Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2011.m1039

Vol. **4**, No. 4, pp. 478-488
November 2011

Orthogonal Polynomials with Respect to Modified Jacobi Weight and Corresponding Quadrature Rules of Gaussian Type

Marija P. Stanić[1,*] and Aleksandar S. Cvetković[2]

[1] *Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac, Radoja Domanovića 12, 34000 Kragujevac, Serbia.*
[2] *Department of Mathematics and Informatics, Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Belgrade, Serbia.*

Received 4 November 2010; Accepted (in revised version) 7 February 2011

Available online 7 November 2011

Abstract. In this paper we consider polynomials orthogonal with respect to the linear functional $\mathcal{L} : \mathcal{P} \to \mathbb{C}$, defined on the space of all algebraic polynomials $\mathcal{P}$ by

$$\mathcal{L}[p] = \int_{-1}^{1} p(x)(1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}\exp(\mathrm{i}\zeta x)\mathrm{d}x,$$

where $\alpha, \beta > -1/2$ are real numbers such that $\ell = |\beta - \alpha|$ is a positive integer, and $\zeta \in \mathbb{R}\backslash\{0\}$. We prove the existence of such orthogonal polynomials for some pairs of α and ζ and for all nonnegative integers ℓ. For such orthogonal polynomials we derive three-term recurrence relations and also some differential-difference relations. For such orthogonal polynomials the corresponding quadrature rules of Gaussian type are considered. Also, some numerical examples are included.

AMS subject classifications: 33C47, 41A55, 65D30

Key words: Orthogonal polynomials, modified Jacobi weight function, recurrence relation, Gaussian quadrature rule.

1. Introduction

In this paper we continue investigation on orthogonality with respect to the exponential modification of classical weight functions, studied in [5–8]. Let us suppose that $\alpha, \beta > -1/2$ are real numbers such that $\ell = |\beta - \alpha|$ is a positive integer, and $\zeta \in \mathbb{R}\backslash\{0\}$. We are concerned with the following measure

$$\mathrm{d}\mu(x) = (1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}\exp(\mathrm{i}\zeta x)\chi_{[-1,1]}(x)\,\mathrm{d}x \tag{1.1}$$

*Corresponding author. *Email addresses:* `stanicm@kg.ac.rs` (M. P. Stanić), `acvetkovic@mas.bg.ac.rs` (A. S. Cvetković)

supported on the interval $[-1,1]$. We investigate the question connected with the existence of a sequence of orthogonal polynomials $\{\pi_n\}_{n\in\mathbb{N}_0}$ with respect to the linear moment functional $\mathcal{L}:\mathcal{P}\to\mathbb{C}$, defined on the space of all algebraic polynomials $\mathcal{P}$ by

$$\mathcal{L}[p]=\int_{-1}^{1} p(x)\,\mathrm{d}\mu(x)=\int_{-1}^{1} p(x)\,(1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}\exp(\mathrm{i}\zeta x)\,\mathrm{d}x. \tag{1.2}$$

The corresponding moments are $\mu_k=\mathcal{L}[x^k]$, $k\in\mathbb{N}_0$.

This paper is organized as follows. In Section 2 the existence of orthogonal polynomials for some parameters α and ζ and for all positive integers $\ell=|\beta-\alpha|$ is proved. Section 3 is devoted to three-term recurrence relations as well as to some differential-difference relations. Finally, in Section 4 the corresponding quadrature rules of Gaussian type are considered. Such quadrature rules are suitable for computation of integrals of highly oscillatory functions of the form $\int_{-1}^{1} f(x)(1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}\mathrm{e}^{\mathrm{i}\zeta x}\,\mathrm{d}x$. Notice that such kind of integrals appears in many branches of applied and computational science, e.g., for determining of the retarded potentials of electromagnetic field of a linear wire antenna (see [9]).

2. Existence of orthogonal polynomials

The measure (1.1) can be written in the following form

$$\mathrm{d}\mu(x)=\begin{cases}(1+x)^{\ell}(1-x^2)^{\alpha-1/2}\exp(\mathrm{i}\zeta x)\chi_{[-1,1]}(x)\,\mathrm{d}x, & \beta>\alpha,\\ (1-x)^{\ell}(1-x^2)^{\beta-1/2}\exp(\mathrm{i}\zeta x)\chi_{[-1,1]}(x)\,\mathrm{d}x, & \alpha>\beta.\end{cases}$$

Therefore, in the sequel we consider the measures

$$\mathrm{d}\mu^{\pm}(x)=(1\pm x)^{\ell}(1-x^2)^{\alpha-1/2}\exp(\mathrm{i}\zeta x)\chi_{[-1,1]}(x)\,\mathrm{d}x,$$

where $\alpha>-1/2$ and ℓ is a positive integer, i.e., we consider the existence of polynomials orthogonal with respect to the linear functionals

$$\mathcal{L}^{\pm,\zeta,\alpha,\ell}(p):=\mathcal{L}^{\pm}(p)=\int_{-1}^{1} p\,\mathrm{d}\mu^{\pm},\qquad p\in\mathcal{P}. \tag{2.1}$$

The moments

$$\mu_k^{\pm}=\int_{-1}^{1} x^k(1\pm x)^{\ell}(1-x^2)^{\alpha-1/2}\exp(\mathrm{i}\zeta x)\,\mathrm{d}x \tag{2.2}$$

can be expressed in terms of Bessel functions J_ν of the order ν (see [10, p. 40]). We restrict our attention only to the case $\zeta>0$, since the corresponding results for $\zeta<0$ can be obtained by a simple conjugation, because $\mu_k^{\pm}(-\zeta)=\overline{\mu_k^{\pm}(\zeta)}$, $k\in\mathbb{N}_0$.

Theorem 2.1. *The moments $\mu_k^{\pm}$, $k \in \mathbb{N}_0$, can be expressed in the form*

$$\mu_k^{\pm} = \frac{A}{(\mathrm{i}\zeta)^{k+\ell}} \sum_{j=0}^{\ell} \binom{\ell}{j} (\pm 1)^j (\mathrm{i}\zeta)^{\ell-j} \left(P_{k+j}^{\alpha}(\zeta) J_{\alpha}(\zeta) + Q_{k+j}^{\alpha}(\zeta) J_{\alpha-1}(\zeta) \right), \tag{2.3}$$

where $A = (2/\zeta)^{\alpha} \sqrt{\pi}\, \Gamma(\alpha + 1/2)$, and P_k^{α} and Q_k^{α} are polynomials in ζ, which satisfy the following four-term recurrence relation

$$y_{k+2} = -(k + 2\alpha + 1) y_{k+1} - \zeta^2 y_k - k\zeta^2 y_{k-1},$$

with the initial conditions $P_0^{\alpha}(\zeta) = 1$, $P_1^{\alpha}(\zeta) = -2\alpha$, $P_2^{\alpha}(\zeta) = 2\alpha(2\alpha+1) - \zeta^2$ and $Q_0^{\alpha}(\zeta) = 0$, $Q_1^{\alpha}(\zeta) = \zeta$, $Q_2^{\alpha}(\zeta) = -(2\alpha + 1)\zeta$, respectively.

Proof. In [8, Theorem 2.1] for $\alpha > -1/2$ and for all $k \in \mathbb{N}_0$ it was proved that

$$\widehat{\mu}_k = \int_{-1}^{1} x^k (1 - x^2)^{\alpha - 1/2} \exp(\mathrm{i}\zeta x)\, \mathrm{d}x = \frac{A}{(\mathrm{i}\zeta)^k} \left(P_k^{\alpha}(\zeta) J_{\alpha}(\zeta) + Q_k^{\alpha}(\zeta) J_{\alpha-1}(\zeta) \right). \tag{2.4}$$

By using the binomial formula, (2.2) and (2.4) we get

$$\begin{aligned}
\mu_k^{\pm} &= \int_{-1}^{1} x^k \sum_{j=0}^{\ell} \binom{\ell}{j} (\pm 1)^j x^j (1 - x^2)^{\alpha - 1/2} \exp(\mathrm{i}\zeta x)\, \mathrm{d}x \\
&= \sum_{j=0}^{\ell} \binom{\ell}{j} (\pm 1)^j \int_{-1}^{1} x^{k+j} (1 - x^2)^{\alpha - 1/2} \exp(\mathrm{i}\zeta x)\, \mathrm{d}x = \sum_{j=0}^{\ell} \binom{\ell}{j} (\pm 1)^j \widehat{\mu}_{k+j} \\
&= \sum_{j=0}^{\ell} \binom{\ell}{j} (\pm 1)^j \frac{A}{(\mathrm{i}\zeta)^{k+j}} \left(P_{k+j}^{\alpha}(\zeta) J_{\alpha}(\zeta) + Q_{k+j}^{\alpha}(\zeta) J_{\alpha-1}(\zeta) \right) \\
&= \frac{A}{(\mathrm{i}\zeta)^{k+\ell}} \sum_{j=0}^{\ell} \binom{\ell}{j} (\pm 1)^j (\mathrm{i}\zeta)^{\ell-j} \left(P_{k+j}^{\alpha}(\zeta) J_{\alpha}(\zeta) + Q_{k+j}^{\alpha}(\zeta) J_{\alpha-1}(\zeta) \right).
\end{aligned}$$

□

In numerical calculation we use formula (2.3) only for $k = 0, 1, 2$, and for $k \geq 3$ we use recurrence relations for the moments $\mu_k^{\pm}$ given in the following lemma.

Lemma 2.1. *Moments $\mu_k^{\pm}$, $k \in \mathbb{N}$, of the linear functional $\mathcal{L}^{\pm}$ satisfy the following recurrence relation*

$$\mu_{k+2}^{\pm} = -\frac{k + \ell + 2\alpha + 1}{\mathrm{i}\zeta} \mu_{k+1}^{\pm} + \left(1 \pm \frac{\ell}{\mathrm{i}\zeta} \right) \mu_k^{\pm} + \frac{k}{\mathrm{i}\zeta} \mu_{k-1}^{\pm}, \tag{2.5}$$

with $\mu_k^{\pm}$, $k = 0, 1, 2$, given by (2.3).

Proof. Starting with $\mu_k^{\pm} - \mu_{k+2}^{\pm} = \int_{-1}^{1} x^k (1 \pm x)^{\ell} (1 - x^2)^{\alpha + 1/2} \mathrm{e}^{\mathrm{i}\zeta x} \mathrm{d}x$, by using integration by parts we get

$$\mu_k^{\pm} - \mu_{k+2}^{\pm} = -\frac{1}{\mathrm{i}\zeta} \left(k\mu_{k-1}^{\pm} \pm \ell \mu_k^{\pm} - (k + \ell + 2\alpha + 1)\mu_{k+1}^{\pm} \right).$$

Now it is easy to get (2.5) from the previous equation. □

Now, we are ready to prove the existence theorem.

Theorem 2.2. *Let $\alpha > -1/2$ be a rational number, ℓ be a positive integer and ζ be a positive zero of the Bessel function $J_{\alpha-1}$. Then, the polynomials $\pi_n^{\pm}$ orthogonal with respect to the linear functionals $\mathcal{L}^{\pm}$, given by* (2.1)*, exist.*

Proof. By using a concept of orthogonality with respect to the linear functional $\mathcal{L}$ (cf. Chihara [1, pp. 5–17]), the necessary and sufficient conditions for the existence of the corresponding orthogonal polynomials π_n, $n \in \mathbb{N}_0$, can be expressed in terms of Hankel determinants as follows:

$$\Delta_n^{\pm} = \begin{vmatrix} \mu_0^{\pm} & \mu_1^{\pm} & \cdots & \mu_{n-1}^{\pm} \\ \mu_1^{\pm} & \mu_2^{\pm} & \cdots & \mu_n^{\pm} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}^{\pm} & \mu_n^{\pm} & \cdots & \mu_{2n-2}^{\pm} \end{vmatrix} \neq 0, \qquad n \in \mathbb{N}.$$

Since $J_{\alpha-1}(\zeta) = 0$, the moments (2.3) reduce to

$$\mu_k^{\pm} = \frac{A}{(\mathrm{i}\zeta)^{k+\ell}} \sum_{j=0}^{\ell} \binom{\ell}{j} (\pm 1)^j (\mathrm{i}\zeta)^{\ell-j} P_{k+j}^{\alpha}(\zeta) J_\alpha(\zeta), \qquad k \in \mathbb{N}_0.$$

From the Hankel determinant we can extract the factor $(AJ_\alpha(\zeta))^n/(\mathrm{i}\zeta)^{n(n+\ell-1)}$. Thus we have

$$\Delta_n^{\pm} = \frac{\big(AJ_\alpha(\zeta)\big)^n}{(\mathrm{i}\zeta)^{n(n+\ell-1)}} H_n^{\pm}, \tag{2.6}$$

where $H_n^{\pm}$ is the determinant of the matrix $\left[h_{ij}^{n,\pm}\right]_{i,j=1}^{n}$ with

$$h_{ij}^{n,\pm} = \sum_{\nu=0}^{\ell} \binom{\ell}{\nu} (\pm 1)^{\nu} (\mathrm{i}\zeta)^{\ell-\nu} P_{i+j-2+\nu}^{\alpha}(\zeta), \qquad i,j = 1, \cdots, n.$$

It is easy to see that all of the determinants $H_n^{\pm}$, $n \in \mathbb{N}$, are polynomials in ζ with rational coefficients because α is a rational number. Since positive non-trivial zeros of Bessel functions $J_{\alpha-1}$, with a rational index $\alpha - 1$, are transcendental numbers (see [11, p. 220]), they cannot be zeros of the polynomial with the rational coefficients unless polynomials are identically equal to zero. Therefore, we have to prove that all of the determinants $H_n^{\pm}$ are not identically equal to zero. For that purpose we prove that $H_n^{\pm}(0) \neq 0$ for all $n \in \mathbb{N}$. Since

$$H_n^{\pm}(0) = \begin{vmatrix} (\pm 1)^{\ell} P_{0+\ell}^{\alpha}(0) & (\pm 1)^{\ell} P_{1+\ell}^{\alpha}(0) & \cdots & (\pm 1)^{\ell} P_{n-1+\ell}^{\alpha}(0) \\ (\pm 1)^{\ell} P_{1+\ell}^{\alpha}(0) & (\pm 1)^{\ell} P_{2+\ell}^{\alpha}(0) & \cdots & (\pm 1)^{\ell} P_{n+\ell}^{\alpha}(0) \\ \vdots & \vdots & \ddots & \vdots \\ (\pm 1)^{\ell} P_{n-1+\ell}^{\alpha}(0) & (\pm 1)^{\ell} P_{n+\ell}^{\alpha}(0) & \cdots & (\pm 1)^{\ell} P_{2n-2+\ell}^{\alpha}(0) \end{vmatrix}$$

and $P_k^\alpha(0) = (-1)^k(2\alpha)_k$, $k \in \mathbb{N}_0$ (see [8, Lemma 2.1]), using the fact that $(2\alpha)_{k+\ell} = \Gamma(2\alpha+k+\ell)/\Gamma(2\alpha)$, we get

$$H_n^+(0)=\frac{1}{(\Gamma(2\alpha))^n}\begin{vmatrix} \Gamma(2\alpha+\ell) & \Gamma(2\alpha+1+\ell) & \cdots & \Gamma(2\alpha+n-1+\ell) \\ \Gamma(2\alpha+1+\ell) & \Gamma(2\alpha+2+\ell) & \cdots & \Gamma(2\alpha+n+\ell) \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma(2\alpha+n-1+\ell) & \Gamma(2\alpha+n+\ell) & \cdots & \Gamma(2\alpha+2n-2+\ell) \end{vmatrix}$$

and $H_n^-(0) = (-1)^{n\ell} H_n^+(0)$.

Since

$$\Gamma(2\alpha+k+\ell) = \int_0^{+\infty} x^{k+\ell+2\alpha-1} \mathrm{e}^{-x}\, \mathrm{d}x,$$

we see that the last determinant is the Hankel determinant for the generalized Laguerre measure $x^{\ell+2\alpha-1}\exp(-x)\chi_{[0,+\infty)}(x)\,\mathrm{d}x$, $\alpha \in \mathbb{Q}$, $\alpha > -1/2$, $\ell \in \mathbb{N}$. Because of the well-known fact about the existence of the sequence of orthogonal polynomials with respect to the generalized Laguerre measure, $H_n^\pm(0)$ must be different from zero.

Accordingly, when $\alpha > -1/2$ is a rational number and ζ is a positive zero of the Bessel function $J_{\alpha-1}$, the sequence of orthogonal polynomials with respect to the linear functional given by (2.1) exists. □

Remark 2.1. By definition of confluent hypergeometric function ${}_1F_1$ for $\mathrm{Re}(b) > \mathrm{Re}(a) > 0$ we have (see formula 9.211, 1. in [4])

$${}_1F_1(a,b;z) = \frac{2^{1-b}\mathrm{e}^{z/2}\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_{-1}^{1}(1-u)^{-a+b-1}(1+u)^{a-1}\mathrm{e}^{zu/2}\mathrm{d}u.$$

So, we get the following explicit formulae for the moments μ_k, $k \in \mathbb{N}_0$:

$$\begin{aligned} \mu_k &= \int_{-1}^{1}(1+x-1)^k(1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}\mathrm{e}^{\mathrm{i}\zeta x}\mathrm{d}x \\ &= \sum_{j=0}^{k}\binom{k}{j}(-1)^{k+j}\int_{-1}^{1}(1-x)^{\alpha-1/2}(1+x)^{\beta+j-1/2}\mathrm{e}^{\mathrm{i}\zeta x}\mathrm{d}x \\ &= \sum_{j=0}^{k}\binom{k}{j}(-1)^{k+j}\frac{2^{\alpha+\beta+j}\mathrm{e}^{-\mathrm{i}\zeta}\Gamma(\beta+j+1/2)\Gamma(\alpha+1/2)}{\Gamma(\alpha+\beta+j+1)} \\ &\qquad \times {}_1F_1(\beta+j+1/2, \alpha+\beta+j+1; 2\mathrm{i}\zeta). \end{aligned}$$

3. Recurrence and differential-difference relation

In this section we suppose that the parameters $\zeta > 0$, $\alpha, \beta > -1/2$, $|\beta-\alpha| = \ell \in \mathbb{N}$, are such that the sequence of (monic) orthogonal polynomials $\{\pi_n\}_{n\in\mathbb{N}_0}$ with respect to the moment functional (1.2), i.e., with respect to the quasi inner-product

$$(p,q) := L[pq] = \int_{-1}^{1} p(x)q(x)w(x)\,\mathrm{d}x, \tag{3.1}$$

exists. The inner-product (3.1) has the property $(zp,q) = (p,zq)$, and because of that the corresponding (monic) polynomials $\{\pi_n\}_{n\in\mathbb{N}_0}$ satisfy the fundamental three-term recurrence relation

$$\pi_{n+1}(x) = (x-\alpha_n)\pi_n(x) - \beta_n\pi_{n-1}(x), \qquad n \in \mathbb{N}, \tag{3.2}$$

with $\pi_0(x) = 1$, $\pi_{-1}(x) = 0$. Recurrence coefficient β_0 can be chosen arbitrary, but as usual it is convenient to take $\beta_0 = \mathcal{L}[1]$.

The recursion coefficients α_n and β_n can be expressed in terms of Hankel determinants as (cf. [3])

$$\alpha_n = \frac{\Delta'^{\pm}_{n+1}}{\Delta^{\pm}_{n+1}} - \frac{\Delta'^{\pm}_{n}}{\Delta^{\pm}_{n}}, \qquad \beta_n = \frac{\Delta^{\pm}_{n+1}\Delta^{\pm}_{n-1}}{(\Delta^{\pm}_{n})^2},$$

where $\Delta^{\pm}_n$ is defined in (2.6) and $\Delta'^{\pm}_n$ is the Hankel determinant $\Delta^{\pm}_{n+1}$ with the penultimate column and the last row removed.

Theorem 3.1. *The weight function $w(x) = (1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}\exp(\mathrm{i}\zeta x)$ satisfies the following Pearson type differential equation*

$$(\phi w)' = \psi w, \qquad \phi(x) = 1-x^2, \qquad \psi(x) = \beta-\alpha-(\alpha+\beta+1)x + \mathrm{i}\zeta(1-x^2).$$

Proof. By direct calculation we get

$$\begin{aligned}(\phi w)' &= \Big((1-x)^{\alpha+1/2}(1+x)^{\beta+1/2}\exp(\mathrm{i}\zeta x)\Big)' \\ &= (1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}\exp(\mathrm{i}\zeta x) \\ &\qquad \times \Big((\beta+1/2)(1-x)-(\alpha+1/2)(1+x)+\mathrm{i}\zeta(1-x^2)\Big) \\ &= \Big(\beta-\alpha-(\alpha+\beta+1)x+\mathrm{i}\zeta(1-x^2)\Big)\,w(x).\end{aligned}$$

This completes the proof of the theorem. □

By using the same arguments as in the proof of Theorem 3.1 in [7], with ϕ and ψ given in Theorem 3.1, it is easy to prove the following theorem.

Theorem 3.2. *For every $n \in \mathbb{N}$, we have*

$$\phi\pi_n' + \psi\pi_n = \sum_{k=n-1}^{n+2} \gamma_n^k \pi_k, \tag{3.3}$$

where

$$
\begin{aligned}
\gamma_n^{n+2} &= -\mathrm{i}\zeta,\\
\gamma_n^{n+1} &= -(n+\alpha+\beta+1) - \mathrm{i}\zeta(\alpha_n+\alpha_{n+1}),\\
\gamma_n^{n} &= \left(\beta-\alpha-(\alpha+\beta+1)\alpha_n + \mathrm{i}\zeta(1-\beta_{n+1}-\beta_n-\alpha_n^2)\right)/2,\\
\gamma_n^{n-1} &= \beta_n(n-1),
\end{aligned}
$$

and α_n *and* β_n *are the three-term recurrence coefficients in* (3.2).

Theorem 3.3. *The monic polynomials orthogonal with respect to the linear functional* (1.2) *satisfy the following differential-difference equation*

$$
\phi\pi_n' = p_1^n\pi_n + q_1^n\pi_{n-1}, \qquad n\in\mathbb{N}, \tag{3.4}
$$

where p_1^n *and* q_1^n *are polynomials of the first degree given by*

$$
\begin{aligned}
p_1^n &= -nx + \frac{\alpha_n}{2}(2n+\alpha+\beta+1) - \frac{\beta-\alpha}{2} - \frac{\mathrm{i}\zeta}{2}(1-\beta_{n+1}+\beta_n-\alpha_n^2),\\
q_1^n &= \beta_n\left(2n+\alpha+\beta+\mathrm{i}\zeta(x+\alpha_n)\right).
\end{aligned}
$$

Proof. By using the three-term recurrence relation (3.2) and (3.3) we get

$$
\begin{aligned}
\phi\pi_n' + \psi\pi_n =& \left[\left(\gamma_n^{n+2}(x-\alpha_{n+1})+\gamma_n^{n+1}\right)(x-\alpha_n) - \gamma_n^{n+2}\beta_{n+1}+\gamma_n^n\right]\pi_n\\
&+ \left[-\beta_n\left(\gamma_n^{n+2}(x-\alpha_{n+1})+\gamma_n^{n+1}\right)+\gamma_n^{n-1}\right]\pi_{n-1}.
\end{aligned}
$$

From the previous equation and the expressions given in Theorem 3.2 and Theorem 3.1, by direct computation we get what is stated. □

Theorem 3.4. *The polynomials* p_1^n *and* q_1^n, $n\in\mathbb{N}$, *which appear in Theorem* 3.3*, satisfy the following recurrence relations*

$$
\begin{aligned}
p_1^{n+1} &= -q_1^n\frac{x-\alpha_n}{\beta_n} + p_1^{n-1} + q_1^{n-1}\frac{x-\alpha_{n-1}}{\beta_{n-1}},\\
q_1^{n+1} &= (x-\alpha_n)\left(p_1^n + q_1^n\frac{x-\alpha_n}{\beta_n} - p_1^{n-1} - q_1^{n-1}\frac{x-\alpha_{n-1}}{\beta_{n-1}}\right) + \phi + q_1^{n-1}\frac{\beta_n}{\beta_{n-1}},
\end{aligned}
$$

respectively.

Proof. The proof follows from Theorem 3.3 and the following equation:

$$
\begin{aligned}
&\phi\pi'_{n+1} - \phi\pi_n = (x-\alpha_n)\phi\pi'_n - \beta_n\phi\pi'_{n-1} \\
=&(x-\alpha_n)(p_1^n\pi_n + q_1^n\pi_{n-1}) - \beta_n(p_1^{n-1}\pi_{n-1} + q_1^{n-1}\pi_{n-2}) \\
=&\left(q_1^n(x-\alpha_n) - p_1^{n-1}\beta_n - q_1^{n-1}\frac{\beta_n}{\beta_{n-1}}(x-\alpha_{n-1})\right)\pi_{n-1} \\
&\quad + (x-\alpha_n)p_1^n\pi_n + q_1^{n-1}\frac{\beta_n}{\beta_{n-1}}\left((x-\alpha_{n-1})\pi_{n-1} - \beta_{n-1}\pi_{n-2}\right) \\
=&-\left(q_1^n\frac{x-\alpha_n}{\beta_n} - p_1^{n-1} - q_1^{n-1}\frac{x-\alpha_{n-1}}{\beta_{n-1}}\right)\left((x-\alpha_n)\pi_n - \beta_n\pi_{n-1}\right) \\
&\quad + \left((x-\alpha_n)p_1^n + q_1^{n-1}\frac{\beta_n}{\beta_{n-1}} + (x-\alpha_n)\left(q_1^n\frac{x-\alpha_n}{\beta_n} - p_1^{n-1} - q_1^{n-1}\frac{x-\alpha_{n-1}}{\beta_{n-1}}\right)\right)\pi_n \\
=&-\left(q_1^n\frac{x-\alpha_n}{\beta_n} - p_1^{n-1} - q_1^{n-1}\frac{x-\alpha_{n-1}}{\beta_{n-1}}\right)\pi_{n+1} \\
&\quad + \left(q_1^{n-1}\frac{\beta_n}{\beta_{n-1}} + (x-\alpha_n)\left(p_1^n + q_1^n\frac{x-\alpha_n}{\beta_n} - p_1^{n-1} - q_1^{n-1}\frac{x-\alpha_{n-1}}{\beta_{n-1}}\right)\right)\pi_n.
\end{aligned}
$$

□

Theorem 3.5. *The following equation*

$$p_1^{n+1} + p_1^n + q_1^n\frac{x-\alpha_n}{\beta_n} = -(\beta-\alpha) + (\alpha+\beta-1)x - \mathrm{i}\zeta(1-x^2)$$

holds for all $n \in \mathbb{N}$.

Proof. It is easy to see from the previous Theorem that the quantity $p_1^{n+1} + p_1^n + q_1^n(x-\alpha_n)/\beta_n$ does not depend on n, i.e.,

$$p_1^{n+1} + p_1^n + q_1^n\frac{x-\alpha_n}{\beta_n} = p_1^2 + p_1^1 + q_1^1\frac{x-\alpha_1}{\beta_1}$$

for all positive integers n. By using formulae for p_1^n, $n = 1, 2$, and q_1^1 from Theorem 3.3, by direct calculation we get

$$
\begin{aligned}
p_1^2 + p_1^1 + q_1^1\frac{x-\alpha_1}{\beta_1} =& -(\beta-\alpha) + (\alpha+\beta-1)x - \mathrm{i}\zeta(1-x^2) \\
&- \frac{1}{2}\left(\alpha_1(\alpha+\beta+1) - \alpha_2(\alpha+\beta+5) + \mathrm{i}\zeta(\alpha_1^2 - \alpha_2^2 + \beta_1 - \beta_3)\right).
\end{aligned}
$$

Starting with explicit formulas for moments μ_k, $k \in \mathbb{N}_0$, given in Remark 2.1, by using functions for symbolic computations implemented in the software package OrthogonalPolynomials (see [2]) we generate three-term recurrence coefficients in symbolic form and with coefficients α_1, α_2, β_1 and β_3 obtained in such a way we get that $\alpha_1(\alpha+\beta+1) - \alpha_2(\alpha+\beta+5) + \mathrm{i}\zeta(\alpha_1^2 - \alpha_2^2 + \beta_1 - \beta_3) = 0$, i.e., we get what is stated. □

Theorem 3.6. *The polynomials* π_n *orthogonal with respect to the linear functional* (1.2) *satisfy the following second order differential equation*

$$\phi q_1^n \pi_n'' + \left(\psi q_1^n - \mathrm{i}\zeta\beta_n\phi\right)\pi_n' - \left(-nq_1^n - \mathrm{i}\zeta\beta_n p_1^n + \frac{q_1^n(A_1^n q_1^n - p_1^n B_1^n)}{\phi}\right)\pi_n = 0,$$

where p_1^n *and* q_1^n *are given in Theorem* 3.3,

$$A_1^n = -\frac{q_1^{n-1}}{\beta_{n-1}}, \qquad B_1^n = p_1^{n-1} + (x - \alpha_{n-1})\frac{q_1^{n-1}}{\beta_{n-1}},$$

α_n *and* β_n *are the three-term recurrence coefficients.*

Proof. In the same way as in proof of Theorem 3.7 in [7], using (3.4) and (3.2) we get $\phi\pi_{n-1}' = A_1^n\pi_n + B_1^n\pi_{n-1}$. From the previous equation and (3.4), after some calculation we get

$$\begin{aligned}\phi q_1^n \pi_n'' &+ \left(q_1^n(\phi' - p_1^n - B_1^n) - \phi(q_1^n)'\right)\pi_n' \\ &- \left((p_1^n)' q_1^n - p_1^n (q_1^n)' + \frac{q_1^n(A_1^n q_1^n - p_1^n B_1^n)}{\phi}\right)\pi_n = 0.\end{aligned}$$

From Theorem 3.3 we have that $(p_1^n)' = -n$, $(q_1^n)' = \mathrm{i}\zeta\beta_n$, and from Theorems 3.1 and 3.5 that

$$\begin{aligned}\phi' - p_1^n - B_1^n &= -2x - p_1^n - p_1^{n-1} - (x - \alpha_{n-1})\frac{q_1^{n-1}}{\beta_{n-1}} \\ &= -2x + \beta - \alpha - (\alpha + \beta - 1)x + \mathrm{i}\zeta(1 - x^2) = \psi.\end{aligned}$$

Now it is easy to get what is stated. □

4. Quadrature rules of Gaussian type

Let us suppose as in the previous Section that parameters $\zeta > 0$, $\alpha, \beta > -1/2$, $|\beta - \alpha| = \ell \in \mathbb{N}$, are such that the sequence of (monic) orthogonal polynomials $\{\pi_n\}_{n\in\mathbb{N}_0}$ with respect to the moment functional (1.2) exists. Knowing three-term recurrence coefficients, by using functions implemented in the software package OrthogonalPolynomials (see [2]) in extended arithmetics we are able to construct the corresponding quadrature rules of Gaussian type:

$$\int_{-1}^{1} f(x)(1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}\mathrm{e}^{\mathrm{i}\zeta x}\,\mathrm{d}x = \sum_{k=1}^{n} w_k^{(n)} f(x_k^{(n)}) + R_n(f), \tag{4.1}$$

where $R_n(f) = 0$ for each polynomial of degree at most $2n - 1$. Such rules can be applied for numerical integration of highly oscillating functions.

We illustrate effectiveness of our Gaussian quadrature rules by an example. We apply formula (4.1) to the following integral ($\alpha = 3/4$, $\ell = 2$, $\beta = \alpha + \ell = 11/4$)

$$I(\zeta) = \int_{-1}^{1} \frac{1}{x - \mathrm{i}/2}(1-x)^{1/4}(1+x)^{9/4} \mathrm{e}^{\mathrm{i}\zeta x} \mathrm{d}x \approx G_n(\zeta) = \sum_{k=1}^{n} \frac{w_k^{(n)}}{x_k^{(n)} - \mathrm{i}/2},$$

for the following two different values of ζ (zeros of $J_{-1/4}(z)$): $\zeta \in \{\zeta_1, \zeta_2\}$, where $\zeta_1 \approx 5.123062742746341$ and $\zeta_3 \approx 1000.990052907274$. The exact values of integrals are

$$\begin{aligned} I(\zeta_1) &= -0.99791002158030877955697778495726461722778728602453936 51\cdots \\ &\quad + \mathrm{i}\, 0.10604523452184338167925254043624475997180507805579746 08\cdots, \\ I(\zeta_2) &= 0.00061277393978481971292877714033237488966393605799029 57\cdots \\ &\quad + \mathrm{i}\, 0.00030676867546404495149488702442303091596590075339646 26\cdots. \end{aligned}$$

In Table 1 the relative errors in Gaussian approximations, $r_n = |(G_n(\zeta_\nu) - I(\zeta_\nu))/I(\zeta_\nu)|$, $\nu = 1, 2$, for $n = 5(5)30$ nodes are given. In numerical construction we use software package [2]. We also apply the corresponding Gauss-Jacobi quadrature formula with respect to the weight function $x \mapsto (1-x)^{1/4}(1+x)^{9/4}$ and give its relative errors r_n^J. It is easy to compare the obtained results. Gauss-Jacobi quadrature is faster for small ζ, but when ζ increases, Gauss-Jacobi formula cannot be applied, and our formula becomes much faster.

Table 1: Relative errors r_n and r_n^J, for $n = 5(5)30$, when $\zeta = \zeta_1$, ζ_2.

ζ	ζ_1		ζ_2	
n	r_n	r_n^J	r_n	r_n^J
5	2.10(−1)	6.44(−4)	3.37(−16)	1.75(4)
10	2.53(−3)	8.56(−6)	3.70(−24)	5.35(3)
15	2.04(−5)	6.78(−8)	2.12(−36)	1.48(4)
20	1.65(−7)	5.43(−10)	5.99(−43)	1.24(4)
25	1.33(−9)	4.38(−12)	8.33(−54)	1.08(4)
30	1.08(−11)	3.54(−14)	1.91(−59)	6.16(3)

Acknowledgments The authors were supported in part by Serbian Ministry of Education and Science (Projects #174015 and III44006).

References

[1] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[2] A. S. Cvetković and G. V. Milovanović, *The Mathematica package "OrthogonalPolynomials"*, Facta Univ. Ser. Math. Inform., 19, (2004), pp. 17–36.

[3] W. Gautschi and G. V. Milovanović, *Polynomials orthogonal on the semicircle*, J. Approx. Theory, 46, (1986) pp. 230–250.

[4] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series, and Products*, sixth edition, Academic Press, London, 2000.

[5] G. V. MILOVANOVIĆ AND A. S. CVETKOVIĆ, *Orthogonal polynomials and Gaussian quadrature rules related to oscillatory weight functions*, J. Comput. Appl. Math., 179, (2005), pp. 263–287.

[6] G. V. MILOVANOVIĆ AND A. S. CVETKOVIĆ, *Orthogonal polynomials related to the oscillatory-Chebyshev weight function*, Bull. Cl. Sci. Math. Nat. Sci. Math., 30, (2005), pp. 47–60.

[7] G. V. MILOVANOVIĆ, A. S. CVETKOVIĆ, AND Z. M. MARJANOVIĆ, *Orthogonal polynomials for the oscillatory-Gegenbauer weight*, Publ. Inst. Math. (Beograd) (N.S.), 84(98), (2008), pp. 49–60.

[8] G. V. MILOVANOVIĆ, A. S. CVETKOVIĆ, AND M. P. STANIĆ, *Orthogonal polynomials for modified Gegenbauer weight and corresponding quadratures*, Appl. Math. Lett., 22(8), (2009), pp. 1189–1194.

[9] S. J. ORFANIDIS, *Electromagnetic Waves and Antennas*, Rutgers University, NJ, 2008, `www.ece.rutgers.edu/~orfanidi/ewa`

[10] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, 1922.

[11] A. B. ŠIDLOVSKIĬ, *Transcendental Numbers*, Nauka, Moscow, 1987 [Translated from the Russian by Neal Koblitz. de Gruyter Studies in Mathematics, 12, Walter de Gruyter & Co., Berlin-New York, 1989].