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EXTRAPOLATION OF THE FINITE ELEMENT METHOD ON GENERAL MESHES

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Abstract. In this paper, we consider the extrapolation method for second order elliptic problems on general meshes and derive a type of finite element expansion which is dependent of the triangulation. It allows to prove the effectiveness of the extrapolation on general meshes and also validates the extrapolation method can be applied on the automatically produced meshes of the general computing domains. Some numerical examples are given to illustrate the theoretical analysis.

Key words. Extrapolation, finite element method, general meshes.

1. Introduction

It is well known that the extrapolation method, which was established by Richardson in 1926, is an efficient procedure for increasing the solution accuracy of many problems in numerical analysis. The effectiveness of this technique relies heavily on the existence of an asymptotic expansion for the error. The application of this approach in finite difference method can be found in the book of Marchuk and Shaidurov [11]. This technique has been well demonstrated in the frame of the finite element method [7, 10, 9, 5].

Usually in the finite element method, we first need to get the error expansion for the solution approximations such as [7, 2, 10, 9, 5]

$$u_h(x) - \pi_h u(x) = c_1(u)h^k + O(h^{k+\delta}), \tag{1}$$

in some norm sense, where c_1 is a function depending on u and independent of h, $\delta > 0$, u_h and $\pi_h u$ are the finite element approximation and interpolation, respectively. Then we can use the extrapolation method ([7, 2, 10, 9])

$$u_h^{\mathrm{extra}} := \frac{2^k u_{h/2} - u_h}{2^k - 1}, \tag{2}$$

which has higher convergence order $O(h^{k+\delta})$ only at the mesh nodes ([7]).

If we want to obtain globally higher order convergence, we must need to apply the higher order interpolation postprocessing operator $\mathcal{Q}_h$ ([7, 9, 5])

$$u_h^{\mathrm{extra}} := \frac{2^k \mathcal{Q}_{h/2} u_{h/2} - \mathcal{Q}_h u_h}{2^k - 1}, \tag{3}$$

which has globally higher convergence order $O(h^{k+\delta})$.

So far there are two types of extrapolation schemes for the finite element method as described above: mesh nodes extrapolation and extrapolation based on the interpolation postprocessing. So, the key for the extrapolation of the finite element method is whether we can get the expansion (1) for the finite element approximation. But, so far the expansion (1) almost need structured meshes ([7, 2, 8, 10, 9, 5]).

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So far, we always study the extrapolation situation under the structured mesh and the mesh condition is the important restrict for the extrapolation method extended to general meshes. In this paper, we first consider the interpolation expansion on general meshes and then derive what kind of needed properties of the meshes to improve the accuracy of the finite element approximations by extrapolation method. For this aim, we derive the definition of the mesh measurement for the finite element extrapolation. And based on the properties of the mesh measurement, we can obtain that the extrapolation method always has effectiveness on general meshes.

For simplicity, we consider the following second order elliptic problem

$$B(u,v) \;=\; \int_\Omega (\mathcal{A}\nabla u\cdot\nabla v+\rho uv)dxdy = f(v),\ \forall v\in\mathcal{V}:=H_0^1(\Omega), \tag{4}$$

where $\mathcal{A}=\{a_{ij}\}_{1\le i,j\le 2}\in\mathcal{R}^{2\times 2}$ is a symmetric positive definite matrix, $\rho\ge 0$ in Ω, $f(\cdot)$ a bounded linear functional in $H^{-1}(\Omega)$, and Ω is a bounded domain in $\mathcal{R}^2$ with Lipschitz boundary $\partial\Omega$. For simplicity, we assume the matrix $\mathcal{A}$ and function ρ are smooth enough.

Let $\mathcal{T}_h$ be the consistent triangulation of the domain Ω in the set of triangular elements and satisfy the following quasi-uniform condition:

$$\exists\sigma>0 \text{ such that } \ h_K/\tau_K>\sigma,\ \ \forall K\in\mathcal{T}_h$$

and

$$\exists\gamma>0,\ \text{such that}\ \ \max\{h/h_K,\ K\in\mathcal{T}_h\}\le\gamma,$$

where h_K is the diameter of K; τ_K is maximum diameter of the inscribed circle in $K\in\mathcal{T}_h$; and $h:=\max\{h_K,K\in\mathcal{T}_h\}$.

The linear finite element space $\mathcal{V}_h$ on $\mathcal{T}_h$ is defined as follows:

$$\mathcal{V}_h=\big\{v\in H^1(\Omega), v|_K\in\mathcal{P}_1(K),\ \forall K\in\mathcal{T}_h\big\}\cap H_0^1(\Omega),$$

where $\mathcal{P}_1=\mathrm{span}\{1,x,y\}$. For our analysis, we need to define the interpolation operator $\pi_h: H^2(\Omega)\mapsto\mathcal{V}_h$ on the mesh $\mathcal{T}_h$ as

$$\pi_h u(Z_i)=u(Z_i),\quad i=1,2,3,$$

where Z_i are the three vertices of element $K\in\mathcal{T}_h$.

Based on the finite element space $\mathcal{V}_h$, we define the Ritz-projection operator $\mathcal{L}_h:\mathcal{V}\mapsto\mathcal{V}_h$ as

$$B(\mathcal{L}_h u, v_h)=f(v_h),\quad \forall v_h\in\mathcal{V}_h. \tag{5}$$

It is known about the convergence rate that

$$\|\mathcal{L}_h u-u\|_0+h\|\mathcal{L}_h u-u\|_1\le Ch^2\|u\|_2, \tag{6}$$

where $\|\cdot\|_0$ denotes the L^2-norm.

In order to use the extrapolation method, we need to refine the mesh $\mathcal{T}_h$ in the regular way. Each element $K\in\mathcal{T}_h$ is subdivided into 4 congruent triangles by connecting the midpoints of its edges (see Figure 3) and we get the finer mesh $\mathcal{T}_{h/2}$. In the similar way, we can define the finite element space $\mathcal{V}_{h/2}$ and the corresponding operators $\pi_{h/2}$, $\mathcal{L}_{h/2}$ on the finer mesh $\mathcal{T}_{h/2}$. It is obviously $\mathcal{V}_h\subset\mathcal{V}_{h/2}$.

Other notations for Sobolev spaces and norms in them (including with fractional orders) are standard and can be found in many sources like [4].

The rest of the paper is organized in the following way. In section 2 we give some useful preliminary lemmas. Interpolation expansions are obtained in section 3. Section 4 is devoted to deriving the asymptotic error expansion of the finite element approximation. The extrapolation method is discussed in Section 5. In section 6, two numerical examples are given to illustrate the validity of our analysis. Finally, we give some concluding remarks in the last section.

2. Some useful notations and preliminary lemmas

We first need to define some notations and give some geometric identities for an arbitrary element K. Let K have vertices $Z_i = (x_i, y_i)$ $(1 \le i \le 3)$ oriented counterclockwise. Let e_i $(1 \le i \le 3)$ denote the edges of the element K; $\mathbf{n}_i$ $(1 \le i \le 3)$ are the unit outward normal vectors; $\mathbf{t}_i = (\cos\theta_i, \sin\theta_i)$ $(1 \le i \le 3)$ are the unit tangent vectors with the counterclockwise orientation and θ_i are its corresponding angle to the x-axes; h_i $(1 \le i \le 3)$ are the edge lengths; H_i $(1 \le i \le 3)$ are the perpendicular heights (see Figure 1). We also need to define the following constants of the element K:

$$l_i = h_i/h, \quad i = 1,2,3, \quad \alpha = |K|/h^2.$$

We also use the periodic relation for the subscripts: $i + 3 = i$.

Let $\partial_i = \partial/\partial \mathbf{t}_i$.

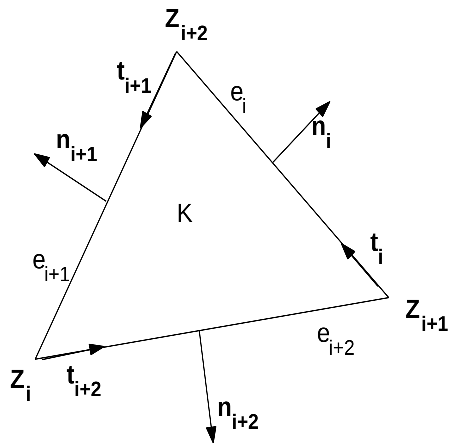


FIGURE 1. The main features of an element K

Now we give some lemmas. They can been found in some papers ([2, 14]) or book ([10, 9]), but for the convenience of readers, we provide proofs here.

Lemma 2.1.

$$\mathbf{t}_i \cdot \mathbf{n}_{i+1} = \frac{2|K|}{h_i h_{i+1}}, \tag{7}$$

$$\mathbf{n}_i \cdot \mathbf{t}_{i+1} = -\frac{2|K|}{h_i h_{i+1}}, \tag{8}$$

$$\mathbf{n}_i = \frac{h_i h_{i+1}}{2|K|}[(\mathbf{n}_i \cdot \mathbf{n}_{i+1})\mathbf{t}_i - \mathbf{t}_{i+1}], \quad i = 1,2,3. \tag{9}$$

Proof. First, we have

$$\frac{1}{2} h_i H_i = |K|, \quad (h_i \mathbf{t}_i) \cdot \mathbf{n}_{i+1} = H_{i+1},$$

then

$$\mathbf{t}_i \cdot \mathbf{n}_{i+1} = \frac{1}{h_i} H_{i+1} = \frac{2|K|}{h_i h_{i+1}}.$$

So, we obtain (7). Similarly we can obtain (8).

Since $\mathbf{t}_i$ and $\mathbf{t}_{i+1}$ are two linear independent vectors, we have two constants β_i and β_{i+1} such that

$$\mathbf{n}_i = \beta_i \mathbf{t}_i + \beta_{i+1}\mathbf{t}_{i+1}.$$

Using equality $\mathbf{t}_{i+1} \cdot \mathbf{n}_{i+1} = 0$ and (7), we have

$$\mathbf{n}_i \cdot \mathbf{n}_{i+1} = \beta_i \mathbf{t}_i \cdot \mathbf{n}_{i+1} = \frac{2\beta_i |K|}{h_i h_{i+1}}.$$

So

$$\beta_i = \frac{h_i h_{i+1}}{2|K|}\mathbf{n}_i \cdot \mathbf{n}_{i+1}.$$

Similarly using equality $\mathbf{t}_i \cdot \mathbf{n}_i = 0$ and (8), we have $\beta_{i+1} = -h_i h_{i+1}/(2|K|)$. This completes the proof. □

Using (9), we can have the following differential property.

Lemma 2.2.

$$\frac{\partial v}{\partial \mathbf{n}_i} \quad = \quad \frac{l_i l_{i+1}}{2\alpha}\big[(\mathbf{n}_i \cdot \mathbf{n}_{i+1})\partial_i v - \partial_{i+1} v\big]. \tag{10}$$

Proof. From (9) we have equality

$$\begin{aligned} \frac{\partial v}{\partial \mathbf{n}_i} = \nabla v \cdot \mathbf{n}_i \quad &= \quad \frac{l_i l_{i+1}}{2\alpha}\nabla v \cdot \big[(\mathbf{n}_i \cdot \mathbf{n}_{i+1})\mathbf{t}_i - \mathbf{t}_{i+1}\big] \\ &= \quad \frac{l_i l_{i+1}}{2\alpha}\big[(\mathbf{n}_i \cdot \mathbf{n}_{i+1})\partial_i v - \partial_{i+1} v\big]. \end{aligned}$$

□

We also need the following integration formula.

Lemma 2.3. *Assume that $v \in C^1(\bar{K})$, then we have*

$$h_{i+1}\int_{e_i} v ds - h_i \int_{e_{i+1}} v ds = \frac{h_1 h_2 h_3}{2|K|}\int_K \partial_{i+2} v dx dy. \tag{11}$$

Proof. With the Green formula, we have

$$\int_K \partial_{i+2} v dx dy = \int_{\partial K} v \mathbf{t}_{i+2} \cdot \mathbf{n} ds,$$

where ∂K is the boundary of the element K. Using equality $\mathbf{t}_{i+2} \cdot \mathbf{n}_{i+2} = 0$, (7), and (8), we have

$$\begin{aligned} \int_K \partial_{i+2} v dx dy \quad &= \quad \int_{e_i} v \mathbf{t}_{i+2} \cdot \mathbf{n}_i ds + \int_{e_{i+1}} v \mathbf{t}_{i+2} \cdot \mathbf{n}_{i+1} ds \\ &= \quad \frac{2h_{i+1}|K|}{h_1 h_2 h_3}\int_{e_i} v ds - \frac{2h_i |K|}{h_1 h_2 h_3}\int_{e_{i+1}} v ds. \end{aligned}$$

From this, (11) follows by multiplication of $h_1 h_2 h_3/(2|K|)$. □

3. Interpolation expansions

First, we need the following one dimensional interpolation expansion which is the application of the Bramble-Hilbert lemma and scaling argument.

Lemma 3.1. *Let $\pi_h u$ be the linear interplant of u on K and e_i be an edge of the element K. Assume that $u \in H^3(K)$ and $a \in W^{1,\infty}(K)$. Then we have*

$$\begin{aligned} & \int_{e_i} a(u - \pi_h u)\frac{\partial v_h}{\partial \mathbf{n}_i} ds \\ = & -\frac{h_i^2}{12}\int_{e_i} a\partial_i^2 u \frac{\partial v_h}{\partial \mathbf{n}_i} ds + O(h^2)\|u\|_{3,K}|v_h|_{1,K}, \quad \forall v_h \in \mathcal{V}_h. \end{aligned} \tag{12}$$

Proof. Let $\hat{e} = [0,1]$ be the reference edge and define the affine transformation F from e_i to $\hat{e}$ and then K to $\widehat{K}$. Define the functions $\hat{u} = u$, $\widehat{\pi}\hat{u} = \pi_h u$.

Consider the following linear functional on $\hat{e}$

$$\Phi(\hat{u}) = \int_{\hat{e}} (\hat{u} - \widehat{\pi}\hat{u}) d\hat{s} + \frac{1}{12}\int_{\hat{e}} \partial_{\hat{x}}^2 \hat{u} d\hat{s}.$$

By the Sobolev imbedding theorem, we know that the functional Φ is bounded

$$|\Phi(\hat{u})| \le C\|\hat{u}\|_{3,\widehat{K}}.$$

A direct computation shows that

$$\Phi(\hat{u}) = 0, \quad \forall \hat{u} \in \mathcal{P}_2(\widehat{K}).$$

Then the Bramle-Hilbert lemma gives

$$|\Phi(\hat{u})| \le C|\hat{u}|_{3,\widehat{K}}.$$

With the inverse map of F, we obtain (12) by some easy calculation. □

Now, let's consider the interpolation error expansion of $B(u - \pi_h u, v_h)$.

Theorem 3.1. *Let $\pi_h u$ be the piecewise linear interpolant of u. If $u \in H^4(\Omega)$, we have the following expansion*

$$\begin{aligned} \int_\Omega \nabla(u - \pi_h u)\cdot \mathcal{A}\nabla v_h dxdy &= -\frac{h^2}{12} W\big(u, v_h, \mathcal{T}_h\big) + \frac{h^2}{12} K\big(u, v_h, \mathcal{T}_h\big) \\ & \quad + O(h^2)\|u\|_3\|v_h\|_1, \quad \forall v_h \in \mathcal{V}_h, \end{aligned} \tag{13}$$

where

$$\begin{aligned} & W(u, v_h, \mathcal{T}_h) \\ = & \sum_{K\in\mathcal{T}_h}\sum_{i=1}^3 l_i^3 \frac{l_{i+1}(\mathbf{n}_i\cdot\mathbf{n}_{i+1})}{2\alpha}\Big((A_i\partial_i^2 u v_h)(Z_{i+2}) - (A_i\partial_i^2 u v_h)(Z_{i+1})\Big) \\ & - \sum_{K\in\mathcal{T}_h}\sum_{i=1}^3 \frac{l_{i+2}^4}{2\alpha}\Big((A_{i+2}\partial_{i+2}^2 u v_h)(Z_{i+2}) - (A_{i+2}\partial_{i+2}^2 u v_h)(Z_{i+1})\Big), \end{aligned} \tag{14}$$

$$\begin{aligned} K(u, v_h, \mathcal{T}_h) &= \sum_{K\in\mathcal{T}_h}\sum_{i=1}^3 l_i^3 \frac{l_{i+1}(\mathbf{n}_i\cdot\mathbf{n}_{i+1})}{2\alpha}\int_{e_i}\partial_i(A_i\partial_i^2 u)v_h ds \\ & \quad - \sum_{K\in\mathcal{T}_h}\sum_{i=1}^3 \frac{l_i^4}{2\alpha}\int_{e_{i+1}}\partial_{i+1}(A_i\partial_i^2 u)v_h ds, \end{aligned} \tag{15}$$

and

$$A_i = \mathbf{n}_i \cdot \mathcal{A} \cdot \mathbf{n}_i. \tag{16}$$

Proof. We need the following inequality for the finite element space $\mathcal{V}_h$ and $K \in \mathcal{T}_h$:

$$|v_h|_{1,\partial K} \leq Ch^{-1/2}\|v_h\|_{1,K}, \quad \forall v_h \in \mathcal{V}_h, \tag{17}$$

and the trace inequality

$$\|u\|_{0,\partial K} \leq Ch^{-1/2}\|u\|_{0,K} + Ch^{1/2}\|u\|_{1,K}, \quad \text{for} \ \ u \in H^1(K). \tag{18}$$

With Green formula we have

$$\begin{aligned}
\int_\Omega \nabla(u-\pi_h u)\cdot\mathcal{A}\nabla v_h dxdy &= \sum_{K\in\mathcal{T}_h}\sum_{i=1}^3\int_{e_i}(u-\pi_h u)\mathbf{n}_i\cdot(\mathcal{A}\nabla v_h)ds\\
&\quad -\sum_{K\in\mathcal{T}_h}\int_K(u-\pi_h u)\nabla\cdot(\mathcal{A}\nabla v_h)dxdy\\
&:= \Pi + O(h^2)\|u\|_2\|v\|_1.
\end{aligned} \tag{19}$$

Let's compute Π as follows

$$\begin{aligned}
\Pi &= \sum_{K\in\mathcal{T}_h}\sum_{i=1}^3\int_{e_i}(u-\pi_h u)\mathbf{n}_i\cdot\mathcal{A}\cdot\mathbf{n}_i\frac{\partial v_h}{\partial\mathbf{n}_i}ds\\
&\quad +\sum_{K\in\mathcal{T}_h}\sum_{i=1}^3\int_{e_i}(u-\pi_h u)\mathbf{n}_i\cdot\mathcal{A}\cdot\mathbf{t}_i\frac{\partial v_h}{\partial\mathbf{t}_i}ds\\
&= \sum_{K\in\mathcal{T}_h}\sum_{i=1}^3\int_{e_i}(u-\pi_h u)A_i\frac{\partial v_h}{\partial\mathbf{n}_i}ds\\
&= -\sum_{K\in\mathcal{T}_h}\sum_{i=1}^3\frac{h_i^2}{12}\int_{e_i}\partial_i^2 uA_i\frac{\partial v_h}{\partial\mathbf{n}_i}ds + O(h^2)\|u\|_3\|v_h\|_1\\
&= -\frac{h^2}{12}\sum_{K\in\mathcal{T}_h}\sum_{i=1}^3\int_{e_i}l_i^2A_i\partial_i^2 u\frac{\partial v_h}{\partial\mathbf{n}_i}ds + O(h^2)\|u\|_3\|v_h\|_1\\
&:= -\frac{h^2}{12}\mathbf{I} + O(h^2)\|u\|_3\|v_h\|_1.
\end{aligned} \tag{20}$$

With Green formula and Lemma 2.2, we have

$$\mathbf{I} = \sum_{K\in\mathcal{T}_h}\sum_{i=1}^3 l_i^2\frac{l_il_{i+1}}{2\alpha}\int_{e_i}A_i\partial_i^2 u\big((\mathbf{n}_i\cdot\mathbf{n}_{i+1})\partial_i v_h - \partial_{i+1}v_h\big)ds. \tag{21}$$

By Lemma 2.3, the following integral formula holds

$$\int_{e_i}A_i\partial_i^2 u\partial_{i+1}v_h ds = \frac{l_i}{l_{i+1}}\int_{e_{i+1}}A_i\partial_i^2 u\partial_{i+1}v_h ds + \frac{l_il_{i+2}}{2\alpha}\int_K\partial_{i+2}(A_i\partial_i^2 u)\partial_{i+1}v_h dxdy.$$

From (21) and integration by parts, we can obtain

$$\begin{aligned}
\mathbf{I} &= \sum_{K\in\mathcal{T}_h}\sum_{i=1}^3\Big[l_i^3\frac{l_{i+1}(\mathbf{n}_i\cdot\mathbf{n}_{i+1})}{2\alpha}\int_{e_i}A_i\partial_i^2 u\partial_i v_h ds - \frac{l_{i+2}^4}{2\alpha}\int_{e_i}A_{i+2}\partial_{i+2}^2 u\partial_i v_h ds\Big]\\
&\quad -\sum_{K\in\mathcal{T}_h}\sum_{i=1}^3 l_{i-1}^3\frac{l_1l_2l_3}{(2\alpha)^2}\int_K\partial_{i+1}(A_{i+2}\partial_{i+2}^2 u)\partial_i v_h dxdy
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{K\in\mathcal{T}_h}\sum_{i=1}^{3} l_i^3 \frac{l_{i+1}(\mathbf{n}_i\cdot\mathbf{n}_{i+1})}{2\alpha}\Big((A_i\partial_i^2 uv_h)(Z_{i+2}) - (A_i\partial_i^2 uv_h)(Z_{i+1}) \\
&\qquad - \int_{e_i} \partial_i(A_i\partial_i^2 u)v_h ds\Big) \\
&\quad - \sum_{k\in\mathcal{T}_h}\sum_{i=1}^{3} \frac{l_{i+2}^4}{2\alpha}\Big((A_{i+2}\partial_{i+2}^2 uv_h)(Z_{i+2}) - (A_{i+2}\partial_{i+2}^2 uv_h)(Z_{i+1}) \\
&\qquad - \int_{e_i} \partial_i(A_{i+2}\partial_{i+2}^2 u)v_h ds\Big) + O(h^2)\|u\|_3\|v_h\|_1. \qquad (22)
\end{aligned}
$$

Combining (20) and (22), we can get the desired result (13). $\square$

Let $\mathcal{N}_h$ denote the set of vertices of the triangulation $\mathcal{T}_h$ and $\omega_j^h := \{K : K \in \mathcal{T}_h \text{ and } Z_j \in K\}$ denote the patch around the node Z_j (see Figure 2). From (14) and assume the local number of Z_j in each triangle $K \in \omega_j$ is i (see Figure 2), we have

$$
\begin{aligned}
W(u,v,\mathcal{T}_h) &= \sum_{K\in\mathcal{T}_h}\sum_{i=1}^{3}\Big(l_i^3\frac{l_{i+1}(\mathbf{n}_i\cdot\mathbf{n}_{i+1})}{2\alpha}(A_i\partial_i^2 uv)(Z_{i+2}) \\
&\qquad -l_{i+1}^3\frac{l_{i+2}(\mathbf{n}_{i+1}\cdot\mathbf{n}_{i+2})}{2\alpha}(A_{i+1}\partial_{i+1}^2 uv)(Z_{i+2})\Big) \\
&\quad - \sum_{K\in\mathcal{T}_h}\sum_{i=1}^{3}\Big(\frac{l_{i+2}^4}{2\alpha}(A_{i+2}\partial_{i+2}^2 uv)(Z_{i+2}) - \frac{l_i^4}{2\alpha}(A_i\partial_i^2 uv)(Z_{i+2})\Big) \\
&= \sum_{K\in\mathcal{T}_h}\sum_{i=1}^{3}\Big(\frac{l_{i+1}^3 l_{i+2}(\mathbf{n}_{i+1}\cdot\mathbf{n}_{i+2})}{2\alpha}(A_{i+1}\partial_{i+1}^2 u)(Z_i) \\
&\qquad - \frac{l_{i+2}^3 l_i(\mathbf{n}_{i+2}\cdot\mathbf{n}_i)}{2\alpha}(A_{i+2}\partial_{i+2} u)(Z_i)\Big)v(Z_i) \\
&\quad - \sum_{K\in\mathcal{T}_h}\sum_{i=1}^{3}\Big(\frac{l_i^4}{2\alpha}(A_i\partial_i^2 u)(Z_i) - \frac{l_{i+1}^4}{2\alpha}(A_{i+1}\partial_{i+1}^2 u)(Z_i)\Big)v(Z_i) \\
&= \sum_{Z_j\in\mathcal{N}_h}\Big[\sum_{K\in\omega_j^h}\Big(\frac{l_{i+1}^3 l_{i+2}(\mathbf{n}_{i+1}\cdot\mathbf{n}_{i+2})}{2\alpha}(A_{i+1}\partial_{i+1}^2 u)(Z_j) \\
&\qquad - \frac{l_{i+2}^3 l_i(\mathbf{n}_{i+2}\cdot\mathbf{n}_i)}{2\alpha}(A_{i+2}\partial_{i+2} u)(Z_j)\Big)\Big]v(Z_j) \\
&\quad - \sum_{Z_j\in\mathcal{N}_h}\Big[\sum_{K\in\omega_j^h}\Big(\frac{l_i^4}{2\alpha}(A_i\partial_i^2 u)(Z_j) - \frac{l_{i+1}^4}{2\alpha}(A_{i+1}\partial_{i+1}^2 u)(Z_j)\Big)\Big]v(Z_j). \qquad (23)
\end{aligned}
$$

Let's define $\mathbf{N}_i = (\cos^2\theta_i, 2\sin\theta_i\cos\theta_i, \sin^2\theta_i)$. Assume $\mathcal{T}_h$ has N nodes and let's define the matrix $\mathbf{Mes}(\mathcal{T}_h) \in \mathcal{R}^{N\times 3}$ and $\mathbf{d}_u \in \mathcal{R}^{N\times 3}$ as follows

$$
\begin{aligned}
\mathbf{Mes}(\mathcal{T}_h)(j,:) &= \sum_{K\in\omega_j^h}\Big(l_{i+1}^3\frac{l_{i+2}(\mathbf{n}_{i+1}\cdot\mathbf{n}_{i+2})}{2\alpha}A_{i+1}(Z_j)\mathbf{N}_{i+1} \\
&\qquad -l_{i+2}^3\frac{l_i(\mathbf{n}_{i+2}\cdot\mathbf{n}_i)}{2\alpha}A_{i+2}(Z_j)\mathbf{N}_{i+2}\Big)
\end{aligned}
$$

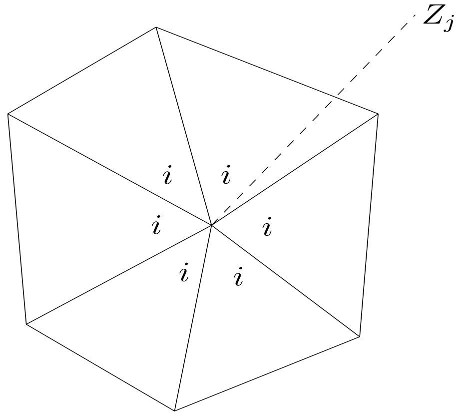


FIGURE 2. The patch ω_j and the local numbers of Z_j in each $K \in \omega_j$

$$-\sum_{K\in\omega_j^h}\Big(\frac{l_i^4}{2\alpha}A_i(Z_j)\mathbf{N}_i - \frac{l_{i+1}^4}{2\alpha}A_{i+1}(Z_j)\mathbf{N}_{i+1}\Big), \tag{24}$$

and

$$\mathbf{d}_u(j,:) \;=\; \Big(\partial_x^2 u(Z_j), \partial_x\partial_y u(Z_j), \partial_y^2 u(Z_j)\Big), \tag{25}$$

where $\mathbf{Mes}(\mathcal{T}_h)(j,:)$ and $\mathbf{d}_u(j,:)$ denote the j-th row of the corresponding matrix.

Corollary 3.1. *For $W(u, v_h, \mathcal{T}_h)$, we have*

$$\big|W(u,v_h,\mathcal{T}_h)\big| \;=\; \Big|\sum_{Z_j\in\mathcal{N}_h}\mathbf{Mes}(\mathcal{T}_h)(j,:)\cdot\mathbf{d}_u(j,:)v_h(Z_j)\Big|, \quad \forall v_h \in \mathcal{V}_h, \tag{26}$$

where the matrix $\mathbf{Mes}(\mathcal{T}_h)$ is defined as (24). Then the following estimate holds

$$\big|W(u,v_h,\mathcal{T}_h)\big| \le Ch^{-1}\|\mathbf{Mes}(\mathcal{T}_h)\|_F\|u\|_{2,\infty}\|v_h\|_0, \quad \forall v_h\in\mathcal{V}_h, \tag{27}$$

where $\|\cdot\|_F$ denote the Frobenius matrix norm. Furthermore, we also have the following estimate for $\|\mathbf{Mes}(\mathcal{T}_h)\|_F$

$$\|\mathbf{Mes}(\mathcal{T}_h)\|_F \le Ch^{-1}. \tag{28}$$

Proof. From (23) and (24), we can easily obtain (26). By using the following relations

$$ch^{-1}\|v_h\|_0 \le \Big(\sum_{Z_j\in\mathcal{N}_h} v_h(Z_j)^2\Big)^{\frac{1}{2}} \le Ch^{-1}\|v_h\|_0, \quad \forall v_h\in\mathcal{V}_h, \tag{29}$$

and Höider inequality, we can obtain (27). The estimate (28) can be directly obtained from the quasi-uniform condition of $\mathcal{T}_h$. $\square$

4. Asymptotic error expansion

In this section, we give the asymptotic error expansion of the finite element approximation by using the suitable interpolation postprocessing method.

In order to obtain asymptotic error expansion, we need to construct the following auxiliary finite element equation:

Find $\psi \in \mathcal{V}$ such that

$$B(\psi, v) = g(v), \quad \forall v \in \mathcal{V}, \tag{30}$$

where

$$g(v) := -\frac{1}{12}W\big(u,\mathcal{P}_{h/2}v,\mathcal{T}_h\big)+\frac{1}{12}K\big(u,\mathcal{P}_{h/2}v,\mathcal{T}_h\big), \tag{31}$$

with $\mathcal{P}_{h/2}$ is the L^2-projection operator with respect to $\mathcal{V}_{h/2}$ defined on the mesh $\mathcal{T}_{h/2}$

$$\big(\mathcal{P}_{h/2}\phi, v_{h/2}\big) = (\phi, v_{h/2}), \qquad \forall v_{h/2}\in\mathcal{V}_{h/2}. \tag{32}$$

First we have the following estimate

$$\sup_{v\in\mathcal{V}}\frac{g(v)}{\|v\|_0}\le Ch^{-1}\big(\|\mathbf{Mes}(\mathcal{T}_h)\|_F+1\big)\big(\|u\|_4+\|u\|_{2,\infty}\big). \tag{33}$$

Then from the regularity of the elliptic problem, the following estimate holds

$$\|\psi\|_2\le C\sup_{v\in\mathcal{V}}\frac{g(v)}{\|v\|_0}\le Ch^{-1}\big(\|\mathbf{Mes}(\mathcal{T}_h)\|_F+1\big)\big(\|u\|_4+\|u\|_{2,\infty}\big). \tag{34}$$

Remark 4.1. *Here, we use the projection operator $\mathcal{P}_{h/2}$ to overcome the difficulty $\mathcal{V}\not\subset L^\infty(\Omega)$.*

Lemma 4.1. *Assume $u\in H^4(\Omega)$, $\mathcal{L}_h u$ and $\pi_h u$ are the finite element approximation and interplant corresponding to $\mathcal{T}_h$. Then we have*

$$\big\|\mathcal{L}_h u-\pi_h u-h^2\mathcal{L}_h\psi\big\|_1\le Ch^2\|u\|_3. \tag{35}$$

Proof. Let $\eta_h := u_h-\pi_h u-h^2\mathcal{L}_h\psi$. By the coercivity of the bilinear form $B(\cdot,\cdot)$, (6), (13), (30) and the property $\mathcal{V}_h\subset\mathcal{V}_{h/2}$, we have

$$\begin{aligned}C_0\|\eta_h\|_1^2 &\le B(\eta_h,\eta_h)=B(u_h-\pi_h u-h^2\mathcal{L}_h\psi,\eta_h)\\ &= B(u-\pi_h u-h^2\mathcal{L}_h\psi,\eta_h)=B(u-\pi_h u,\eta_h)-h^2B(\mathcal{L}_h\psi,\eta_h)\\ &= B(u-\pi_h u,\eta_h)-h^2B(\psi,\eta_h)\le\int_\Omega\rho(u-\pi_h u)\eta_h\,dxdy+Ch^2\|u\|_3\|\eta_h\|_1\\ &\le Ch^2\|u\|_3\|\eta_h\|_1.\end{aligned}$$

This is the desired result (35). ☐

In order to do the extrapolation, we always need to obtain the asymptotic expansion of the finite element approximation. For this aim, it is needed to do the higher order interpolation postprocessing ([9, 5]). For the general meshes, since they are not obtained by the regular refinement from the structured meshes, the reasonable postprocessing method is the type of recovery method for linear element ([17, 18]). Let us define $\mathcal{Q}_h$ as the recovery operator $\mathcal{Q}_h:\mathcal{V}_h\longmapsto\mathcal{V}_h\times\mathcal{V}_h$ where $\mathcal{Q}_h\mathcal{L}_h u$ is some type of approximation to the gradient of the exact solution ∇u by the finite element approximation $\mathcal{L}_h u$. Here we assume that the operator $\mathcal{Q}_h$ has the following properties

$$\|\mathcal{Q}_h v_h\|_0 \le C|v_h|_1, \qquad \forall v_h\in\mathcal{V}_h, \tag{36}$$

$$\|\mathcal{Q}_h u-\nabla u\|_0 \le Ch^2\|u\|_3, \tag{37}$$

$$\mathcal{Q}_h u = \mathcal{Q}_h(\pi_h u). \tag{38}$$

Theorem 4.1. *Assume $\mathcal{L}_h u$ and $\pi_h u$ are the finite element approximation and interplant corresponding to $\mathcal{T}_h$. Then we have*

$$\big\|\mathcal{Q}_h\mathcal{L}_h u-\nabla u-h^2\nabla\psi\big\|_0\le Ch^2\big(1+\|\mathbf{Mes}(\mathcal{T}_h)\|_F\big)\big(\|u\|_4+\|u\|_{2,\infty}\big). \tag{39}$$

Proof. By (34), (35) and the properties (36)-(38) of the interpolation postprocessing operator $\mathcal{Q}_h$, we have

$$\begin{aligned}\big\|\mathcal{Q}_h\mathcal{L}_hu-\nabla u-h^2\nabla\psi\big\|_0 \leq&\|\mathcal{Q}_h(\mathcal{L}_hu-\pi_hu-h^2\mathcal{L}_h\psi)\|_0+\big\|\mathcal{Q}_h\pi_hu-\nabla u\big\|_0\\&+h^2\big\|\mathcal{Q}_h(\mathcal{L}_h\psi-\pi_h\psi)\big\|_0+h^2\big\|\mathcal{Q}_h\pi_h\psi-\nabla\psi\big\|_0\\ \leq& Ch^2\big(\|u\|_4+\|u\|_{2,\infty}\big)+Ch^3\|\psi\|_2\\ \leq& Ch^2\big(1+\|\mathbf{Mes}(\mathcal{T}_h)\|_F\big)\big(\|u\|_4+\|u\|_{2,\infty}\big).\end{aligned}$$

Thus we complete the proof. □

In order to use the extrapolation method, we need to refine the mesh $\mathcal{T}_h$ in the regular way (please read Section 2).

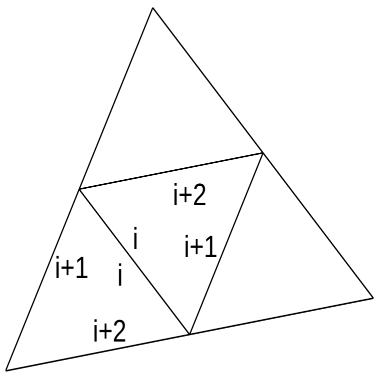


FIGURE 3. The elements of $\mathcal{T}_{h/2}$ in an element $K\in\mathcal{T}_h$

For the relation between $\mathcal{T}_h$ and $\mathcal{T}_{h/2}$, we have the following lemma.

Lemma 4.2. *If $\mathcal{T}_{h/2}$ is obtained from $\mathcal{T}_h$ by the regular refinement, we have*

$$\|\mathbf{Mes}(\mathcal{T}_h)\|_F = \|\mathbf{Mes}(\mathcal{T}_{h/2})\|_F, \tag{40}$$

$$W(u,v_{h/2},\mathcal{T}_{h/2}) = W(u,v_{h/2},\mathcal{T}_h), \tag{41}$$

$$K(u,v_{h/2},\mathcal{T}_{h/2}) = K(u,v_{h/2},\mathcal{T}_h). \tag{42}$$

Proof. For every new nodes Z_j produced by the regular refinement, it is easy to check that

$$\mathbf{Mes}(\mathcal{T}_{h/2})(j,:)=(0,0,0). \tag{43}$$

Thus, we can obtain (40) and (41) can also be derived from (43). Similarly, we can obtain (42) by the property of the new edges by regular refinement (Figure 3). □

Similarly to Theorem 4.1, based on Lemma 4.2, we can obtain the following asymptotic error expansion of the finite element approximation $\mathcal{L}_{h/2}u$.

Lemma 4.3. *Assume $u\in H^4(\Omega)$, $\mathcal{L}_{h/2}u$ and $\pi_{h/2}u$ are the finite element approximation and interplant corresponding to $\mathcal{T}_{h/2}$. Then we have*

$$\Big\|\mathcal{L}_{h/2}u-\pi_{h/2}u-\Big(\frac{h}{2}\Big)^2\mathcal{L}_{h/2}\psi\Big\|_1\leq Ch^2\|u\|_3, \tag{44}$$

$$\Big\|\mathcal{Q}_{h/2}\mathcal{L}_{h/2}u-\nabla u-\Big(\frac{h}{2}\Big)^2\nabla\psi\Big\|_0\leq Ch^2\big(1+\|\mathbf{Mes}(\mathcal{T}_h)\|_F\big)\big(\|u\|_4+\|u\|_{2,\infty}\big). \tag{45}$$

Proof. Let

$$\eta_{h/2} := \mathcal{L}_{h/2}u - \pi_{h/2}u - \Big(\frac{h}{2}\Big)^2\mathcal{L}_{h/2}\psi.$$

By the coercivity of the bilinear form $B(\cdot,\cdot)$, (6), (13) and (30), similarly to Theorem 4.1 we have

$$\begin{aligned}
C_0\|\eta_{h/2}\|_1^2 \le B(\eta_{h/2},\eta_{h/2}) &= B\big(\mathcal{L}_{h/2}u - \pi_{h/2}u - \Big(\frac{h}{2}\Big)^2\mathcal{L}_{h/2}\psi, \eta_{h/2}\big)\\
&= B\big(u - \pi_{h/2}u - \Big(\frac{h}{2}\Big)^2\mathcal{L}_{h/2}\psi, \eta_{h/2}\big)\\
&= B(u-\pi_{h/2}u, \eta_{h/2}) - \Big(\frac{h}{2}\Big)^2 B\big(\mathcal{L}_{h/2}\psi, \eta_{h/2}\big)\\
&\le Ch^2\|u\|_3\|\eta_{h/2}\|_1.
\end{aligned}$$

This is the desired result (44). Based on (44) and (40)-(42) and the properties (36)-(38) of the interpolation postprocessing operator $\mathcal{Q}_{h/2}$, we have

$$\begin{aligned}
&\Big\|\mathcal{Q}_{h/2}\mathcal{L}_{h/2}u - \nabla u - \Big(\frac{h}{2}\Big)^2\nabla\psi\Big\|_0\\
\le&\Big\|\mathcal{Q}_{h/2}(\mathcal{L}_{h/2}u - \pi_{h/2}u - \Big(\frac{h}{2}\Big)^2\mathcal{L}_{h/2}\psi)\Big\|_0 + \big\|\mathcal{Q}_{h/2}\pi_{h/2}u - \nabla u\big\|_0\\
&+\Big(\frac{h}{2}\Big)^2\Big\|\mathcal{Q}_{h/2}(\mathcal{L}_{h/2}\psi - \pi_{h/2}\psi)\Big\|_0 + \Big(\frac{h}{2}\Big)^2\Big\|\mathcal{Q}_{h/2}\pi_{h/2}\psi - \nabla\psi\Big\|_0\\
\le& Ch^2\|u\|_3 + Ch^3\|\psi\|_2\\
\le& Ch^2\big(1+\|\mathbf{Mes}(\mathcal{T}_{h/2})\|_F\big)\big(\|u\|_4+\|u\|_{2,\infty}\big)\\
=& Ch^2\big(1+\|\mathbf{Mes}(\mathcal{T}_{h})\|_F\big)\big(\|u\|_4+\|u\|_{2,\infty}\big).
\end{aligned}$$

This is the result (45) and we complete the proof. □

Based on the asymptotic error expansions (39) and (45), we can define the extrapolation scheme as

$$\mathcal{L}_h^{\text{extra}}u := \frac{4\mathcal{Q}_{h/2}\mathcal{L}_{h/2}u - \mathcal{Q}_h\mathcal{L}_h u}{3}. \tag{46}$$

Theorem 4.2. *We have the following error estimate for the extrapolation solution defined in (46)*

$$\big\|\mathcal{L}_h^{\text{extra}}u - \nabla u\big\|_0 \le Ch^2\big(1+\|\mathbf{Mes}(\mathcal{T}_h)\|_F\big)\big(\|u\|_4+\|u\|_{2,\infty}\big). \tag{47}$$

Proof. By Theorem 4.1 and Lemma 4.3, we have

$$\begin{aligned}
&\big\|\mathcal{L}_h^{\text{extra}}u - \nabla u\big\|_0 = \big\|4\mathcal{Q}_{h/2}\mathcal{L}_{h/2}u - \mathcal{Q}_h\mathcal{L}_h u - 3\nabla u\big\|_0\\
=&\Big\|4\Big(\mathcal{Q}_{h/2}\mathcal{L}_{h/2}u - \nabla u - \Big(\frac{h}{2}\Big)^2\nabla\psi\Big) - \big(\mathcal{Q}_h\mathcal{L}_h u - \nabla u - h^2\nabla\psi\big)\Big\|_0\\
\le&4\Big\|\mathcal{Q}_{h/2}\mathcal{L}_{h/2}u - \nabla u - \Big(\frac{h}{2}\Big)^2\nabla\psi\Big\|_0 + \Big\|\big(\mathcal{Q}_h\mathcal{L}_h u - \nabla u - h^2\nabla\psi\big)\Big\|_0\\
\le& Ch^2\big(1+\|\mathbf{Mes}(\mathcal{T}_h)\|_F\big)\big(\|u\|_4+\|u\|_{2,\infty}\big).
\end{aligned}\tag{48}$$

Thus we complete the proof. □

The estimate (47) shows that the effectiveness of extrapolation method depends on the estimate $\|\mathbf{Mes}(\mathcal{T}_h)\|_F$. So we call $\|\mathbf{Mes}(\mathcal{T}_h)\|_F$ as the extrapolation measurement of the mesh $\mathcal{T}_h$ ([14]).

From (28), (40) and Theorem 4.2, the extrapolation method can arrive $O(h^2)$ convergence rate if we refine the mesh in the regular way from any initial mesh $\mathcal{T}_{h_0}$. Especially, we have the following corollary.

Corollary 4.1. *Assume that $u \in H^4(\Omega)$, $\mathcal{T}_h$ is produced by refining the triangulation $\mathcal{T}_{h_0}$ in the regular way and $h_0 = O(1)$. Then we have the following extrapolation estimate*

$$\left\|\mathcal{L}_h^{\mathrm{extra}} u - \nabla u\right\|_0 \leq Ch^2\big(\|u\|_4 + \|u\|_{2,\infty}\big). \tag{49}$$

Proof. (49) can be obtained easily from (28), (40), (47) and $h_0 = O(1)$. $\square$

5. Numerical results

In this section, we show the effectiveness of the extrapolation method on general meshes by two numerical examples. First, we present an example to test the convergence order of the extrapolation on general meshes. In the second example, we will check the influence of the regularity of exact solutions on the extrapolation method.

5.1. Convergence order of extrapolation scheme.

In this subsection, we solve the following second order elliptic problem

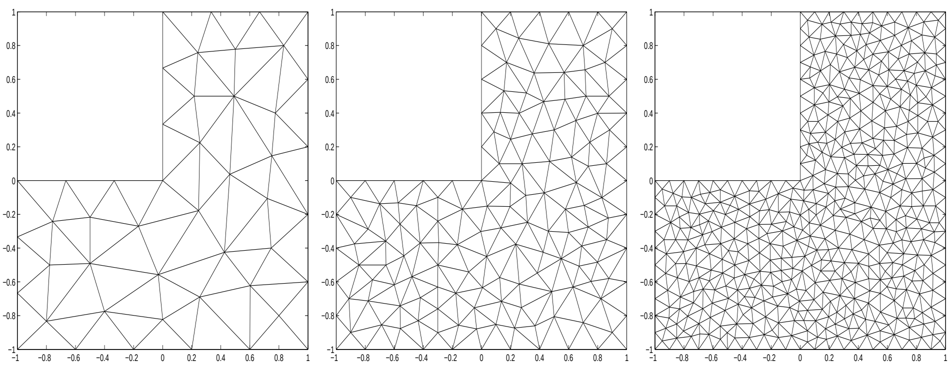
$$-\nabla(\mathcal{A}\nabla u) + \rho u = f, \quad \text{in } \Omega, \tag{50}$$

$$u = u_D, \quad \text{on } \partial\Omega, \tag{51}$$

where

$$\mathcal{A} = \begin{pmatrix} e^{x^2+1} & e^{xy} \\ e^{xy} & e^{y^2} \end{pmatrix}, \tag{52}$$

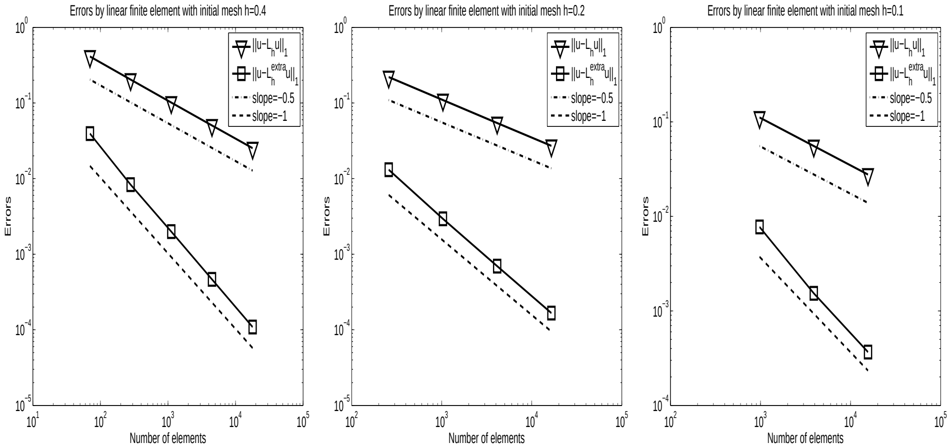
$\rho = x^2 + y^2$ and $\Omega = [-1,1]\times[-1,1]\backslash[-1,0]\times[0,1]$. The function f and boundary condition u_D are chosen such that the exact solution is $u = e^{xy}$. Figure 4 shows the three initial meshes with size $h = 0.4$, $h = 0.2$ and $h = 0.1$ for our numerical tests. The corresponding numerical results are presented in Figure 5.



$h_0 = 0.4$ $h_0 = 0.2$ $h_0 = 0.1$

FIGURE 4. The initial meshes for the L shape domain

From numerical results showed in Figure 5, we find that the extrapolation method can improve the convergence order form first to second on the general initial meshes. This confirms the theoretical result in Theorem 4.2.



$h_0 = 0.4$ $h_0 = 0.2$ $h_0 = 0.1$

FIGURE 5. The numerical results for equation (50)-(51)

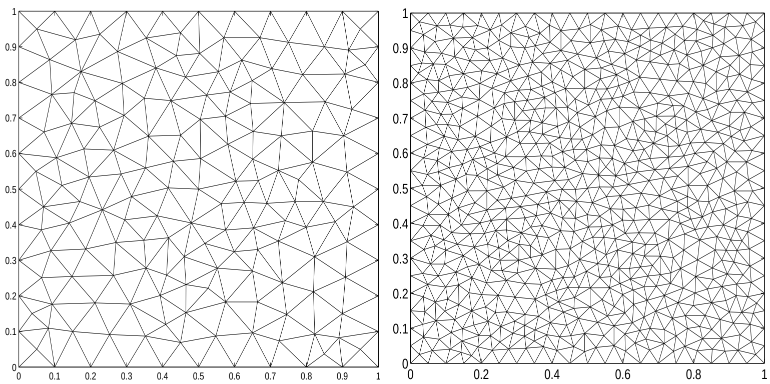
5.2. Regularity check of extrapolation. In this subsection, we solve the following model problem

$$-\Delta u = f, \qquad \text{in } \Omega, \tag{53}$$

$$u = u_D, \quad \text{on } \partial\Omega, \tag{54}$$

where $\Omega = [0, 1] \times [0, 1]$. We chose different functions f and u_D such that the exact solutions are $u = (x^2 + y^2)^{4/3}$, $u = (x^2 + y^2)^{5/6}$ and $u = (x^2 + y^2)^{1/3}$, respectively. It is easy to know the three exact functions belong to $H^{3+1/6-\epsilon}(\Omega)$, $H^{2+1/6-\epsilon}(\Omega)$ and $H^{1+1/3-\epsilon}(\Omega)$.

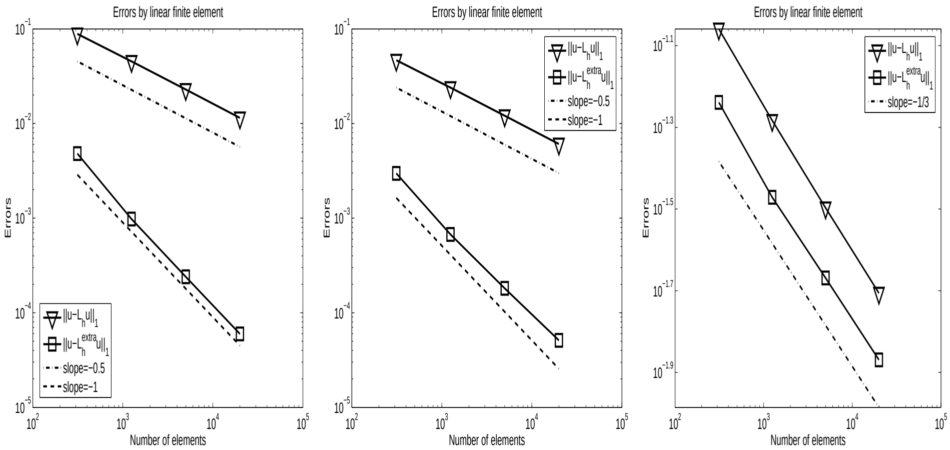
We compute these three examples with linear finite element and extrapolation method to check the influence of the regularity on the extrapolation. Here, we adopt the initial meshes showed in Figure 6 to produce two mesh sequences by the regular refinement.



$h_0 = 0.1$ $h_0 = 0.05$

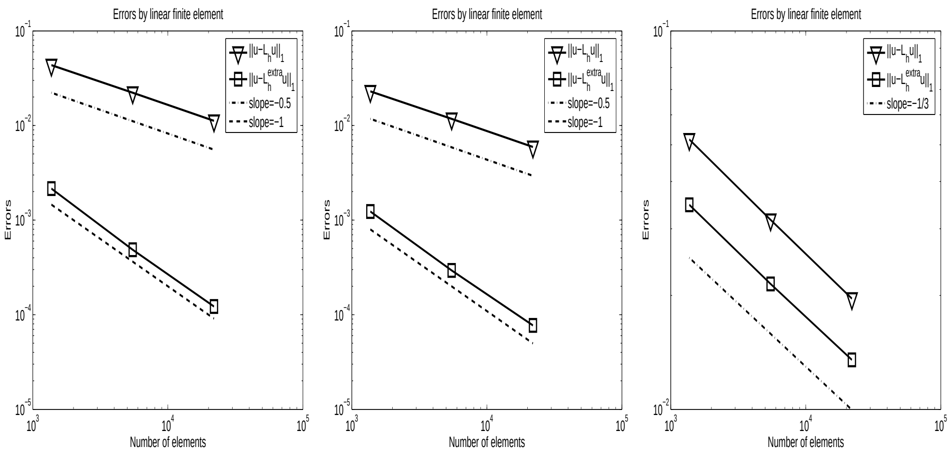
FIGURE 6. The initial meshes for the square domain

Figures 7 and 8 show the errors of the extrapolation method for the initial meshes $h = 0.1$ and $h = 0.05$, respectively.



$u = (x^2 + y^2)^{4/3}$ $u = (x^2 + y^2)^{5/6}$ $u = (x^2 + y^2)^{1/3}$

FIGURE 7. Numerical results for equation (53)-(54) with initial mesh size $h_0 = 0.1$



$u = (x^2 + y^2)^{4/3}$ $u = (x^2 + y^2)^{5/6}$ $u = (x^2 + y^2)^{1/3}$

FIGURE 8. Numerical results for equation (53)-(54) with initial mesh size $h_0 = 0.1$

Figures 7 and 8 shows that even when the exact solution belongs to $H^{2+1/6}(\Omega)$, the extrapolation method can improve the convergence order obviously. But when the exact solution $u \in H^{1+1/3}(\Omega)$, extrapolation scheme can not improve the convergence order.

6. Concluding remarks

In this paper, we analyze the extrapolation method for the second order elliptic problems by linear finite element on general meshes. Based on our analysis, we find the extrapolation can improve the convergence order on the meshes produced by regular refinement on general initial meshes. This means the extrapolation method can be applied on the general domains by the totally automatical triangulation and dose not need to construct the meshes artificially. Based on the definition $\|\mathbf{Mes}(\mathcal{T}_h)\|_F$, extrapolation method can improve the convergence order when it is applied on the classical so-called superconvergence meshes (structured meshes) ([1, 2, 3, 5, 7, 9, 10, 12, 13, 15, 16, 17, 18]).

The idea presented in this paper can be extended to other types of problems and this may be our future work.

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