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Finite-Difference Methods for a Class of Strongly Nonlinear Singular Perturbation Problems

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Abstract. The paper is concerned with strongly nonlinear singularly perturbed boundary value problems in one dimension. The problems are solved numerically by finite-difference schemes on special meshes which are dense in the boundary layers. The Bakhvalov mesh and a special piecewise equidistant mesh are analyzed. For the central scheme, error estimates are derived in a discrete L^1 norm. They are of second order and decrease together with the perturbation parameter ε. The fourth-order Numerov scheme and the Shishkin mesh are also tested numerically. Numerical results show ε-uniform pointwise convergence on the Bakhvalov and Shishkin meshes.

AMS subject classifications: 65L10, 65L12

Key words: Boundary-value problem, singular perturbation, finite differences, Bakhvalov and piecewise equidistant meshes, L^1 stability.

Dedicated to Professor Yucheng Su on the Occasion of His 80th Birthday

1. Introduction

We consider the following singularly perturbed boundary value problem:

$$-\varepsilon^2(k(u)u')' + c(x,u) = 0, \quad x \in I := [0,1], \quad u(0) = \alpha, \quad u(1) = \beta, \tag{1.1}$$

where ε is a small positive parameter, α and β are given constants, and the functions k and c are sufficiently smooth and satisfy

$$k^* \geq k(u) \geq k_* > 0, \quad c_u(x,u) \geq c_* > 0, \quad x \in I, \quad u \in \mathbb{R}. \tag{1.2}$$

This problem has a unique solution, u_ε, for which the following estimates hold true:

$$|u_\varepsilon^{(j)}(x)| \leq M\left(1 + \varepsilon^{-j}e^{-\gamma x/\varepsilon} + \varepsilon^{-j}e^{\gamma(x-1)/\varepsilon}\right), \quad x \in I, \quad j = 0,1,2,3,4, \tag{1.3}$$

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with a constant γ in the interval $(0, \sqrt{c_*/k^*})$. Here and throughout the paper, M is a generic positive constant independent of ε. Thus, estimates (1.3) show that the solution has in general two boundary layers whose width is $\mathcal{O}\left(\varepsilon \ln \frac{1}{\varepsilon}\right)$. This result can be proved as follows. For $K(u) = \int^u k(s)\, \mathrm{d}s$, it holds that $K_u(u) \geq k_* > 0$, so the inverse function K^{-1} exists. We can therefore introduce the substitution $v = K(u)$ to transform (1.1) to

$$-\varepsilon^2 v'' + g(x,v) = 0, \quad x \in I, \quad v(0) = K(\alpha), \quad v(1) = K(\beta), \tag{1.4}$$

where $g(x,v) = c(x, K^{-1}(v))$. Then from $g_v(x,v) = c_u(x, K^{-1}(v))/k(u)$, we get that $g_v(x,v) > \gamma^2$. This implies that problem (1.4) has a unique solution, v_ε, and it is well known that its derivatives can be estimated by the right-hand side of (1.3). These estimates immediately transfer to u_ε.

Problems similar to (1.1), as well as the more general ones with $k = k(x,u)$, arise in applications to chemistry as models of catalytic reactions accompanied by a change in volume [3,14,17,19]. Some numerical methods for those problems have been considered in [14, 17], but no complete error-analysis has been given. This is finally done in the present paper. The special case $k(u) \equiv 1$ describes the standard reaction-diffusion problem which has been discussed very often. Earlier papers, like [2, 13], typically consider the condition $c_u(x,u) \geq c_* > 0$, which is also assumed here. This condition is relaxed in [7,8,12,15]. Of other more recent papers on numerical methods for singularly perturbed semilinear reaction-diffusion problems, let us mention [5] and [6]. These papers deal with *a posteriori* error estimates in the maximum norm; paper [6] is a 2D generalization of [5].

The numerical method proposed by Wang [18] for (1.1) in the non-perturbed case $\varepsilon = 1$ is the fourth-order Numerov scheme applied to (1.4). Wang considers the situation when K^{-1} can be found explicitly. Since this is not always easy to do, we discretize here the original problem after rewriting the differential equation in (1.1) as

$$-\varepsilon^2 K(u)'' + c(x,u) = 0. \tag{1.5}$$

The method we discuss in detail is the central finite-difference scheme applied on meshes of Bakhvalov and piecewise equidistant types. It is well known in the semilinear case $k(u) \equiv 1$ that the central scheme is ε-uniformly stable in the maximum norm. Here, because of the strong nonlinearity of the problem, it is much easier to use a discrete L^1 norm to prove stability uniform in ε. Stability of finite-difference approximations of quasilinear singular perturbation problems is often proved in this norm, see [1] for instance. Solutions of such problems may have interior layers with *a priori* unknown locations. This is not the case in the present problem, but, in addition to the strong nonlinearity, there is another reason for using the L^1 norm. If $w(x) = \exp(-\gamma x/\varepsilon)$ is the exponential boundary-layer function, then $\|w\|_1$ is of order ε, thus small values of ε increase accuracy in L^1 norm. Such higher L^1-accuracy is important in the catalytic-reaction applications when calculating the so-called efficiency factor, see [17].

ε-uniform stability in L^1 norm implies convergence results in the same norm, the errors being estimated by

$$E_B := MN^{-2}\left(\varepsilon + e^{-mN}\right) \quad \text{on the Bakhvalov mesh} \tag{1.6}$$

and

$$E_S := MN^{-2}\varepsilon\left(\ln\frac{1}{\varepsilon}\right)^3 \quad \text{on the special piecewise equidistant mesh,} \tag{1.7}$$

where the generic constant M is assumed to be independent also of the number of mesh steps, N, and where m is a another generic positive constant that does not depend on either ε or N. We see that E_B behaves almost as $N^{-2}\varepsilon$ since the term $\exp(-mN)$ is very small. This term is present in the error estimate because of the technique used in the proof. $E_S \to 0$ as $\varepsilon \to 0$, slowed down by the logarithmic factors resulting from the way the special piecewise equidistant mesh is constructed. This mesh can be viewed as a modification of the Shishkin mesh. Its two transition points are given in terms of the quantity $\varepsilon \ln \frac{1}{\varepsilon}$ instead of the more usual $\varepsilon \ln N$ which we find in the standard Shishkin-type meshes. Such a piecewise equidistant mesh has been used in [9] for instance. In the method presented here, L^1-accuracy is better on the modified Shishkin mesh than on the standard one, whose L^1-errors do not decrease as $\varepsilon \to 0$. On the other hand, the standard Shishkin mesh gives better maximum-norm errors.

The theoretical results are illustrated in Section 3 by numerical experiments, performed on test-problems which are similar to the above-mentioned application problems. The numerical results for the Bakhvalov and standard Shishkin meshes show moreover the presence of ε-uniform pointwise convergence. The fourth-order Numerov scheme is also tested numerically.

2. The numerical method

Let I^N be any discretization mesh with points x_i, $i = 0,1,\cdots,N$, $0 = x_0 < x_1 < \cdots < x_N = 1$, and steps $h_i = x_i - x_{i-1}$, $i = 1,2,\cdots,N$. Let also $\hbar_i = (h_i + h_{i+1})/2$, $i = 1,2,\cdots,N-1$. By $\{w_i\}$ we denote an arbitrary mesh function defined on $I^N \setminus \{0,1\}$ and identified with the column-vector $w^N = [w_1, w_2, \cdots, w_{N-1}]^T$. For any mesh function, we formally set $w_0 = \alpha$ and $w_N = \beta$. Let also $\delta^N = [1,1,\cdots,1]^T$ and $u_\varepsilon^N = [u_\varepsilon(x_1), u_\varepsilon(x_2), \cdots, u_\varepsilon(x_N)]^T$. We use the maximum vector norm, $\|w^N\|_\infty = \max_{1\le i\le N-1}|w_i|$, and the discrete L^1 norm, $\|w^N\|_1 = \sum_{i=1}^{N-1}\hbar_i|w_i|$. $\|\cdot\|_1$ is used to denote also the matrix norm induced by vector norm $\|\cdot\|_1$.

The discretization of the equation (1.5) is

$$T^N w_i := -\varepsilon^2 D'' K(w_i) + c(x_i, w_i) = 0, \quad i = 1,2,\cdots,N-1, \tag{2.1}$$

where D'' is the standard central scheme approximating the second derivative,

$$D'' w_i = \frac{1}{\hbar_i}\left(\frac{w_{i+1}-w_i}{h_{i+1}} - \frac{w_i - w_{i-1}}{h_i}\right).$$

Theorem 2.1. *Under the condition* (1.2)*, the discrete problem* (2.1) *has a unique solution and it holds that*

$$\|w^N - v^N\|_1 \le c_*^{-1}\|T^N w^N - T^N v^N\|_1 \tag{2.2}$$

for any two mesh functions w^N and v^N.

Proof. Let us introduce the matrix $A = DFD^{-1}$, where $D = \text{diag}(\hbar_1, \hbar_2, \cdots, \hbar_{N-1})$ and where F is the Fréchet derivative of the discrete operator T^N at an arbitrary mesh function, $F = (T^N)'(w^N)$. It is easy to see that A is an L-matrix. Moreover, $A^T \delta^N \geq c_* \delta^N$, where the inequality is to be understood componentwise. Then A is an M-matrix and $\|A^{-1}\|_1 \leq 1/c_*$. This implies that (2.2) holds true and that (2.1) has a unique solution. □

We consider next two special discretization meshes, the Bakhvalov mesh and the special piecewise equidistant mesh. It would be possible to analyze a general class of meshes like in [10], but this is not our interest here. By working with the two mesh types separately, we point out what the similarities and differences between them are. For simplicity, the meshes are described below on the subinterval $[0, 1/2]$ since the mesh points are symmetrical on the other half of the interval I. It is assumed that N is even and sufficiently big independently of ε, and also that $x_{N/2} = 1/2$.

Bakhvalov meshes

Bakhvalov meshes [2] (see [13] and [10] as well) are generated by a suitable function λ which appropriately redistributes equidistantly spaced points, $x_i = \lambda(t_i)$, $t_i = i/N$, $i = 0, 1, \cdots, N/2$. As a representative of this type of meshes, we choose here the one introduced in [13],

$$\lambda(t) = \begin{cases} \varphi(t) := \varepsilon a t/(q-t) & \text{if } 0 \leq t \leq \tau, \\ \psi(t) := \varphi'(\tau)(t-\tau) + \varphi(\tau) & \text{if } \tau \leq t \leq 1/2. \end{cases}$$

Let the mesh generated this way be denoted by B^N. a and q are positive parameters that can be used to control the density of the mesh in the layer. They satisfy $q < 1/2$ and $a\varepsilon < q$. $\tau \in (0, q)$ is a solution of the equation $\psi(1/2) = 1/2$. When $\varepsilon \to 0$, τ behaves like $q - m\sqrt{\varepsilon}$.

Theorem 2.2. *Under the condition* (1.2)*, the solution w_ε^N of the discrete problem* (2.1) *on the mesh B^N satisfies*

$$\|w_\varepsilon^N - u_\varepsilon^N\|_1 \leq E_B,$$

where E_B is defined in (1.6).

Proof. The technique used in this proof is a variation on the theme from [2] and [13]. Because of (2.2), it suffices to prove $\|T^N u_\varepsilon^N\|_1 \leq E_B$. We show below that

$$\Sigma := \sum_{i=1}^{N/2} \hbar_i |T^N u_\varepsilon(x_i)| \leq E_B$$

and then the result follows since the error can be estimated analogously on the second half of the mesh. Using (1.3) and taking into account that the second boundary-layer term is ε-uniformly bounded on the interval $[0, x_{N/2+1}]$, we get

$$\Sigma \leq M\left(\Sigma_0 + \Sigma_1^{N/2}\right), \quad \text{where} \quad \Sigma_0 = \sum_{i=1}^{N/2} \varepsilon^2 h_{i+1}(h_{i+1} - h_i + h_{i+1}^2)$$

and

$$\Sigma_j^\ell = \sum_{i=j}^{\ell} h_{i+1}[\varepsilon^{-1}(h_{i+1} - h_i) + \varepsilon^{-2}h_{i+1}^2]e^{-\gamma x_{i-1}/\varepsilon}, \quad 1 \le j \le \ell \le N/2.$$

Since $h_{i+1} \le \lambda'(t_{i+1})/N \le M/N$ and $h_{i+1} - h_i \le M\varphi''(\tau)/N^2 \le M/(N^2\sqrt{\varepsilon})$, it follows that

$$\Sigma_0 \le M\varepsilon^{3/2}N^{-2} \le E_B.$$

Therefore, it remains to be proved that

$$\Sigma_1^{N/2} \le E_B. \tag{2.3}$$

Let J be such an index that $t_J \ge \tau > t_{J-1}$. Then,

$$\Sigma_1^{J-3} \le M\sum_{i=1}^{J-3} N^{-3}\varepsilon(q - t_{i+1})^{-6}e^{-\gamma\varphi(t_{i-1})/\varepsilon} \le M\varepsilon N^{-2} \le E_B.$$

To prove this, we use the fact that for $1 \le i \le J-3$ it holds that $t_{i+1} < q - 1/N$, implying $3(q - t_{i+1}) \ge q - t_{i-1}$. Also,

$$\Sigma_{J+1}^{N/2} \le MN^{-2}\varepsilon^{-2}e^{-\gamma\varphi(\tau)/\varepsilon} \le M\varepsilon N^{-2} \le E_B.$$

Thus, to complete the proof of (2.3), we need to show that $\Sigma_{J-2}^J \le E_B$. When $N\sqrt{\varepsilon} \ge 1$, this follows from

$$\begin{aligned}\Sigma_{J-2}^J &\le MN^{-3}\varepsilon^{-2}e^{-\gamma\varphi(\tau-3/N)/\varepsilon} \le MN^{-3}\varepsilon^{-2}e^{-mN/(N\sqrt{\varepsilon}+1)}\\ &\le MN^{-3}\varepsilon^{-2}e^{-m/\sqrt{\varepsilon}} \le E_B.\end{aligned}$$

On the other hand, when $N\sqrt{\varepsilon} \le 1$, we use the following form of the consistency error:

$$|T^N u_\varepsilon(x_i)| \le 2\varepsilon^2 \max_{x_{i-1}\le x\le x_{i+1}} |K(u_\varepsilon(x))''| \tag{2.4}$$

to prove for $i = J-2, J-1, J$ that

$$|T^N u_\varepsilon(x_i)| \le M\left(\varepsilon^2 + e^{-mN/(N\sqrt{\varepsilon}+1)}\right) \le M\left(\varepsilon N^{-2} + e^{-mN}\right) \le E_B.$$

This completes the proof of this theorem. □

The special piecewise equidistant mesh

We use here a mesh of the following kind:

$$x_i = ih, \;\; i = 0, 1, \cdots, J, \quad \text{and} \quad x_i = \sigma + (i-J)H, \;\; i = J+1, J+2, \cdots, N/2,$$

where

$$h = \frac{\sigma}{J} = \frac{-a\varepsilon\ln\varepsilon}{\gamma J} \quad \text{and} \quad H = \frac{1-2\sigma}{N-2J}$$

(ε is assumed sufficiently small, so that $\sigma < 1/2$). In the above definitions, a is a positive parameter, γ is the same constant as in (1.3), and J is an integer such that $J \geq 6$ and $Q = 2J/N$ is kept fixed, $Q < 1$ and $1/Q \leq M$. In this mesh, denoted by S_ε^N, σ is the transition point between the fine and the coarse equidistant parts. σ is more usually defined with $\ln N$ instead of $-\ln\varepsilon$.

Theorem 2.3. *Under the condition* (1.2)*, the solution w_ε^N of the discrete problem* (2.1) *on the mesh S_ε^N with $a \geq 3$ satisfies*

$$\|w_\varepsilon^N - u_\varepsilon^N\|_1 \leq E_S,$$

where E_S is defined in (1.7).

Proof. Using the same notation as in the previous proof, we have $\Sigma_0 \leq M\varepsilon^2/N^2 \leq E_S$. Also,

$$\Sigma_1^{J-1} \leq \sum_{i=1}^{J-1} h^3\varepsilon^{-2} \leq E_S$$

and

$$\Sigma_{J+1}^{N/2} \leq \sum_{i=J+1}^{N/2} H^3\varepsilon^{-2}e^{-\gamma\sigma/\varepsilon} \leq M\frac{\varepsilon^{a-2}}{N^2} \leq E_S.$$

It remains to estimate the consistency error at $x_J = \sigma$. If $N\sqrt{\varepsilon} \geq 1$, we use

$$\Sigma_J^J \leq \left(\frac{H^2}{\varepsilon} + \frac{H^3}{\varepsilon^2}\right)e^{-\gamma(\sigma-h)/\varepsilon} \leq \frac{M}{N^2}\varepsilon^{a(1-1/J)-3/2} \leq M\frac{\varepsilon}{N^2} \leq E_S.$$

On the other hand, when $N\sqrt{\varepsilon} \leq 1$, we apply (2.4) to get

$$|T^N u_\varepsilon(x_J)| \leq M\left(\varepsilon^2 + e^{-\gamma(\sigma-h)/\varepsilon}\right) \leq M\varepsilon^2 \leq M\frac{\varepsilon}{N^2} \leq E_S.$$

This completes the proof of Theorem 2.3. □

3. Numerical experiments and conclusions

The choice of test-problems for numerical experiments is motivated by the following problem which models a catalytic chemical reaction accompanied with a change in volume:

$$-\varepsilon^2\left(\frac{u'}{1+\theta u}\right)' + u = 0, \quad x \in I, \quad -u'(0) = 0, \quad u(1) = 1, \tag{3.1}$$

where θ is a positive constant (we consider below $\theta = 1$), see [3, 14, 17, 19]. In this problem, $c_* = 1$ and we can take $k^* = 1$ and $k_* = 1/(1+\theta)$ since 0 and 1 are respectively lower and upper solutions of (3.1). Due to the left boundary condition, the solution u_ε satisfies here an estimate milder than (1.3),

$$|u_\varepsilon^{(j)}(x)| \leq M\left(1 + \varepsilon^{1-j}e^{-\gamma x/\varepsilon} + \varepsilon^{-j}e^{\gamma(x-1)/\varepsilon}\right), \quad x \in I, \quad j = 0, 1, \cdots,$$

Table 1: Problem (3.3) solved by central scheme on mesh B^N.

ε	10^{-3}, 10^{-6}, 10^{-9}		10^{-3}		10^{-6}		10^{-9}	
N	$E_\infty(N)$	$R_\infty(N)$	$\tilde{E}_1(N)$	$R_1(N)$	$\tilde{E}_1(N)$	$R_1(N)$	$\tilde{E}_1(N)$	$R_1(N)$
32	3.3–3	—	1.4–2	—	1.4–2	—	2.4–2	—
64	8.3–4	2.0	3.3–3	2.1	3.3–3	2.1	3.3–3	2.8
128	2.1–4	2.0	8.2–4	2.0	8.2–4	2.0	8.2–4	2.0
256	5.2–5	2.0	2.1–4	2.0	2.1–4	2.0	2.1–4	2.0
512	1.3–5	2.0	5.1–5	2.0	5.1–5	2.0	5.1–5	2.0

where $0 < \gamma < 1$. We would like to analyze separately the two different boundary-layer behaviors and we consider two test-problems with known solutions:

$$-\varepsilon^2\left(\frac{u'}{1+u}\right)' + u = f_1(x), \quad x \in I, \quad -u'(0) = 0, \quad u(1) = \beta_1, \tag{3.2}$$

$$-\varepsilon^2\left(\frac{u'}{1+u}\right)' + u = f_2(x), \quad x \in I, \quad u(0) = 1, \quad u(1) = \beta_2. \tag{3.3}$$

f_i and β_i, $i = 1,2$, are defined by the respective exact solutions, which are of the form $u_\varepsilon(x) = \mu e^{-x/\varepsilon} + e^x - 1$, where $\mu = \varepsilon$ for (3.2) and $\mu = 1$ for (3.3). The two problems are constructed for convenience so that there is no layer at $x = 1$ and the discretization meshes are modified not to condense close to this point.

Mesh B^N is used with $a = 1$ and $q = 0.5$ and S_ε^N with $a = 3$, $\gamma = 1$, and $J = N/2$. We compare these meshes to the standard Shishkin mesh, denoted by S^N, with the transition point $\sigma = a\varepsilon \ln N$ (cf. [11]; more accessible sources [10] and [4] for instance). Presented in the tables are errors $E_\infty(N)$ and $\tilde{E}_1(N) = E_1(N)/\varepsilon$, where $E_s(N) := \|w_\varepsilon^N - u_\varepsilon^N\|_s$, $s = 1, \infty$. The corresponding numerical rates of convergence, $R_s(N) := \log_2[E_s(N/2)/E_s(N)]$, are also provided.

Results are given first for problem (3.3), which is harder to solve numerically. Tables 1 and 3 show that errors $E_1(N)$ on B^N and S_ε^N decrease together with ε, as predicted by Theorems 2.2 and 2.3. This is not the case on mesh S^N, see Table 4. Except for one entry in Table 1, the values of $\tilde{E}_1(N)$ are fixed when $\varepsilon \to 0$.

As for Theorem 2.3 and mesh S_ε^N, the influence of $\ln(1/\varepsilon)$-factors in E_S can be observed in Table 3.

Additionally, mesh B^N produces ε-uniform pointwise convergence, see Table 1. Table 4 shows that errors $E_\infty(N)$ on S^N are also ε-uniform. The rate of convergence is lower on S^N than on B^N, which is not surprising (cf. [16]), but the errors are actually somewhat better on S^N. This is only because S^N is much less dense near $x = 0$. ε-uniform pointwise convergence cannot be expected of S_ε^N and Table 2 indeed shows that $E_\infty(N)$ errors become somewhat worse when ε decreases.

Overall, mesh B^N gives the best results and we do not present results on S_ε^N and S^N in Tables 5-7. We apply to (3.3) also a non-equidistant generalization of the Numerov (Hermite) scheme, of the type analyzed in [16]. Table 5 confirms our expectation of ε-uniform pointwise convergence whose rate is four. With the exception of $N = 16$, the

Table 2: Problem (3.3) solved by central scheme on mesh S_ε^N, error $E_\infty(N)$.

ε	10^{-3}		10^{-6}		10^{-9}	
N	$E_\infty(N)$	$R_\infty(N)$	$E_\infty(N)$	$R_\infty(N)$	$E_\infty(N)$	$R_\infty(N)$
32	3.2–3	—	1.4–2	—	2.1–2	—
64	7.9–4	2.0	3.2–3	2.2	6.7–3	1.7
128	2.0–4	2.0	7.9–4	2.0	1.8–4	1.9
256	4.9–5	2.0	2.0–4	2.0	4.4–4	2.0
512	1.2–5	2.0	4.9–5	2.0	1.1–4	2.0

Table 3: Problem (3.3) solved by central scheme on mesh S_ε^N, error $\tilde{E}_1(N)$.

ε	10^{-3}		10^{-6}		10^{-9}	
N	$\tilde{E}_1(N)$	$R_1(N)$	$\tilde{E}_1(N)$	$R_1(N)$	$\tilde{E}_1(N)$	$R_1(N)$
32	1.0–2	—	5.1–2	—	9.0–2	—
64	2.9–3	1.8	1.0–2	2.4	2.7–2	1.7
128	7.6–4	1.9	2.9–3	1.8	6.5–3	2.1
256	1.9–4	2.0	7.6–4	1.9	1.7–3	1.9
512	4.8–5	2.0	1.9–4	2.0	4.3–4	2.0

Table 4: Problem (3.3) solved by central scheme on mesh S^N.

ε	10^{-3}, 10^{-6}, 10^{-9}		10^{-3}		10^{-6}		10^{-9}	
N	$E_\infty(N)$	$R_\infty(N)$	$\tilde{E}_1(N)$	$R_1(N)$	$\tilde{E}_1(N)$	$R_1(N)$	$\tilde{E}_1(N)$	$R_1(N)$
32	7.9–4	—	3.8–3	—	9.6–1	—	9.5+2	—
64	2.8–4	1.5	1.2–3	1.7	6.1–2	4.0	6.0+1	4.0
128	9.7–5	1.6	3.8–4	1.6	4.1–3	3.9	3.7+0	4.0
256	3.2–5	1.6	1.3–4	1.6	3.6–4	3.5	2.3–1	4.0
512	1.0–5	1.7	3.9–5	1.7	5.4–5	2.7	1.5–2	4.0

errors are better than those obtained by the central scheme and presented in Table 1. In Table 6, we see the same qualitative behavior of $\tilde{E}_1(N)$ as in Table 1, but the values stabilize later, starting from $N = 256$. This is also when the Table 6 results become better than the corresponding ones in Table 1.

Finally, we consider problem (3.2). We report that equidistant meshes produce unsatisfactory results and, therefore, a special mesh is needed for this problem too. Mesh B^N gives maximum errors that decrease linearly when $\varepsilon \to 0$, see Table 7. However, Tables 1 and 7 indicate that there is no significant difference between problems (3.3) and (3.2) in the $\tilde{E}_1(N)$ values. It is important to mention that the results are only first-order ε-uniformly accurate when the left boundary condition in (3.2) is discretized as $-D_+w_0 := (w_0 - w_1)/h_1$. The results presented in Table 7 are obtained with a second-order scheme for $-u'(0) = 0$,

$$\frac{w_0 - w_1}{h_1} + \frac{h_1}{2\varepsilon^2}(w_0 + 1)[w_0 - f_1(0)] = 0$$

(where $f_1(0) = 0$). This approximation results from the expansion

$$D_+u_\varepsilon(x_0) \approx u_\varepsilon'(0) + \frac{h_1}{2}u_\varepsilon''(0)$$

Table 5: Problem (3.3) solved by Numerov scheme on mesh B^N, error $E_\infty(N)$.

ε	10^{-3}		10^{-6}		10^{-9}	
N	$E_\infty(N)$	$R_\infty(N)$	$E_\infty(N)$	$R_\infty(N)$	$E_\infty(N)$	$R_\infty(N)$
32	2.4–3	—	3.0–3	—	3.1–3	—
64	1.2–4	4.3	1.2–4	4.6	1.2–4	4.7
128	7.4–6	4.1	7.5–6	4.1	7.5–6	4.1
256	4.6–7	4.0	4.6–7	4.0	4.6–7	4.0
512	2.9–8	4.0	2.9–8	4.0	2.9–8	4.0

Table 6: Problem (3.3) solved by Numerov scheme on mesh B^N, error $\tilde{E}_1(N)$.

ε	10^{-3}		10^{-6}		10^{-9}	
N	$\tilde{E}_1(N)$	$R_1(N)$	$\tilde{E}_1(N)$	$R_1(N)$	$\tilde{E}_1(N)$	$R_1(N)$
32	5.9–2	—	1.1+2	—	1.2+5	—
64	1.3–3	5.6	4.0–1	8.0	5.0+2	7.9
128	5.2–5	4.6	1.2–4	11.7	9.9–2	12.3
256	3.1–6	4.1	3.1–6	5.3	3.1–6	15.0
512	1.9–7	4.0	1.9–7	4.0	1.9–7	4.0

Table 7: Problem (3.2) solved by central scheme on mesh B^N.

ε	10^{-3}, 10^{-6}, 10^{-9}		10^{-3}		10^{-6}		10^{-9}	
N	$E_\infty(N)$	$R_\infty(N)$	$\tilde{E}_1(N)$	$R_1(N)$	$\tilde{E}_1(N)$	$R_1(N)$	$\tilde{E}_1(N)$	$R_1(N)$
32	5.2–3	—	2.2–2	—	2.2–2	—	3.2–2	—
64	1.3–3	2.0	5.4–3	2.0	5.4–3	2.0	5.4–3	2.6
128	3.3–4	2.0	1.3–3	2.0	1.3–3	2.0	1.3–3	2.0
256	8.2–5	2.0	3.4–4	2.0	3.4–4	2.0	3.4–4	2.0
512	1.4–5	2.5	4.9–5	2.8	4.9–5	2.8	4.9–5	2.8

in which $u_\varepsilon''(0)$ is then expressed from the differential equation in (3.2), taking into account that $u_\varepsilon'(0) = 0$. The stability of the central scheme with the above discretization used at x_0 can be analyzed using the technique applied in [14].

We can therefore conclude that for problems like (3.1), a special discretization mesh is needed which condenses near both endpoints $x = 0$ and $x = 1$. Mesh B^N would be our first choice, but S_ε^N or S^N are also possible to use. S_ε^N gives better errors in the discrete L^1 norm, whereas S^N produces better maximum-norm errors.

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