac{\eta_{n+1} - \eta_n}{\Delta t}, \mathbf{U}_{n+1/2}^h) + (\tilde{q}_{n+1/2}^h - p_{n+1/2}, \nabla \cdot \mathbf{U}_{n+1/2}^h) \\ &\quad + (\chi_{n+1/2}, \mathbf{J}_{n+1/2}^h) + (\mathbf{U}_{n+1/2}^h \times \mathbf{B}_{n+1/2} - \chi_{n+1/2}, \nabla\Phi_{n+1/2}^h) \\ &\quad - R_n^h(\mathbf{U}_{n+1/2}^h) - \tau_n^{(1)}(\mathbf{U}_{n+1/2}^h) + 2\tau_n^{(2)}(\Phi_{n+1/2}^h). \end{aligned} \tag{76}$$

Let ε', $\varepsilon'' > 0$ be Young's inequality constants to be fixed later. Apply duality estimate on $H_0^1 \times W_2^{-1}$ and Cauchy-Schwarz with Young to get

$$\begin{aligned} &|N^{-1}(\frac{\eta_{n+1} - \eta_n}{\Delta t}, \mathbf{U}_{n+1/2}^h) + (p_{n+1/2} - \tilde{q}_{n+1/2}^h, \nabla \cdot \mathbf{U}_{n+1/2}^h) \\ &\quad + (\chi_{n+1/2}, \mathbf{J}_{n+1/2}^h) + (\mathbf{U}_{n+1/2}^h \times \mathbf{B}_{n+1/2}, \nabla\Phi_{n+1/2}^h) - (\chi_{n+1/2}, \nabla\Phi_{n+1/2}^h)| \\ &\leq CM^2 N^{-2}||\frac{\eta_{n+1} - \eta_n}{\Delta t}||_{-1}^2 + CM^2||p_{n+1/2} - \tilde{q}_{n+1/2}^h||^2 \\ &\quad + C||\chi_{n+1/2}||^2 + C||\mathbf{B}_{n+1/2}||_{0,\infty}^2||\mathbf{U}_{n+1/2}^h||^2 + \frac{1}{\varepsilon' M^2}|\mathbf{U}_{n+1/2}^h|_1^2 \\ &\quad + \frac{1}{\varepsilon''}|\Phi_{n+1/2}^h|_1^2 + \frac{1}{2}||\mathbf{J}_{n+1/2}^h||^2. \end{aligned} \tag{77}$$

The convective terms are bounded in a similar way as in the proof of Lemma 3.5 noting that $R(\mathbf{v})$ is $R_h^n(\mathbf{v})$ with $\mathbf{u}$, $\mathbf{u}^h$, η, $\mathbf{U}^h$ replaced by $\mathbf{u}_{n+1/2}$, $\mathbf{u}_{n+1/2}^h$, $\eta_{n+1/2}$,

$\mathbf{U}^h_{n+1/2}$. We conclude without further proof

$$|R^h_n(\mathbf{U}^h_{n+1/2})| \le CM^2N^{-2}(|\mathbf{u}_{n+1/2}|_1^2 + |\eta_{n+1/2}|_1^2)|\eta_{n+1/2}|_1^2 \\ + \frac{1}{\varepsilon' M^2}|\mathbf{U}^h_{n+1/2}|_1^2 + CM^6N^{-4}(|\mathbf{u}_{n+1/2}|_1^4 + |\eta_{n+1/2}|_1^4)||\mathbf{U}^h_{n+1/2}||^2. \tag{78}$$

Bounding the time-consistency error remains.

Lemma 4.7. *Let* $\mathbf{u}$ *satisfy the regularity assumptions of Theorem 4.3. Then, for any* $\varepsilon' > 0$ *and any integer* $n \ge 0$

$$|\tau^{(1)}_n(\mathbf{U}^h_{n+1/2})| \le \frac{1}{\varepsilon' M^2}|\mathbf{U}^h_{n+1/2}|_1^2 + CM^2N^{-2}\Delta t^3 E^{(1)}_{n+1} \tag{79}$$

where

$$E^{(1)}_{n+1} := C||\mathbf{u}_{ttt}||^2_{L^2(t_n,t_{n+1};W_2^{-1})} + C||\mathbf{u}||^2_{l^\infty(H^2)}||\mathbf{u}_{tt}||^2_{L^2(t_n,t_{n+1};L^2)} \\ + C||\mathbf{u}_t||^2_{l^\infty(L^2)}||\mathbf{u}_t||^2_{L^2(t_n,t_{n+1},H^2)} + C\overline{B}^2||\mathbf{j}_{tt}||^2_{L^2(t_n,t_{n+1};W-1,2)} \\ + C||\mathbf{j}||^2_{l^\infty(H^1)}||\mathbf{B}_{tt}||^2_{L^2(t_n,t_{n+1};L^2)} + C||\mathbf{B}_t||^2_{l^\infty(H^1)}||\mathbf{j}_t||^2_{L^2(t_n,t_{n+1};L^2)}. \tag{80}$$

Proof. Duality estimate on $W_2^{-1} \times H_0^1$ and Young give

$$N^{-1}|(\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t} - \mathbf{u}_t(t_{n+1/2}), \mathbf{U}^h_{n+1/2})| \\ \le CM^2N^{-2}||\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t} - \mathbf{u}_t(\cdot, t_{n+1/2})||_{-1} \\ + CM^2N^{-2}||\mathbf{u}_t(\cdot,t_{n+1/2}) - (\mathbf{u}_t)_{n+1/2}||_{-1} + \frac{1}{\varepsilon' M^2}|\mathbf{U}^h_{n+1/2}|_1. \tag{81}$$

For the remaining terms, details are provided in Appendix A. Taylor-expansion about $t_{n+1/2}$ with integral remainder gives

$$\frac{1}{2}(\mathbf{u}_{n+1}\cdot\nabla\mathbf{u}_{n+1}, \mathbf{v}) + \frac{1}{2}(\mathbf{u}_n\cdot\nabla\mathbf{u}_n, \mathbf{v}) = (\mathbf{u}(\cdot,t_{n+1/2})\cdot\nabla\mathbf{u}(\cdot,t_{n+1/2}), \mathbf{v}) \\ + \frac{1}{2}\int_{t_{n+1/2}}^{t_{n+1}}(t_{n+1}-t)\frac{d^2}{dt^2}(\mathbf{u}(\cdot,t)\cdot\nabla\mathbf{u}(\cdot,t),\mathbf{v})dt \\ + \frac{1}{2}\int_{t_n}^{t_{n+1/2}}(t-t_n)\frac{d^2}{dt^2}(\mathbf{u}(\cdot,t)\cdot\nabla\mathbf{u}(\cdot,t),\mathbf{v})dt. \tag{82}$$

Add/subtract $(\mathbf{u}_{n+1/2}\cdot\nabla\mathbf{u}(\cdot,t_{n+1/2}),\mathbf{v})$ and apply (82) to get

$$\hat{\tau}^{(1)}_{n+1}(\mathbf{U}^h_{n+1}) = (\mathbf{u}_{n+1/2}\cdot\nabla(\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot,t_{n+1/2})), \mathbf{U}^h_{n+1/2}) \\ + (\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot,t_{n+1/2})\cdot\nabla\mathbf{u}(\cdot,t_{n+1/2}), \mathbf{U}^h_{n+1/2}) \\ - \frac{1}{2}\int_{t_{n+1/2}}^{t_{n+1}}(t_{n+1}-t)\int(\mathbf{u}_{tt}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}_{tt} + 2\mathbf{u}_t\cdot\nabla\mathbf{u}_t)\cdot\mathbf{U}^h_{n+1/2}dt \\ - \frac{1}{2}\int_{t_n}^{t_{n+1/2}}(t-t_n)\int(\mathbf{u}_{tt}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}_{tt} + 2\mathbf{u}_t\cdot\nabla\mathbf{u}_t)\cdot\mathbf{U}^h_{n+1/2}dt \tag{83}$$

Majorize (83) with (15) and Hölder's inequality (in time) applied to (83) to get

$$
\begin{aligned}
|\hat{\tau}^{(1)}_{n+1}(\mathbf{U}^h_{n+1/2})| &\le C||\mathbf{u}||_{l^\infty(H^2)}||\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})||\ |\mathbf{U}^h_{n+1/2}|_1 \\
&+ \frac{C\Delta t^{3/2}}{\sqrt{t_{n+1/2}}}||\mathbf{u}||_{l^\infty(H^2)}(\int_{t_n}^{t_{n+1}} t\ ||\mathbf{u}_{tt}(\cdot, t)||^2 dt)^{1/2}|\mathbf{U}^h_{n+1/2}|_1 \\
&+ \frac{C\Delta t^{3/2}}{\sqrt{t_{n+1/2}}}||\mathbf{u}_t||_{l^\infty(L^2)}(\int_{t_n}^{t_{n+1}} t\ ||\mathbf{u}_t(\cdot, t)||_2^2 dt)^{1/2}|\mathbf{U}^h_{n+1/2}|_1 \qquad (84)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\hat{\hat{\tau}}^{(1)}_{n+1}(\mathbf{U}^h_{n+1}) &= -(\mathbf{j}_{n+1/2}, (\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2})) \times \mathbf{U}^h_{n+1/2}) \\
&- (\mathbf{j}_{n+1/2} - \mathbf{j}(\cdot, t_{n+1/2}), \mathbf{B}(\cdot, t_{n+1/2}) \times \mathbf{U}^h_{n+1/2}) \\
&+ \frac{1}{2}\int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1} - t) \int (\mathbf{j}_{tt} \cdot (\mathbf{B} \times \mathbf{U}^h_{n+1/2}) + \mathbf{j} \cdot (\mathbf{B}_{tt} \times \mathbf{U}^h_{n+1/2}) \\
&\quad \ldots + 2\mathbf{j}_t \cdot (\mathbf{B}_t \times \mathbf{U}^h_{n+1/2}))dt \\
&+ \frac{1}{2}\int_{t_n}^{t_{n+1/2}} (t - t_n) \int (\mathbf{j}_{tt} \cdot (\mathbf{B} \times \mathbf{U}^h_{n+1/2}) + \mathbf{j} \cdot (\mathbf{B}_{tt} \times \mathbf{U}^h_{n+1/2}) \\
&\quad \ldots + 2\mathbf{j}_t \cdot (\mathbf{B}_t \times \mathbf{U}^h_{n+1/2}))dt \qquad (85)
\end{aligned}
$$

Majorize (85) with duality estimate on $H^1_0 \times W_2^{-1}$ and Hölder's inequality (in time and space) to get

$$
\begin{aligned}
|\hat{\hat{\tau}}^{(1)}_{n+1}(\mathbf{U}^h_{n+1/2})| &\le C\overline{B}||\mathbf{j}_{n+1/2} - \mathbf{j}(\cdot, t_{n+1/2})||_{-1}|\mathbf{U}^h_{n+1/2}|_1 \\
&+ C||\mathbf{j}||_{l^\infty(L^\infty)}||\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2})||_{-1}|\mathbf{U}^h_{n+1/2}|_1 \\
&+ \frac{C\Delta t^{3/2}}{\sqrt{t_{n+1/2}}}\overline{B}(\int_{t_n}^{t_{n+1}} t\ ||\mathbf{j}_{tt}(\cdot, t)||_{-1}^2 dt)^{1/2}|\mathbf{U}^h_{n+1}|_1 \\
&+ \frac{C\Delta t^{3/2}}{\sqrt{t_{n+1/2}}}||\mathbf{j}||_{l^\infty(L^(H^1))}(\int_{t_n}^{t_{n+1}} t\ ||\mathbf{B}_{tt}(\cdot, t)||^2 dt)^{1/2}|\mathbf{U}^h_{n+1}|_1 \\
&+ \frac{C\Delta t^{3/2}}{\sqrt{t_{n+1/2}}}||\mathbf{B}_t||_{l^\infty(H^1)}(\int_{t_n}^{t_{n+1}} t\ ||\mathbf{j}_t(\cdot, t)||^2 dt)^{1/2}|\mathbf{U}^h_{n+1/2}|_1 \qquad (86)
\end{aligned}
$$

Estimates (81), (82), (84), (86), along with Young's inequality and application of assumed regularity (to bring weighting factor t outside of integrals) and estimates (62), (63), (64) for $\mathbf{u}$, $\mathbf{j}$, and $\mathbf{B}$ prove (79). □

Lemma 4.8. *Let* $\mathbf{u}$ *satisfy the regularity assumptions of Theorem 4.3. Then, for any* $\varepsilon'' > 0$ *and any integer* $n \ge 0$

$$
|\tau^{(2)}_n(\Phi^h_{n+1/2})| \le \frac{1}{\varepsilon''}|\Phi^h_{n+1/2}|_1^2 + C\Delta t^3 E^{(2)}_{n+1} \qquad (87)
$$

where

$$
\begin{aligned}
E^{(2)}_{n+1} &:= C\overline{B}^2||\mathbf{u}_{tt}||^2_{L^2(t_n,t_{n+1};L^2)} + C||\mathbf{u}||^2_{l^\infty(H^2)}||\mathbf{B}_{tt}||^2_{L^2(t_n,t_{n+1};L^2)} \\
&+ C||\mathbf{B}_t||^2_{l^\infty(H^1)}||\mathbf{u}_t||^2_{L^2(t_n,t_{n+1};H^1)} \qquad (88)
\end{aligned}
$$

Proof. We estimate $\tau^{(2)}(\mathbf{v})$ in a manner similar to $\hat{\hat{\tau}}^{(1)}(\mathbf{v})$ in the proof of Lemma 4.7. Application of Taylor's theorem with integral remainder gives

$$
\begin{aligned}
\tau^{(2)}_{n+1}(\mathbf{\Phi}^h_{n+1}) &= ((\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2})) \times \mathbf{u}_{n+1/2}, \nabla \mathbf{\Phi}^h_{n+1/2}) \\
&+ (\mathbf{B}(\cdot, t_{n+1/2}) \times (\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})), \nabla \mathbf{\Phi}^h_{n+1/2}) \\
&- \frac{1}{2}\int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1} - t) \int ((\mathbf{B} \times \mathbf{u}_{tt}) \cdot \nabla \Phi^h_{n+1/2} + (\mathbf{B}_{tt} \times \mathbf{U}^h) \cdot \nabla \Phi^h_{n+1/2} \\
&\quad \ldots + 2(\mathbf{B}_t \times \mathbf{u}_t) \cdot \nabla \Phi^h_{n+1/2}) dt \\
&- \frac{1}{2}\int_{t_n}^{t_{n+1/2}} (t - t_n) \int ((\mathbf{B} \times \mathbf{u}_{tt}) \cdot \nabla \Phi^h_{n+1/2} + (\mathbf{B}_{tt} \times \mathbf{U}^h) \cdot \nabla \Phi^h_{n+1/2} \\
&\quad \ldots + 2(\mathbf{B}_t \times \mathbf{u}_t) \cdot \nabla \Phi^h_{n+1/2}) dt
\end{aligned} \tag{89}
$$

Majorize (89) with duality estimate on $H^1_0 \times W^{-1}_2$ and Hölder's inequality (in time and space) to get

$$
\begin{aligned}
|\tau^{(2)}_{n+1}(\mathbf{\Phi}^h_{n+1/2})| &\le C\overline{B}||\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})||\ |\mathbf{\Phi}^h_{n+1/2}|_1 \\
&+ C||\mathbf{u}||_{l^\infty(L^\infty)}||\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2})||\ |\mathbf{\Phi}^h_{n+1/2}|_1 \\
&+ \frac{C\Delta t^{3/2}}{\sqrt{t_{n+1/2}}}\overline{B}(\int_{t_n}^{t_{n+1}} t\ ||\mathbf{u}_{tt}(\cdot, t)||^2 dt)^{1/2} |\mathbf{\Phi}^h_{n+1}|_1 \\
&+ \frac{C\Delta t^{3/2}}{\sqrt{t_{n+1/2}}}||\mathbf{u}||_{l^\infty(L^\infty)}(\int_{t_n}^{t_{n+1}} t\ ||\mathbf{B}_{tt}(\cdot, t)||^2 dt)^{1/2} |\mathbf{\Phi}^h_{n+1}|_1 \\
&+ \frac{C\Delta t^{3/2}}{\sqrt{t_{n+1/2}}}||\mathbf{B}_t||_{l^\infty(H^1)}(\int_{t_n}^{t_{n+1}} t\ ||\mathbf{u}_t(\cdot, t)||_1^2 dt)^{1/2} |\mathbf{\Phi}^h_{n+1/2}|_1.
\end{aligned} \tag{90}
$$

Estimate (90), along with Young and application of assumed regularity (to bring weighting factor t outside of integrals) and estimates (63), (64) for $\mathbf{u}$ and $\mathbf{B}$, proves (87). □

Apply estimates (77), (78), (79), (87) to (76). Set ε', $\varepsilon'' = 4$ and absorb all like-terms from the right into left-hand-side of (74). Approximation (11) gives $|\eta_n|_1 \le C|\mathbf{u}_n|_1^2$. Sum the resulting inequality on both sides from $n = 0$ to $n = K-1$.

Apply the estimate (79), (87), (11). Multiply by N to get

$$||\mathbf{U}_K^h||^2 + M^{-2}N\Delta t\sum_{n=0}^{K-1}|\mathbf{U}_{n+1/2}^h|_1^2 + N\Delta t\sum_{n=0}^{K-1}(||\mathbf{J}_{n+1/2}^h||^2 + |\Phi_{n+1/2}^h|_1^2)$$
$$\le C\Delta t\sum_{n=0}^{K-1}(M^6N^{-3}|\mathbf{u}_{n+1/2}|_1^4 + ||\mathbf{B}_{n+1/2}||_{0,\infty}^2)||\mathbf{U}_{n+1/2}^h||^2$$
$$+ ||\mathbf{U}_0^h||^2 + CM^2N^{-1}\Delta t\sum_{n=0}^{K-1}(||\frac{\eta_{n+1}-\eta_n}{\Delta t}||_{-1}^2 + ||\mathbf{u}||_{L^\infty(H^1)}^2|\eta_{n+1/2}|^2)$$
$$+ CM^2N\Delta t\sum_{n=0}^{K-1}||\tilde{q}_{n+1/2}^h - p_{n+1/2}||^2 + CN\Delta t\sum_{n=0}^{K-1}||\chi_{n+1/2}||^2$$
$$+ M^2N^{-1}\Delta t^4\sum_{n=0}^{K-1}E_{n+1}^{(1)} + N\Delta t^4\sum_{n=0}^{K-1}E_{n+1}^{(2)}. \tag{91}$$

Write

$$G_K := \exp(\sum_{n=0}^{K}\frac{\kappa_n}{1-\Delta t\kappa_n}), \quad \kappa_n := M^6N^{-3}|\mathbf{u}_{n+1/2}|_1^4 + ||\mathbf{B}_{n+1/2}||_{0,\infty}^2.$$

Suppose that $\Delta t\kappa_n < 1$. Apply Gronwall Lemma 4.6 to (91) to get

$$||\mathbf{U}_K^h||^2 + \frac{N\Delta t}{M^2}\sum_{n=0}^{K-1}|\mathbf{U}_{n+1/2}^h|_1^2 + N\Delta t\sum_{n=0}^{K-1}(||\mathbf{J}_{n+1/2}^h||^2 + |\Phi_{n+1/2}^h|_1^2)$$
$$\le G_K||\mathbf{U}_0^h||^2 + G_KM^2N^{-1}\Delta t^4\sum_{n=0}^{K-1}E_{n+1}^{(1)} + G_KN\Delta t^4\sum_{n=0}^{K-1}E_{n+1}^{(2)}$$
$$+ G_KM^2N^{-1}\Delta t\sum_{n=0}^{K-1}(||\frac{\eta_{n+1}-\eta_n}{\Delta t}||_{-1}^2 + ||\mathbf{u}||_{L^\infty(H^1)}^2|\eta_{n+1/2}|^2)$$
$$+ G_KM^2N\Delta t\sum_{n=0}^{K-1}||\tilde{q}_{n+1/2}^h - p_{n+1/2}||^2 + G_KN\Delta t\sum_{n=0}^{K-1}||\chi_{n+1/2}||^2. \tag{92}$$

Fix $k \ge 0$. Recall that (19), (11)(a) give

$$|\eta|_1 \le Ch^k||\mathbf{u}||_{k+1}. \tag{93}$$

Fix $k' \ge 2$. Then together we estimate (62), we get

$$||\frac{\eta_{n+1}-\eta_n}{\Delta t}||_{-1}^2 \le Ch^{2k'}\Delta t^{-1}||\mathbf{u}_t||_{L^2(t_n,t_{n+1};H^{k'-1})}^2. \tag{94}$$

Fix $s \ge -1$. Estimate (11)(b) gives

$$\inf_{\tilde{q}^h\in Q^h}||p_{n+1/2} - \tilde{q}_{n+1/2}^h|| \le Ch^{s+1}||p_{n+1/2}||_{s+1}. \tag{95}$$

Fix $r \ge 0$. Then estimates (19), (20), and (11)(a)(c)

$$||\chi_{n+1/2}|| \le C(h^r||\phi_{n+1/2}||_{r+1} + h^k\overline{B}||\mathbf{u}_{n+1/2}||_k). \tag{96}$$

Apply estimates (93), (94), (95), (96) to (92) to get

$$
\begin{aligned}
&||\mathbf{U}_K^h||^2 + \frac{N\Delta t}{M^2}\sum_{n=0}^{K-1}|\mathbf{U}_{n+1/2}^h|_1^2 \\
&+ N\Delta t\sum_{n=0}^{K-1}(||\mathbf{J}_{n+1/2}^h||^2 + |\Phi_{n+1/2}^h|_1^2) \le G_K||\mathbf{E}_{u,0}||^2 + (F_{\Delta t}^h)^2.
\end{aligned} \tag{97}
$$

where $F_{\Delta t}^h$ is given in (56). The triangle inequality $||\mathbf{E}_{u,n}|| \le ||\mathbf{U}_n^h|| + ||\eta_n||$, $||E_{\phi,n}|| \le ||\Phi_n^h|| + ||\zeta_n||$, $||\mathbf{E}_{j,n}|| \le ||\mathbf{J}_n^h|| + ||\chi_n||$ along with estimates (19), (20), (11) proves (57), (58), (59). Bound the right-hand-side of (73) with Cauchy-Schwarz, Young, and estimate (87). Absorb like-terms into the left-hand-side to get

$$
|\Phi_{n+1/2}^h|_1^2 \le C\overline{B}^2||\mathbf{E}_{u,n+1/2}||^2 + C|\zeta_{n+1/2}|_1^2 + CE_{n+1}^{(2)}. \tag{98}
$$

Apply triangle inequality $||E_{\phi,n}|| \le ||\Phi_n^h|| + ||\zeta_n||$ along with estimates (97), (19), (20), (11) to prove (60).

Estimates (11) and (62), (63), (64) applied to the results in Theorem 4.3 prove the estimate in Corollary 4.5.

5. Numerical results

We consider two distinct numerical experiments in this section. First, we confirm the converge rate as h, $\Delta t \to 0$ for the fully discrete, simplified MHD model (49), (50), (51), (52). Second, we consider the effect of an applied magnetic field to a conducting flow in a channel past a step by illustrating the damping effect of $B \neq 0$.

We use the FreeFem++ software for each of our simulations. We utilize Taylor-Hood mixed FE's (piecewise quadratics for velocity and piecewise linear pressure) for the discretization. We apply Newton iterations to solve the nonlinear system with a $||\mathbf{u}^{(j)} - \mathbf{u}^{(j-1)}||_1 < 10^{-8}$ as a stopping criterion.

Experiment 1 - Convergence Analysis: For the first experiment, $\Omega = [0,\pi]^2$, $t_0 = 0$, $T = 1$, $Re = 25$, $B = (0,0,1)$. The true solution $(\mathbf{u}, p, \phi)$ is given by

$$
\begin{array}{rclrcl}
\psi(x,y) &=& \cos(2x)\cos(2y), & \mathbf{u}(x,y,t) &=& \left(-\dfrac{\partial\psi(x,y)}{\partial y}, \dfrac{\partial\psi(x,y)}{\partial x}\right)e^{-5t} \\
p(x,y,t) &=& 0, & \phi(x,y,t) &=& (\psi(x,y) + x^2 - y^2)e^{-5t}.
\end{array} \tag{99}
$$

We obtain $\mathbf{f}$, $\mathbf{u}|_{\partial\Omega}$ from the true solution. We solve one large coupled system for $\mathbf{u}$, p, ϕ. A uniform triangular mesh is used. We set $Re = 25$ for this experiment. Results are compiled in Tables 1 and 2 for $M = 20$, $N = M^2/Re = 16$ and $M = 200$, $N = M^2/Re = 1600$ respectively . Write $|||\cdot|||_{0,\infty} = \max_n ||\cdot||$ and $|||\cdot|||_{0,2} = (\Delta t\sum_n ||\cdot||^2)^{1/2}$. The rates of convergence suggested in Table 1 correlate with theoretical rate $\mathcal{O}(h^2 + \Delta t^2)$ predicted in Corollary 4.5 for $\mathbf{u}$ corresponding discrete norms. Note that the $\mathcal{O}(h^3)$ convergence suggested in L^2 is expected and can be shown by extension of the estimates in Theorem 4.3 via a duality argument. Similar results are reported in Table 2 for different error-measures, including the electric potential ϕ.

Experiment 2: Flow in channel over a step is a classic benchmark test. It is well-known that there exists a critical fluid Reynolds number $Re_c > 0$ so that the vortex developed in the wake of the step will detach from the step and be carried downstream for any $Re > Re_c$. In this test, we show how the SMHD

TABLE 1. Convergence rate data for the first experiment, $H = 20$

h	Δt	$\|\|\|\mathbf{E}_{u,n}\|\|\|_{0,\infty}$	rate	$\|\|\|\nabla\mathbf{E}_{u,n+1/2}\|\|\|_{0,2}$	rate
1/10	1/40	3.727e-2	−−	3.659e-1	−−
1/20	1/80	4.477e-3	3.05	8.111e-2	2.17
1/40	1/160	4.498e-4	3.32	1.439e-2	2.49
1/80	1/320	4.564e-5	3.30	2.789e-3	2.36
1/160	1/640	4.805e-6	3.24	6.285e-4	2.15

h	Δt	$\|\|\|\mathbf{E}_{j,n+1/2}\|\|\|_{0,2}$	rate
1/10	1/40	5.471e-2	−−
1/20	1/80	1.295e-2	2.08
1/40	1/160	3.326e-3	1.96
1/80	1/320	8.451e-4	1.98
1/160	1/640	2.126e-4	1.99

TABLE 2. Convergence rate data for the first experiment, $H = 200$

h	Δt	$\|\|\|\mathbf{E}_{u,n+1/2}\|\|\|_{0,2}$	rate	$\|\|\|\nabla\mathbf{E}_{u,n+1/2}\|\|\|_{0,2}$	rate
1/10	1/100	3.559e-1	−−	1.488e-0	−−
1/20	1/200	3.772e-2	3.23	5.546e-1	1.42
1/40	1/400	4.159e-3	3.18	2.311e-1	1.26
1/80	1/800	5.299e-4	2.97	7.135e-2	1.69
1/160	1/1600	5.069e-5	3.39	1.424e-2	2.32

h	Δt	$\|\|\|\nabla\mathbf{E}_{\phi,n+1/2}\|\|\|_{0,2}$	rate
1/10	1/100	3.575e-1	−−
1/20	1/200	3.873e-2	2.02
1/40	1/400	4.890e-3	2.98
1/80	1/800	9.416e-4	2.37
1/160	1/1600	2.167e-4	2.12

model accurately models the damping effect of the magnetic field by suppressing the shedding of these vortices.

We consider here a $[0, 40] \times [0, 10]$ channel. The step is square with width 1. The front of the step is located at $x = 5$. For velocity boundary conditions, let $\mathbf{u}|_{x=0} = 0.04y(10 - y)$ at the in-flow, do-nothing $(-M^{-2}\nabla\mathbf{u} \cdot \hat{n} + p\hat{n})|_{x=40} = 0$ at the out-flow, and no-slip $\mathbf{u} = 0$ otherwise. For the potential, let $\phi|_{y=10} = 0$ and $\nabla\phi \cdot \hat{n} = 0$ otherwise. We set $Re = M^2/N = 1800$, $M = 1000$, so that $N \approx 555.6$. We compute the SMHD solution with $\mathbf{B} = (0, 0, 1)$ as well as the NSE solution ($\mathbf{B} \equiv 0$) for comparison. We use $\Delta t = 0.005$ in both cases. Both problems are solved on a non-uniform mesh generated with the Delaunay-Vornoi algorithm, shown in Figure 1. Notice the mesh refinement along the step. The SMHD problem contained 58046 degrees of freedom. We show the streamlines for the velocity field for the corresponding NSE and SMHD solutions in Figures 2, 3 (domain restricted to [0,20] x [0,4]). Notice that as time evolves, vortices are created and separate for the NSE flow. As expected, the magnetic field suppresses this shedding for the SMHD flow so that the vortex in the wake of the step is elongated without separation.

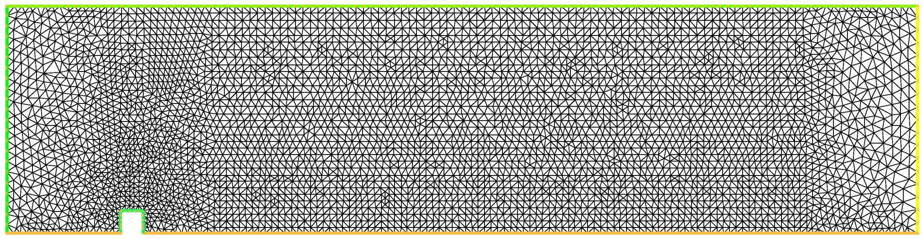

FIGURE 1. Experiment 2: Sample mesh - 8761 triangles.

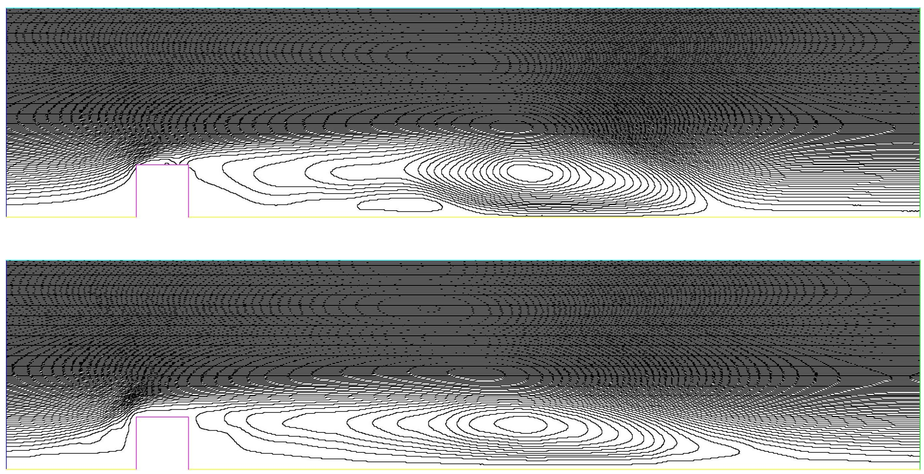

FIGURE 2. Experiment 2: Streamlines at $t = 40$, $Re = 1800$ (top) NS solution, $\mathbf{B} = (0, 0, 0)$, (bottom) SMHD solution, $\mathbf{B} = (0, 0, 1)$, $M = 1000$.

6. Conclusions

We introduced a FE method for quasi-static MHD equation at small Re_m. We decomposed the approximation into two parts. In the first part, we presented the stability and error analysis of semi-discrete approximation. In the second part, we presented the stability and error analysis of fully-discrete approximation, CN in-time. We also conducted two numerical experiments to verify the effectiveness of the proposed model. We confirm the theoretical rate of convergence derived in this report in the first experiment. In the second experiment, we investigate how an applied magnetic field affects the dynamics of the classic flow-past-a-step problem in a channel. As expected, we show with the SMHD model that the magnetic field suppresses the shedding so that the vortex in the wake of the step is elongated without separation. For future work, note that we studied a fully coupled method between fluid velocity and electric potential. To improve speed and assuage memory requirements, we are investigating the existence and effectiveness of an *uncoupled*, approximation of SMHD. Lastly, we are investigating the integration of LES models to SMHD.

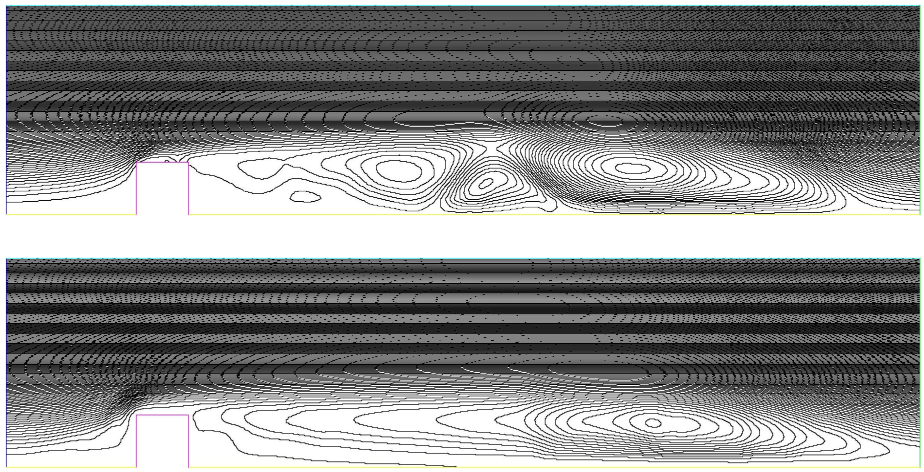

FIGURE 3. Experiment 2: Streamlines at $t = 60$, $Re = 1800$ (top) NS solution, $\mathbf{B} = (0, 0, 0)$, (bottom) SMHD solution, $\mathbf{B} = (0, 0, 1)$, $M = 1000$.

Acknowledgments We thank William Layton for the illuminating discussions and feedback. This work was partially supported by National Science Foundation Grant Division of Mathematical Sciences 080385.

Appendix A. Derivation of intermediate estimates

Proof of Estimate (62). Fix $n \geq 0$. Then, for $k \geq 0$

$$\begin{aligned} |\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t}|_k^2 &= \int \left| \Delta t^{-1} \int_{t_n}^{t_{n+1}} D^k \mathbf{u}_t(\cdot, t) dt \right|^2 \\ &\leq \Delta t^{-2} \int \left(\int_{t_n}^{t_{n+1}} dt \int_{t_n}^{t_{n+1}} \left| D^k \mathbf{u}_t(\cdot, t) \right|^2 dt \right) \leq \Delta t^{-1} \int_{t_n}^{t_{n+1}} |\mathbf{u}_t(\cdot, t)|_k^2 dt. \end{aligned}$$

Similar proof for $k = -1$ applied to definition of W_2^{-1}-norm. □

Proof of Estimate (63). Fix $n \geq 0$ and $k \geq 0$. A Taylor-expansion with integral remainder gives

$$\begin{aligned} |\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})|_k^2 \leq C \int & \left| \int_{t_n}^{t_{n+1/2}} (t - t_n) D^k \mathbf{u}_{tt}(\cdot, t) dt \right|^2 \\ & + C \int \left| \int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1} - t) D^k \mathbf{u}_{tt}(\cdot, t) dt \right| \end{aligned} \tag{100}$$

where, for any $r \in \mathbb{R}$,

$$\begin{aligned}
&\int \left| \int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1} - t) D^k \mathbf{u}_{tt}(\cdot, t) dt \right|^2 \\
&\le C \int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1} - t)^2 dt \int_{t_{n+1/2}}^{t_{n+1}} ||\mathbf{u}_{tt}(\cdot, t)||_k^2 dt \\
&\le C \Delta t^3 \int_{t_{n+1/2}}^{t_{n+1}} \frac{1}{t^r} \left(t^r \; ||\mathbf{u}_{tt}(\cdot, t)||_k^2 \right) dt \\
&\le \frac{C \Delta t^3}{(t_{n+1/2})^r} \int_{t_{n+1/2}}^{t_{n+1}} t^r \; ||\mathbf{u}_{tt}(\cdot, t)||_k^2 dt. \qquad (101)
\end{aligned}$$

and similarly on the time interval $(t_n, t_{n+1/2})$ when $n > 0$. If $n = 0$, then

$$\begin{aligned}
\int \left| \int_0^{\Delta t/2} t \; D^k \mathbf{u}(\cdot, t) dt \right|^2 &\le C \int_0^{\Delta t/2} dt \int_0^{\Delta t/2} t^2 \; ||\mathbf{u}_{tt}(\cdot, t)||_k^2 dt \\
&\le C \Delta t \int_0^{\Delta t/2} t^2 ||\mathbf{u}_{tt}(\cdot, t)||_k^2 dt. \qquad (102)
\end{aligned}$$

Note that $\sqrt{t_{n+1/2}} = \sqrt{\Delta t/2}$ when $n = 0$. Then estimates (101), (102) applied to (100) give

$$|\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})|_k^2 \le \frac{C \Delta t^3}{(t_{n+1/2})^2} \int_{t_n}^{t_{n+1}} t^2 ||\mathbf{u}_{tt}(\cdot, t)||_k^2 dt. \qquad (103)$$

□

Proof of Estimate (64). Fix $n \ge 0$. First add/subtract $\mathbf{u}_t(\cdot, t_{n+1/2})$ and apply the triangle inequality to get

$$\begin{aligned}
&||\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} - (\mathbf{u}_t)_{n+1/2}||_k^2 \\
&\le ||\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} - \mathbf{u}_t(\cdot, t_{n+1/2})||_k^2 + ||\mathbf{u}_t(\cdot, t_{n+1/2}) - (\mathbf{u}_t)_{n+1/2}||_k^2 \qquad (104)
\end{aligned}$$

Following a similar method used to derive (100), we get

$$||\mathbf{u}_t(\cdot, t_{n+1/2}) - (\mathbf{u}_t)_{n+1/2}||_k^2 \le \frac{C \Delta t^3}{(t_{n+1/2})^2} \int_{t_n}^{t_{n+1}} t^2 \; ||\partial_t^{(3)} \mathbf{u}(\cdot, t)||_k^2 dt. \qquad (105)$$

Additionally,

$$\begin{aligned}
&||\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} - \mathbf{u}_t(\cdot, t_{n+1/2})||_k^2 \\
&= \left\| \int_{t_n}^{t_{n+1/2}} (t - t_n) \mathbf{u}_{ttt}(\cdot, t) dt + \int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1} - t) \mathbf{u}_{ttt}(\cdot, t) dt \right\|_k^2 \\
&\le \frac{C \Delta t^3}{(t_{n+1/2})^2} \int_{t_n}^{t_{n+1}} t^2 ||\mathbf{u}_{ttt}(\cdot, t)||_k^2 dt. \qquad (106)
\end{aligned}$$

Apply (105) and (106) to (104) to get

$$||\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} - (\mathbf{u}_t)_{n+1/2}||_k^2 \le \frac{C \Delta t^3}{(t_{n+1/2})^2} \int_{t_n}^{t_{n+1}} t^2 ||\mathbf{u}_{ttt}(\cdot, t)||_k^2 dt. \qquad (107)$$

□

Proof of Estimate (84). Fix $n \geq 0$. Then, for any $r \in \mathbb{R}$, and for either $i = 0$ or 1,

$$\int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1} - t)(\mathbf{u}_{tt} \cdot \nabla \mathbf{u}, \mathbf{v})dt \leq C \int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1} - t)||\mathbf{u}||_2||\mathbf{u}_{tt}||_{1-i}|\mathbf{v}|_i$$

$$\leq C||\mathbf{u}||_{L^\infty(t_n,t_{n+1};H^2)}(\int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1} - t)^2 dt)^{1/2}(\int_{t_{n+1/2}}^{t_{n+1}} ||\mathbf{u}_{tt}||_{1-i}^2 dt)^{1/2}|\mathbf{v}|_i$$

$$\leq \frac{C\Delta t^{3/2}}{(t_{n+1/2})^{r/2}}||\mathbf{u}||_{L^\infty(t_n,t_{n+1};H^2)}(\int_{t_{n+1/2}}^{t_{n+1}} t^r \; ||\mathbf{u}_{tt}||_{1-i}^2 dt)^{1/2}|\mathbf{v}|_i. \tag{108}$$

A similar estimate holds when time interval is shifted to (t_n, t_{n+1}) except when $n = 0$. In this case

$$\int_0^{\Delta t/2} t(\mathbf{u}_{tt} \cdot \nabla \mathbf{u}, \mathbf{v})dt \leq C \int_0^{\Delta t/2} t||\mathbf{u}||_2||\mathbf{u}_{tt}||_{1-i}|\mathbf{v}|_i$$

$$\leq C||\mathbf{u}||_{L^\infty(0,\Delta t;H^2)}(\int_0^{\Delta t/2} dt)^{1/2}(\int_0^{\Delta t/2} t^2 \; ||\mathbf{u}_{tt}||_{1-i}^2 dt)^{1/2}|\mathbf{v}|_i$$

$$\leq C\Delta t^{1/2}||\mathbf{u}||_{L^\infty(0,\Delta t;H^2)}(\int_0^{\Delta t/2} t^2 \; ||\mathbf{u}_{tt}||_{1-i}^2 dt)^{1/2}|\mathbf{v}|_i. \tag{109}$$

Note that $\sqrt{t_{n+1/2}} = \sqrt{\Delta t/2}$ when $n = 0$. Therefore, (108), (109) combine to give, for $n \geq 0$

$$\int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1} - t)(\mathbf{u}_{tt} \cdot \nabla \mathbf{u}, \mathbf{v})dt + \int_{t_n}^{t_{n+1/2}} (t - t_n)(\mathbf{u}_{tt} \cdot \nabla \mathbf{u}, \mathbf{v})dt$$

$$\leq \frac{C\Delta t^{3/2}}{t_{n+1/2}}||\mathbf{u}||_{L^\infty(t_n,t_{n+1};H^2)}(\int_{t_n}^{t_{n+1}} t^2 \; ||\mathbf{u}_{tt}||_{1-i}^2 dt)^{1/2}|\mathbf{v}|_i. \tag{110}$$

Now recall that $(\mathbf{u} \cdot \nabla \mathbf{u}_{tt}, \mathbf{v}) = -(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{u}_{tt})$ since $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{v} = 0$. Then again a similar argument used to derive (109) proves

$$\int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1} - t)(\mathbf{u} \cdot \nabla \mathbf{u}_{tt}, \mathbf{v})dt + \int_{t_n}^{t_{n+1/2}} (t - t_n)(\mathbf{u} \cdot \nabla \mathbf{u}_{tt}, \mathbf{v})dt$$

$$\leq \frac{C\Delta t^{3/2}}{t_{n+1/2}}||\mathbf{u}||_{L^\infty(t_n,t_{n+1};H^2)}(\int_{t_n}^{t_{n+1}} t^2 \; ||\mathbf{u}_{tt}||_{1-i}^2 dt)^{1/2}|\mathbf{v}|_i. \tag{111}$$

Once again, following a similar argument used to derive (109) proves, for $n \geq 0$,

$$\int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1} - t)(\mathbf{u}_t \cdot \nabla \mathbf{u}_t, \mathbf{v})dt + \int_{t_n}^{t_{n+1/2}} (t - t_n)(\mathbf{u}_t \cdot \nabla \mathbf{u}_t, \mathbf{v})dt$$

$$\leq \frac{C\Delta t^{3/2}}{t_{n+1/2}}||\mathbf{u}_t||_{L^\infty(t_n,t_{n+1};L^2)}(\int_{t_n}^{t_{n+1}} t^2 \; ||\mathbf{u}_t||_{3-i}^2 dt)^{1/2}|\mathbf{v}|_i. \tag{112}$$

□

References

[1] S.H. Aydin, A.I. Nesliturk, and M. Tezer-Sezgin. Two-level finite element method with a stabilizing subgrid for the incompressible MHD equations. *Inter. J. Num. Meth. Fluids*, 62(2):188–210, 2009.

[2] L. Barleon, V. Casal, and L. Lenhart. MHD flow in liquid-metal-cooled blankets. *Fusion Engineering and Design*, 14(3-4):401–412, 1991.

[3] P.A. Davidson. *An Introduction to Magnetohydrodynamics*. Cambridge University Press, United Kingdom, 2001.
[4] G.P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, volume I. Springer-Verlag, New York, 1994.
[5] V. Girault and P.A. Raviart. *Finite Element Methods for Navier-Stokes Equations*. Springer-Verlag, Berlin, 1986.
[6] M.D Gunzburger, A. Meir, and J. Peterson. On the existence, uniqueness and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics. *Math. Comput.*, 56:523–563, 1991.
[7] H. Hashizume. Numerical and experimental research to solve MHD problem in liquid-blanket system. *Fusion Engineering and Design*, 81(8-14):1431–1438, 2006.
[8] R.D. Hazeltine and J.D Meiss. *Plasma Confinement*. Addison-Wesley Publishing Company, Redwood city, California, 1992.
[9] J.G Heywood and R. Rannacher. Finite element approximation of the nonstationary Navier-Stokes problem, IV. Error analysis for second-order time discretization. *SIAM J. Numer. Anal.*, 19:275–311, 1990.
[10] W.F. Hughes and F.J. Young. *The Electromagnetics of Fluids*. Wiley, New York, 1966.
[11] H. Lass. *Vector and Tensor Analysis*. McGraw-Hill, New York, 1950.
[12] W. Layton, H.W.J. Lenferink, and J. Peterson. A two-level Newton, finite element algorithm for approximation of electrically conducting incompressible fluid flows. *Comput. Math. Appl.*, 28(5):21–31, 1994.
[13] W. Layton, A.J Meir, and P.G. Schmidt. A two-level discretization method for the stationary MHD equations. *Electron. Trans. Numer. Anal.*, 6:198–210, 1997.
[14] W. Layton and L. Tobiska. A two-level method with backtracking for the Navier-Stokes equations. *SIAM J. Numer. Anal.*, 35:2035–2054, 1998.
[15] A.J Meir and P.G. Schmidt. Analysis and finite element simulation of MHD flows, with an application to seawater drag reduction. In J.C.S. Meng and et. al., editors, *Proceedings of an International Symposium on Seawater Drag Reduction*, pages 401–406, Englewood Cliffs, 1998. Naval Warfare Center.
[16] A.J Meir and P.G. Schmidt. Analysis and finite element simulation of MHD flows, with an application to liquid metal processing. In N. El-Kaddah and et. al., editors, *Fluid Flow Phenomena in Metals Processing*, pages 561–569, Newport, 1999. TMS.
[17] A.J Meir and P.G. Schmidt. Analysis and numerical approximation of stationary MHD flow problem with nonideal boundary. *SIAM J. Numer. Anal.*, 36(4):1304–1332, 1999.
[18] H.A. Navarro, L. Cabezas-Govez, R.C. Silva, and A.N. Montagnoli. A generalized alternating-direction implicit scheme for incompressible magnetohydrodynamics viscous flows at low magnetic Reynolds number. *Appl. Math. Comput.*, 189:1601–1613, 2007.
[19] J. Peterson. On the finite element approximation of incompressible flows of an electrically conducting fluid. *Numer. Methods Partial Differential Equations*, 4:57–68, 1988.
[20] R. Samulyak, J. Du, J. Glimm, and Z. Xu. A numerical algorithm for MHD of free surface flows at low magnetic Reynolds numer. *J. Comput. Phys.*, 226:1532–1549, 2007.
[21] J.A. Shercliff. *A textbook of Mahnetohydrodynamics*. Pergamon, Oxford, 1965.
[22] S. Smolentsev, S. Cuevas, and A. Beltran. Induced electric current-based formulation in computations at low magnetic Reynolds number magnetohydrodynamic flows. *J. Comput. Phys*, 229:1558–1572, 2010.
[23] J. Wesson and D.J. Campbell. *Tokamaks*. Oxford University Press, New York, third edition, 2004.
[24] M. Wiedmer. Finite element approximation for equations of magnetohydrodynamics. *Math. Comput.*, 69(229):83–101, 1999.

Department of Mathematics, Mugla University, Mugla, 48000, TURKEY
E-mail: ngamze@mu.edu.tr

301 Thackeray Hall, University of Pittsburgh, Pittsburgh, PA 15260, USA
E-mail: rni1@pitt.edu
URL: http://www.pitt.edu/∼rni1