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A Posteriori Error Estimates of a Combined Mixed Finite Element and Discontinuous Galerkin Method for a Kind of Compressible Miscible Displacement Problems

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Abstract. A kind of compressible miscible displacement problems which include molecular diffusion and dispersion in porous media are investigated. The mixed finite element method is applied to the flow equation, and the transport one is solved by the symmetric interior penalty discontinuous Galerkin method. Based on a duality argument, employing projection estimates and approximation properties, a posteriori residual-type hp error estimates for the coupled system are presented, which is often used for guiding adaptivity. Comparing with the error analysis carried out by Yang (Int. J. Numer. Meth. Fluids, 65(7) (2011), pp. 781–797), the current work is more complicated and challenging.

AMS subject classifications: 65M12, 65M60

Key words: A posteriori error, discontinuous Galerkin method, compressible miscible displacement, mixed finite element, duality argument.

1 Introduction

We consider the following single-phase, miscible displacement problem of one compressible fluid by another in porous media:

$$d(c)\frac{\partial p}{\partial t}+\nabla\cdot\boldsymbol{u}=d(c)\frac{\partial p}{\partial t}-\nabla\cdot(a(c)\nabla p)=q, \qquad (x,t)\in\Omega\times J, \tag{1.1a}$$

$$\phi\frac{\partial c}{\partial t}+b(c)\frac{\partial p}{\partial t}+\boldsymbol{u}\cdot\nabla c-\nabla\cdot(\boldsymbol{D}(\boldsymbol{u})\nabla c)=(\widehat{c}-c)q, \qquad (x,t)\in\Omega\times J, \tag{1.1b}$$

$$\boldsymbol{u}\cdot\boldsymbol{n}=0, \qquad (x,t)\in\partial\Omega\times J, \tag{1.1c}$$

$$\boldsymbol{D}(\boldsymbol{u})\nabla c\cdot\boldsymbol{n}=0, \qquad (x,t)\in\partial\Omega\times J, \tag{1.1d}$$

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$$p(x,0)=p_0(x), \qquad x\in\Omega, \tag{1.1e}$$
$$c(x,0)=c_0(x), \qquad x\in\Omega, \tag{1.1f}$$

where Ω is a polygonal and bounded domain in $\mathbb{R}^d$ $(d=2$ or $3)$ with the boundary $\partial\Omega$, $J=(0,T]$, $\boldsymbol{n}$ denotes the unit outward normal vector to $\partial\Omega$; $\boldsymbol{u}(x,t)$ represents the Darcy velocity of the mixture and $p(x,t)$ is the fluid pressure in the fluid mixture; $c(x,t)$ is the solvent concentration of interested species measured in amount of species per unit volume of the fluid mixture, $\phi(x)$ is the effective porosity of the medium and is bounded above and below by positive constants, $\boldsymbol{D}(\boldsymbol{u})$ denotes a diffusion or dispersion tensor which has contributions from molecular diffusion and mechanical dispersion. $\boldsymbol{D}(\boldsymbol{u})=d_m\boldsymbol{I}+|\boldsymbol{u}|\big(\alpha_l\boldsymbol{E}(\boldsymbol{u})+\alpha_t(\boldsymbol{I}-\boldsymbol{E}(\boldsymbol{u}))\big)$, where $\boldsymbol{E}(\boldsymbol{u})$ is the tensor that projects onto the $\boldsymbol{u}$ direction, whose (i,j) component is $(\boldsymbol{E}(\boldsymbol{u}))_{i,j}=u_iu_j/|\boldsymbol{u}|^2$; d_m is the molecular diffusivity and is assumed to be strictly positive; α_l and α_t are the longitudinal and transverse dispersion respectively, and are assumed to be nonnegative. The imposed external total flow rate q is a sum of sources and sinks. That is to say, $q=q^++q^-$, where $q^+=\max(q,0)$, $q^-=\min(q,0)$. The notation $\widehat{c}$ denotes the specified c_w at source $(q>0)$ and the resident concentration at sinks $(q<0)$. It is supposed that $a(c)$, $b(c)$ and $d(c)$ are bounded.

Discontinuous Galerkin method (DG) belongs to a class of non-conforming methods (see [9, 12, 13, 15–19]) and they solve the differential equations by piecewise polynomial functions over a finite element space without any requirement on inter-element continuity-however, continuity on inter-element boundaries together with boundary conditions is weakly enforced through the bilinear form. DG methods are very attractive for practical numerical simulations because of their physical and numerical properties.

For the compressible miscible displacement problems, there are some literature about the DG approximations. In [4,6,7], a priori error for the compressible problem of dispersion-free $(\boldsymbol{D}(\boldsymbol{u})=d_m\boldsymbol{I})$ has been analysed. The authors have derived a priori error estimates of a discontinuous Galerkin approximation and a combined mixed finite element and discontinuous Galerkin method for a kind of compressible miscible displacement problems in [20, 21], respectively. But they only deal with a priori errors for the miscible displacement problem. Comparatively, the literature about a posteriori error for the miscible displacement problem is even scare. A posteriori error indicator is useful for adaptivity. A posteriori error of a discontinuous Galerkin scheme for the compressible miscible displacement problems with molecular diffusion and dispersion is presented in [22]. In this paper, a combined mixed finite element and symmetric interior penalty discontinuous Galerkin method is used to solve the completely compressible case with no restrictions on the diffusion/dispersion tensor. Based on a duality argument, employing projection estimates and approximation properties, a posteriori residual-type hp error estimates are obtained. Comparing with the error analysis of [22], the current work is more complicated and challenging.

The paper is organized as follows. In Section 2, we introduce a combined mixed finite element and discontinuous Galerkin method. Explicit a posteriori error estimates are presented in Section 3.

2 A combined mixed finite element and discontinuous Galerkin method

2.1 Notations

Let $\mathcal{T}_h$ be a family of quasi-uniform and possibly non-conforming finite element partitions of Ω composed of triangles or quadrilaterals if $d=2$, or tetrahedra, prisms or hexahedra if $d=3$. Let h_E be the diameter of the element E and Γ_h be the set of all interior edges or faces for $\mathcal{T}_h$. We denote by $h=\max_{E\in\mathcal{T}_h} h_E$ the maximal element diameter over all elements with the common edge or face $\gamma=\partial E_i\cap\partial E_j\in\Gamma_h$.

Throughout the paper, we denote by K, K_0, K_1 generic positive constants that are independent of h, r and k, but might depend on the solution of PDEs. The usual Sobolev inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|_{m,\Omega}$ on Ω are used. Similar notations are applied for the element E and face or edge γ. For the sake of convenience, we denote $\|\cdot\|=\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_g=\|\cdot\|_{0,g}$ $(g=E,\gamma)$. The notations dx and dt in $\int\cdot dx$ and $\int\cdot dt$ are omitted. That is to say, we use $\int_g\cdot$ $(g=E,\gamma,\Omega)$ and $\int_0^t\cdot$ to represent the integral in space $\int\cdot dx$ and the time integral $\int\cdot dt$, respectively. We also introduce the notation $L^I(L^I(\cdot)):=L^I(J;L^I(\cdot))$, where I maybe takes 2, ∞, etc.

For $s\geq 0$, we define the following broken Sobolev space

$$H^s(\mathcal{T}_h)=\big\{v\in L^2(\Omega):v|_E\in H^s(E),\ E\in\mathcal{T}_h\big\}.$$

Let $E_i\in\mathcal{T}_h$, $E_j\in\mathcal{T}_h$ and $\gamma=\partial E_i\cap\partial E_j\in\Gamma_h$ with $\boldsymbol{n}$ exterior to E_i. For $v\in H^s(\mathcal{T}_h)$, $s>1/2$, the average $\{v\}$ of v on γ and the jump $[v]$ of v across γ are defined as follows:

$$\{v\}=\frac{1}{2}\big((v|_{E_i})|_\gamma+(v|_{E_j})|_\gamma\big),\qquad [v]=(v|_{E_i})|_\gamma-(v|_{E_j})|_\gamma.$$

We set the discontinuous finite element space:

$$D_r(\mathcal{T}_h)\equiv\big\{v\in L^2(\Omega):v|_E\in \mathrm{P}_r(E),\ E\in\mathcal{T}_h\big\},$$

where $\mathrm{P}_r(E)$ denotes the space of polynomials of total degree less than or equal to r on E.

Next, define the spaces

$$\begin{aligned}
&V\equiv H(\mathrm{div};\Omega)\equiv\big\{\boldsymbol{u}\in(L^2(\Omega))^d:\mathrm{div}\boldsymbol{u}\in L^2(\Omega)\big\},\\
&V^0=H_0(\mathrm{div};\Omega)=\big\{\boldsymbol{u}\in H(\mathrm{div};\Omega),\ \boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega}=0\big\},\\
&W\equiv L^2(\Omega).
\end{aligned}$$

Let the approximation subspace $V_k(\mathcal{T}_h)\times W_k(\mathcal{T}_h)$ of $V\times W$ be the kth $(k\geq 0)$ order Raviart-Thomas space (RT_k) [11] of the partition $\mathcal{T}_h$. We set $V_k^0(\mathcal{T}_h)=V_k(\mathcal{T}_h)\cap V^0$.

2.2 A combined MFE/DG scheme

Define the bilinear form $B(\boldsymbol{u};c,\psi)$ and the linear functional $L(c,\psi)$ as follows:

$$\begin{aligned}B(\boldsymbol{u};c,\psi)=&\sum_{E\in\mathcal{T}_h}\int_E \boldsymbol{D}(\boldsymbol{u})\nabla c\cdot\nabla\psi-\sum_{\gamma\in\Gamma_h}\int_\gamma\{\boldsymbol{D}(\boldsymbol{u})\nabla c\cdot\boldsymbol{n}\}[\psi]\\&-\sum_{\gamma\in\Gamma_h}\int_\gamma\{\boldsymbol{D}(\boldsymbol{u})\nabla\psi\cdot\boldsymbol{n}\}[c]+\sum_{E\in\mathcal{T}_h}\int_E(\boldsymbol{u}\cdot\nabla c)\psi+J_0^\sigma(c,\psi),\\L(c,\psi)=&\int_\Omega(\hat{c}-c)q\psi.\end{aligned}$$

Here,

$$J_0^\sigma(c,\psi)=\sum_{\gamma\in\Gamma_h}\frac{r^2\sigma_\gamma}{h_\gamma}\int_\gamma[c][\psi]$$

is the interior penalty term, where σ_γ is a constant value on the edge or face γ and is bounded below by $\sigma_*>0$ and above by σ^*, h_γ denotes the size of γ.

A combined MFE/DG approximating to (1.1a)-(1.1f) which solves the flow equation by the mixed finite element method and the concentration one by a symmetric interior penalty discontinuous Galerkin method, is written as follows: finding $\boldsymbol{U}\in L^\infty(J;V_k^0(\mathcal{T}_h))$, $P\in L^\infty(J;W_k(\mathcal{T}_h))$ and $C\in L^\infty(J;D_r(\mathcal{T}_h))$, s.t.

$$\Big(d(C)\frac{\partial P}{\partial t},w\Big)+(\nabla\cdot\boldsymbol{U},w)=(q,w),\qquad \forall w\in W_k(\mathcal{T}_h),\tag{2.1a}$$

$$(\alpha(C)\boldsymbol{U},\boldsymbol{v})-(\nabla\cdot\boldsymbol{v},P)=0,\qquad \forall \boldsymbol{v}\in V_k^0(\mathcal{T}_h),\tag{2.1b}$$

$$\Big(\phi\frac{\partial C}{\partial t},\psi\Big)+\Big(b(C)\frac{\partial P}{\partial t},\psi\Big)+B(\boldsymbol{U};C,\psi)=L(C,\psi),\qquad \forall\psi\in D_r(\mathcal{T}_h),\tag{2.1c}$$

$$(C_0,\psi)=(c_0,\psi),\qquad \forall\psi\in D_r(\mathcal{T}_h),\tag{2.1d}$$

$$(\nabla\cdot\boldsymbol{v},P_0)=(\nabla\cdot\boldsymbol{v},p_0),\qquad \forall \boldsymbol{v}\in V_k^0(\mathcal{T}_h),\tag{2.1e}$$

where $\alpha(C)=1/a(C)$.

Lemma 2.1 (Consistency)**.** *If $(p, c, \boldsymbol{u})$ is the solution to the Eqs.* (1.1a)-(1.1f) *and is essentially bounded, then for $\forall t\in J$, the following equalities are satisfied*

$$\Big(d(c)\frac{\partial p}{\partial t},w\Big)+(\nabla\cdot\boldsymbol{u},w)=(q,w),\qquad \forall w\in W_k(\mathcal{T}_h),\tag{2.2a}$$

$$(\alpha(c)\boldsymbol{u},\boldsymbol{v})-(\nabla\cdot\boldsymbol{v},p)=0,\qquad \forall \boldsymbol{v}\in V_k^0(\mathcal{T}_h),\tag{2.2b}$$

$$\Big(\phi\frac{\partial c}{\partial t},\psi\Big)+\Big(b(c)\frac{\partial p}{\partial t},\psi\Big)+B(\boldsymbol{u};c,\psi)=L(c,\psi),\qquad \forall\psi\in D_r(\mathcal{T}_h).\tag{2.2c}$$

3 Error estimates of a combined MFE/DG approximation

At first, we make the following assumptions:

Assumption 3.1. Let the integers λ and ω be the regularity orders of functions c and p and they take values λ_E and ω_E on the element E, respectively. $\lambda\geq 2$ and $\omega\geq 2$. Let $(p,\boldsymbol{u},c)$ be the solution of (1.1a)-(1.1f) which satisfies the following regularity requirements: $p\in L^2(J;H^{\omega}(\mathcal{T}_h))$, $\partial p/\partial t\in L^2(J;H^{\omega-1}(\mathcal{T}_h))$, $\boldsymbol{u}\in L^2(J;H^{\omega-1}(\mathcal{T}_h))$, $c\in L^2(J;H^{\lambda}(\mathcal{T}_h))$, $\partial c/\partial t\in L^2(J;H^{\lambda-1}(\mathcal{T}_h))$, $p_0\in W_k(\mathcal{T}_h)$ and $c_0\in D_r(\mathcal{T}_h)$.

Assumption 3.2. Functions p, ∇p, c and ∇c are essentially bounded.

Then, we shall introduce the following residuals

$$
\begin{aligned}
&R_{I1}=\alpha(C)\boldsymbol{U}+\nabla P, \qquad R_{I2}=q-d(C)\frac{\partial P}{\partial t}-\nabla\cdot\boldsymbol{U},\\
&R_{I3}=q(\widehat{C}-C)-b(C)\frac{\partial P}{\partial t}-\phi\frac{\partial C}{\partial t}-\boldsymbol{U}\nabla C+\nabla\cdot\big(\boldsymbol{D}(\boldsymbol{U})\nabla C\big),\\
&R_{B0}=p_0-P_0, \qquad R_{B1}=c_0-C_0,\\
&R_{B2}=\begin{cases}[C], & x\in\Gamma_h,\\ 0, & x\in\partial\Omega,\end{cases} \qquad R_{B3}=\begin{cases}[\boldsymbol{D}(\boldsymbol{U})\nabla C\cdot\boldsymbol{n}], & x\in\Gamma_h,\\ -\boldsymbol{D}(\boldsymbol{U})\nabla C\cdot\boldsymbol{n}, & x\in\partial\Omega.\end{cases}
\end{aligned}
$$

3.1 The projection estimates and approximation properties

Two projections shall be introduced, which are crucial in establishing the error estimates for the flow equation. The first one is the standard L^2 projection $\mathcal{P}_h:H^{\omega}(\Omega)\rightarrow W_h$, defined by

$$(p-\mathcal{P}_h p,w)=0, \quad \forall w\in W_k(\mathcal{T}_h).$$

It is obvious that

$$(\nabla\cdot\boldsymbol{v},p-\mathcal{P}_h p)=0, \quad \forall\boldsymbol{v}\in V_k^0(\mathcal{T}_h).$$

The error of the projection satisfies

$$\|p-\mathcal{P}_h p\|\leq K\sum_{E\in\mathcal{T}_h}\frac{h^{\min(k+1,\omega_E)}}{k^{\omega_E}}\|p\|_{\omega_E,E}, \tag{3.1}$$

where k is the order of the RT_k spaces.

The second projection $\Pi_h:(H^{\omega-1}(\Omega))^d\rightarrow V_k^0(\mathcal{T}_h)$ is the usual Raviart-Thomas projection [3]. It is defined by

$$
\begin{aligned}
&(\nabla\cdot(\boldsymbol{u}-\Pi_h\boldsymbol{u}),w)=0, \quad \forall w\in W_k(\mathcal{T}_h),\\
&\nabla\cdot\Pi_h\boldsymbol{u}=\mathcal{P}_h\nabla\cdot\boldsymbol{u}.
\end{aligned}
$$

It preserves the discrete normal trace $((\boldsymbol{u}-\Pi_h\boldsymbol{u})\cdot\boldsymbol{n},\chi)_\gamma=0, \forall\chi\in\mathrm{P}_k(\gamma)$ and its error satisfies

$$\|\boldsymbol{u}-\Pi_h\boldsymbol{u}\|\leq K\sum_{E\in\mathcal{T}_h}\frac{h^{\min(k+1,\omega_E-1)}}{k^{\omega_E-1}}\|\boldsymbol{u}\|_{\omega_E-1,E}, \tag{3.2}$$

where k is the order of the RT_k spaces.

We assume that the following approximation properties hold, which can be proved using the techniques in [1,2]. For $E\in\mathcal{T}_h$, $c\in H^\lambda(\mathcal{T}_h)$, there exists a constant K depending on λ but independent of c, r, and h_E, and there exist $\widetilde{c}\in \mathrm{P}_r(E)$ (where $\mathrm{P}_r(E)$ denotes the polynomial of degree less than or equal to r on E), such that for $0\le j\le\lambda$ and for $\mu=\min(r+1,\lambda)$,

$$\|c-\widetilde{c}\|_{j,E}\le K\frac{h_E^{\mu-j}}{r^{\lambda-j}}\|c\|_{\lambda,E},\qquad \lambda\ge 0, \tag{3.3a}$$

$$\|c-\widetilde{c}\|_{\delta,\partial E}\le K\frac{h_E^{\mu-\delta-1/2}}{r^{\lambda-\delta-1/2}}\|c\|_{\lambda,E},\qquad \lambda>\frac{1}{2}+\delta,\quad \delta=0,1. \tag{3.3b}$$

And we shall also use the following inverse inequalities, which can be derived by the method in [10,12].

Lemma 3.1. *Let $E\in\mathcal{T}_h$, $v\in \mathrm{P}_r(E)$. Then there exists a constant K independent of v, r and h_E, such that*

$$\begin{aligned}&\|v\|_{\partial E}\le Krh_E^{-1/2}\|v\|_E,\\&\|\nabla v\cdot\boldsymbol{n}\|_{\partial E}\le Krh_E^{-1/2}\|\nabla v\|_E.\end{aligned}$$

3.2 A posteriori error estimate for the pressure in the flow equation

Denote $\boldsymbol{E_u}=\boldsymbol{u}-\boldsymbol{U}$, $E_p=p-P$, $E_c=c-C$. Subtracting (2.1a)-(2.1b) from (2.2a)-(2.2b) respectively, we get

$$\Big(d(c)\frac{\partial p}{\partial t}-d(C)\frac{\partial P}{\partial t},w\Big)+(\nabla\cdot\boldsymbol{E_u},w)=0,\qquad \forall w\in W_k(\mathcal{T}_h), \tag{3.4a}$$

$$\big(\alpha(c)\boldsymbol{u}-\alpha(C)\boldsymbol{U},\boldsymbol{v}\big)-(\nabla\cdot\boldsymbol{v},E_p)=0,\qquad \forall\boldsymbol{v}\in V_k^0(\mathcal{T}_h). \tag{3.4b}$$

Let the functions ξ and $\boldsymbol{\varphi}$ satisfy the duality problem

$$-\frac{\partial\big(d(c)\xi\big)}{\partial t}+\nabla\cdot\boldsymbol{\varphi}=E_p,\qquad (x,t)\in\Omega\times J, \tag{3.5a}$$

$$\boldsymbol{\varphi}=-a(c)\nabla\xi,\qquad (x,t)\in\Omega\times J, \tag{3.5b}$$

$$\boldsymbol{\varphi}\cdot\mathbf{n}=0,\qquad (x,t)\in\partial\Omega\times J, \tag{3.5c}$$

$$\xi(x,T)=0,\qquad x\in\Omega. \tag{3.5d}$$

For ξ and $\boldsymbol{\varphi}$, the following variational problem holds.

$$\Big(-\frac{\partial\big(d(c)\xi\big)}{\partial t},w\Big)+(\nabla\cdot\boldsymbol{\varphi},w)=(E_p,w),\qquad \forall w\in W_k(\mathcal{T}_h), \tag{3.6a}$$

$$(\alpha(c)\boldsymbol{\varphi},\boldsymbol{v})-(\xi,\nabla\cdot\boldsymbol{v})=0,\qquad \forall\boldsymbol{v}\in V_k^0(\mathcal{T}_h). \tag{3.6b}$$

Assume that the dual problem (3.5a)-(3.5d) satisfies the stability estimate

$$\max_{0\leq t\leq T}\|\xi(\cdot,t)\|^2+\int_0^T\|\xi\|^2_{H^2(\Omega)}\leq K\int_0^T\|E_p\|^2. \tag{3.7}$$

We obtain the error estimation for the pressure in the flow problem as follows.

Theorem 3.1. *Let* $(\xi,\boldsymbol{\varphi})$ *be the solution of* (3.5a)-(3.5d). *Under Assumptions* 3.1 *and* 3.2, *there exists a positive constant K independent of the mesh size h and the order k of the RT space, such that*

$$\|E_p\|^2_{L^2(L^2(\Omega))}\leq K\sum_{E\in\mathcal{T}_h}\eta_E^2+\frac{K_0}{2(K_1+1)}\sum_{E\in\mathcal{T}_h}\|E_c\|^2_{L^2(L^2(E))}, \tag{3.8}$$

where

$$\eta_E^2=\frac{h^{\min(4,2k+2)}}{k^4}\|R_{I2}\|^2_{L^2(L^2(E))}+\frac{h^{\min(4,2k+2)}}{k^4}\|R_{B0}\|^2_E+\frac{h^{\min(2,2k+2)}}{k^2}\|R_{I1}\|^2_{L^2(L^2(E))}.$$

Proof. Let $w=E_p$ in (3.6a) and $\boldsymbol{v}=\boldsymbol{E_u}$ in (3.6b). By using Eqs. (3.5a), (3.5c)-(3.5d), and the integration by parts, we get

$$\begin{aligned}
\|E_p\|^2_{L^2(L^2(\Omega))}&=\int_0^T(E_p,E_p)\\
&=\int_0^T\Big(-\frac{\partial\big(d(c)\xi\big)}{\partial t}+\nabla\cdot\boldsymbol{\varphi},E_p\Big)-\int_0^T(\alpha(c)\boldsymbol{\varphi},\boldsymbol{E_u})+\int_0^T(\xi,\nabla\cdot\boldsymbol{E_u})\\
&=\int_0^T\Big(\frac{\partial E_p}{\partial t},d(c)\xi\Big)+\Big(d(c_0)R_{B0},\xi(\cdot,0)\Big)+\int_0^T(\nabla\cdot\boldsymbol{\varphi},E_p)\\
&\quad-\int_0^T(\boldsymbol{\varphi},\alpha(c)\boldsymbol{E_u})+\int_0^T(\xi,\nabla\cdot\boldsymbol{E_u}).
\end{aligned}$$

In (3.4a)-(3.4b), take $w=\mathcal{P}_h\xi\in W_k(\mathcal{T}_h)$, $v=\Pi_h\boldsymbol{\varphi}\in V_k^0(\mathcal{T}_h)$ and combine them with the above equality to get

$$\begin{aligned}
\|E_p\|^2_{L^2(L^2(\Omega))}=&-\int_0^T\Big(\big(d(c)-d(C)\big)\frac{\partial P}{\partial t},\xi\Big)+\big(d(c_0)R_{B0},\xi(\cdot,0)\big)+\int_0^T\big(\nabla\cdot(\boldsymbol{\varphi}-\Pi_h\boldsymbol{\varphi}),E_p\big)\\
&+\int_0^T\big((\alpha(c)-\alpha(C))\boldsymbol{U},\boldsymbol{\varphi}\big)+\int_0^T\big(\nabla\cdot\boldsymbol{E_u},\xi-\mathcal{P}_h\xi\big)\\
&+\int_0^T\Big(\big(d(c)\frac{\partial p}{\partial t}-d(C)\frac{\partial P}{\partial t},\xi-\mathcal{P}_h\xi\big)-\big(\alpha(c)\boldsymbol{u}-\alpha(C)\boldsymbol{U},\boldsymbol{\varphi}-\Pi_h\boldsymbol{\varphi}\big)\Big).
\end{aligned}$$

Note that $\Pi_h\boldsymbol{\varphi}\cdot\boldsymbol{n}|_{\partial\Omega}=0$ for $\Pi_h\boldsymbol{\varphi}\in V_k^0(\mathcal{T}_h)$. Integrate by parts and use the boundary condition (3.5c) to obtain

$$\begin{aligned}\int_0^T\big(\nabla\cdot(\boldsymbol{\varphi}-\Pi_h\boldsymbol{\varphi}),E_p\big)=&\int_0^T\Big(-\sum_{E\in\mathcal{T}_h}(\boldsymbol{\varphi}-\Pi_h\boldsymbol{\varphi},\nabla E_p)_E+\sum_{\gamma\in\Gamma_h}\big((\boldsymbol{\varphi}-\Pi_h\boldsymbol{\varphi})\cdot\boldsymbol{n},[E_p]\big)_\gamma\\&+\sum_{\gamma\in\partial\Omega}\big((\boldsymbol{\varphi}-\Pi_h\boldsymbol{\varphi})\cdot\boldsymbol{n},E_p\big)_\gamma\Big)\\=&-\int_0^T\sum_{E\in\mathcal{T}_h}(\boldsymbol{\varphi}-\Pi_h\boldsymbol{\varphi},\nabla E_p)_E,\end{aligned}$$

where we have used the fact that the sum over interior edges vanishes. To see this, note first that $[p]=0$ on each interior edge. Second, $[P]$ is belonging to the space of polynomials of degree less than or equal k over the interior edge γ. Thus, $(\boldsymbol{\varphi}-\Pi_h\boldsymbol{\varphi})\cdot\boldsymbol{n}$ is orthogonal to it according to the definition of Π_h.

Recalling (2.1e) and $\nabla\cdot V_k^0(\mathcal{T}_h)=W_k(\mathcal{T}_h)$, we find $\big(R_{B0},\mathcal{P}_h\xi(\cdot,0)\big)=0$. By virtue of the residual notations and the relation $\boldsymbol{u}=-a(c)\nabla p$, we obtain

$$\begin{aligned}\|E_p\|^2_{L^2(L^2(\Omega))}=&-\int_0^T\Big(\big(d(c)-d(C)\big)\frac{\partial P}{\partial t},\xi\Big)+\big(d(c_0)R_{B0},(\xi-\mathcal{P}_h\xi)(\cdot,0)\big)\\&+\int_0^T\big((\alpha(c)-\alpha(C)\big)\boldsymbol{U},\boldsymbol{\varphi}\big)+\int_0^T\big(R_{I2},\xi-\mathcal{P}_h\xi\big)+\int_0^T\big(R_{I1},\boldsymbol{\varphi}-\Pi_h\boldsymbol{\varphi}\big).\end{aligned}\tag{3.9}$$

To bound the items on the right side of the above equation, we proceed as follows. By virtue of the projection estimates (3.1)-(3.2), the equality (3.5b), the stability estimate (3.7), and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}\int_0^T\big(\big(a(c)-a(C)\big)\boldsymbol{U},\boldsymbol{\varphi}\big)\le&K\|\boldsymbol{U}\|_{L^\infty(L^\infty(\Omega))}\int_0^T\sum_{E\in\mathcal{T}_h}\|a(c)-a(C)\|_E\cdot\sum_{E\in\mathcal{T}_h}\|\nabla\xi\|_E\\\le&\Big(\sum_{E\in\mathcal{T}_h}\|E_c\|^2_{L^2(L^2(E))}\Big)^{\frac12}\cdot\Big(\frac{K_0}{4(K_1+1)}\sum_{E\in\mathcal{T}_h}\|E_p\|^2_{L^2(L^2(E))}\Big)^{\frac12},\\\big(d(c_0)R_{B0},(\xi-\mathcal{P}_h\xi)(\cdot,0)\big)\le&K\sum_{E\in\mathcal{T}_h}\|R_{B0}\|_E\cdot\frac{h^{\min(2,k+1)}}{k^2}\sum_{E\in\mathcal{T}_h}\|\xi(\cdot,0)\|_{2,E}\\\le&K\Big(\sum_{E\in\mathcal{T}_h}\|E_p\|^2_{L^2(L^2(E))}\Big)^{\frac12}\cdot\Big(\frac{h^{\min(4,2k+2)}}{k^4}\sum_{E\in\mathcal{T}_h}\|R_{B0}\|^2_E\Big)^{\frac12},\\-\int_0^T\Big(\big(d(c)-d(C)\big)\frac{\partial P}{\partial t},\xi\Big)\le&K\Big\|\frac{\partial P}{\partial t}\Big\|_{L^\infty(L^\infty(\Omega))}\int_0^T\sum_{E\in\mathcal{T}_h}\|d(c)-d(C)\|_E\cdot\sum_{E\in\mathcal{T}_h}\|\xi\|_E\\\le&\Big(\sum_{E\in\mathcal{T}_h}\|E_c\|^2_{L^2(L^2(E))}\Big)^{\frac12}\cdot\Big(\frac{K_0}{4(K_1+1)}\sum_{E\in\mathcal{T}_h}\|E_p\|^2_{L^2(L^2(E))}\Big)^{\frac12},\end{aligned}$$

$$
\begin{aligned}
\int_0^T (R_{I2},\xi-\mathcal{P}_h\xi) &\le K\int_0^T \Big(\sum_{E\in\mathcal{T}_h}\|R_{I2}\|_E\cdot\frac{h^{\min(2,k+1)}}{k^2}\sum_{E\in\mathcal{T}_h}\|\xi\|_{2,E}\Big)\\
&\le K\Big(\sum_{E\in\mathcal{T}_h}\|E_p\|^2_{L^2(L^2(E))}\Big)^{\frac{1}{2}}\cdot\Big(\frac{h^{\min(4,2k+2)}}{k^4}\sum_{E\in\mathcal{T}_h}\|R_{I2}\|^2_{L^2(L^2(E))}\Big)^{\frac{1}{2}},\\
\int_0^T \big(R_{I1},\boldsymbol{\varphi}-\Pi_h\boldsymbol{\varphi}\big) &\le K\int_0^T \Big(\sum_{E\in\mathcal{T}_h}\|R_{I1}\|_E\cdot\frac{h^{\min(1,k+1)}}{k}\sum_{E\in\mathcal{T}_h}\|\boldsymbol{\varphi}\|_{1,E}\Big)\\
&\le K\int_0^T \Big(\sum_{E\in\mathcal{T}_h}\|R_{I1}\|_E\cdot\frac{h^{\min(1,k+1)}}{k}\sum_{E\in\mathcal{T}_h}\|\xi\|_{2,E}\Big)\\
&\le K\Big(\frac{h^{\min(2,2k+2)}}{k^2}\sum_{E\in\mathcal{T}_h}\|R_{I1}\|^2_{L^2(L^2(E))}\Big)^{\frac{1}{2}}\cdot\Big(\sum_{E\in\mathcal{T}_h}\|E_p\|^2_{L^2(L^2(E))}\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Then, combining all the above inequalities yields the estimate (3.8). $\square$

3.3 A posteriori error estimate for the velocity in the flow equation

With an additional assumption, similar to a saturation assumption, a similar estimate applies to the error of the velocity as well. This assumption is well motivated by the standard a priori estimates of the Raviart-Thomas spaces, which suggest that the pressure, the velocity, and the divergence of the velocity all converge with the same order accuracy. We describe the assumption as follows. There exists a constant K_f, independent of h and k, such that

$$\|\nabla\cdot\boldsymbol{E_u}\|\le K_f\|\boldsymbol{E_u}\|. \tag{3.10}$$

Let the functions ξ and $\boldsymbol{\varphi}$ satisfy the duality problem

$$
\begin{aligned}
&-\frac{\partial\big(d(c)\xi\big)}{\partial t}+\nabla\cdot\boldsymbol{\varphi}=0, && (x,t)\in\Omega\times J, && \text{(3.11a)}\\
&\boldsymbol{\varphi}+a(c)\nabla\xi=-\boldsymbol{E_u}, && (x,t)\in\Omega\times J, && \text{(3.11b)}\\
&\boldsymbol{\varphi}\cdot\mathbf{n}=0, && (x,t)\in\partial\Omega\times J, && \text{(3.11c)}\\
&\xi(x,T)=0, && x\in\Omega. && \text{(3.11d)}
\end{aligned}
$$

By eliminating $\boldsymbol{\varphi}$, (3.11b) can be rewritten as

$$\frac{\partial\big(d(c)\xi\big)}{\partial t}+\nabla\cdot(a(c)\nabla\xi)=-\nabla\cdot\boldsymbol{E_u}.$$

For ξ and $\boldsymbol{\varphi}$, the following variational problem holds.

$$
\begin{aligned}
&\Big(-\frac{\partial\big(d(c)\xi\big)}{\partial t},w\Big)+(\nabla\cdot\boldsymbol{\varphi},w)=0, \quad \forall w\in W_k(\mathcal{T}_h), && \text{(3.12a)}\\
&(\alpha(c)\boldsymbol{\varphi},\boldsymbol{v})-(\xi,\nabla\cdot\boldsymbol{v})=(-\boldsymbol{E_u},\boldsymbol{v}), \quad \forall\boldsymbol{v}\in V_k^0(\mathcal{T}_h). && \text{(3.12b)}
\end{aligned}
$$

Assume that the dual problem (3.11a)-(3.11d) satisfies the following stability estimate

$$\max_{0\le t\le T}\|\xi(\cdot,t)\|^2+\int_0^T\|\xi\|^2_{H^2(\Omega)}\le K\int_0^T\|\nabla\cdot\boldsymbol{E_u}\|^2.$$

Apply (3.10) to the above inequality to get

$$\max_{0\le t\le T}\|\xi(\cdot,t)\|^2+\int_0^T\|\xi\|^2_{H^2(\Omega)}\le K\int_0^T\|\boldsymbol{E_u}\|^2. \tag{3.13}$$

We obtain the error estimation for the velocity in the flow equation as follows.

Theorem 3.2. *Let* $(\xi,\boldsymbol{\varphi})$ *be the solution of* (3.11a)-(3.11d)*. Under Assumptions* 3.1 *and* 3.2*, there exists a positive constant K independent of the mesh size h and the order k of the RT space, such that*

$$\|E\boldsymbol{u}\|^2_{L^2(L^2(\Omega))}\le K\sum_{E\in\mathcal{T}_h}\eta_E^2+\frac{K_0}{2(K_1+1)}\sum_{E\in\mathcal{T}_h}\|E_c\|^2_{L^2(L^2(E))}, \tag{3.14}$$

where η_E *is the same as the one in Theorem 3.1.*

Proof. Let $w=E_p$ in (3.12a) and $\boldsymbol{v}=\boldsymbol{E_u}$ in (3.12b). By using equations (3.4a)-(3.4b), (3.11b)-(3.11d), and the integration by parts, we get

$$\begin{aligned}\|\boldsymbol{E_u}\|^2_{L^2(L^2(\Omega))}&=\int_0^T\big(-\big(\alpha(c)\boldsymbol{\varphi},\boldsymbol{E_u}\big)+\big(\xi,\nabla\cdot\boldsymbol{E_u}\big)\big)\\&=\int_0^T\big(-\big(\alpha(c)\boldsymbol{\varphi},\boldsymbol{E_u}\big)+\big(\xi,\nabla\cdot\boldsymbol{E_u}\big)\big)+\int_0^T\Big(-\frac{\partial\big(d(c)\xi\big)}{\partial t},E_p\Big)+\int_0^T(\nabla\cdot\boldsymbol{\varphi},E_p)\\&=\int_0^T\Big(\frac{\partial E_p}{\partial t},d(c)\xi\Big)+\big(d(c_0)R_{B0},\xi(\cdot,0)\big)+\int_0^T\big(\nabla\cdot\boldsymbol{\varphi},E_p\big)\\&\quad-\int_0^T(\boldsymbol{\varphi},\alpha(c)\boldsymbol{E_u})+\int_0^T\big(\xi,\nabla\cdot\boldsymbol{E_u}\big).\end{aligned}$$

Let $\mathcal{P}_h\xi\in W_k(\mathcal{T}_h)$, $\Pi_h\boldsymbol{\varphi}\in V_k^0(\mathcal{T}_h)$. Similar to the previous subsection, using the boundary condition (3.11c), we obtain

$$\begin{aligned}\|\boldsymbol{E_u}\|^2_{L^2(L^2(\Omega))}=&-\int_0^T\Big(\big(d(c)-d(C)\big)\frac{\partial P}{\partial t},\xi\Big)+\big(d(c_0)R_{B0},(\xi-\mathcal{P}_h\xi)(\cdot,0)\big)\\&+\int_0^T\big((\alpha(c)-\alpha(C))\boldsymbol{U},\boldsymbol{\varphi}\big)+\int_0^T\big(R_{I2},\xi-\mathcal{P}_h\xi\big)+\int_0^T\big(R_{I1},\boldsymbol{\varphi}-\Pi_h\boldsymbol{\varphi}\big).\end{aligned}$$

Note that the items on the right side of the above equation is exactly the same as the ones on the right-hand side of (3.9). Similar to the previous subsection, using the projection estimates (3.1)-(3.2), the equality (3.11b), the stability estimate (3.13) and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}\|\boldsymbol{E_u}\|^2_{L^2(L^2(\Omega))}\le&\Big(\Big(\frac{K_0}{4(K_1+1)}\sum_{E\in\mathcal{T}_h}\|E_c\|^2_{L^2(L^2(E))}\Big)^{\frac12}+\Big(\frac{h^{\min(4,2k+2)}}{k^4}\sum_{E\in\mathcal{T}_h}\|R_{B0}\|^2_E\Big)^{\frac12}\\&+\Big(\frac{K_0}{4(K_1+1)}\sum_{E\in\mathcal{T}_h}\|E_c\|^2_{L^2(L^2(E))}\Big)^{\frac12}+\Big(\frac{h^{\min(4,2k+2)}}{k^4}\sum_{E\in\mathcal{T}_h}\|R_{I2}\|^2_{L^2(L^2(E))}\Big)^{\frac12}\\&+\Big(\frac{h^{\min(2,2k+2)}}{k^2}\sum_{E\in\mathcal{T}_h}\|R_{I1}\|^2_{L^2(L^2(E))}\Big)^{\frac12}\Big)\cdot\|\boldsymbol{E_u}\|_{L^2(L^2(\Omega))}.\end{aligned}$$

The desired result (3.14) is obtained. ◻

3.4 A posteriori error estimate for the transport equation

Subtract (2.1c) from (2.2c) to get

$$\begin{aligned}&\Big(\phi\frac{\partial(c-C)}{\partial t},\psi\Big)+\Big(b(c)\frac{\partial p}{\partial t},\psi\Big)-\Big(b(C)\frac{\partial P}{\partial t},\psi\Big)+\sum_{E\in\mathcal{T}_h}\int_E\boldsymbol{D}(\boldsymbol{u})\nabla c\cdot\nabla\psi\\&\quad-\sum_{\gamma\in\Gamma_h}\int_\gamma\{\boldsymbol{D}(\boldsymbol{u})\nabla c\cdot\boldsymbol{n}\}[\psi]-\sum_{\gamma\in\Gamma_h}\int_\gamma\{\boldsymbol{D}(\boldsymbol{u})\nabla\psi\cdot\boldsymbol{n}\}[c]+\sum_{E\in\mathcal{T}_h}\int_E(\boldsymbol{u}\cdot\nabla c)\psi\\&\quad+\sum_{\gamma\in\Gamma_h}\frac{r^2\sigma_\gamma}{h_\gamma}\int_\gamma[c][\psi]-\sum_{E\in\mathcal{T}_h}\int_E\boldsymbol{D}(\boldsymbol{U})\nabla C\cdot\nabla\psi+\sum_{\gamma\in\Gamma_h}\int_\gamma\{\boldsymbol{D}(\boldsymbol{U})\nabla C\cdot\boldsymbol{n}\}[\psi]\\&\quad+\sum_{\gamma\in\Gamma_h}\int_\gamma\{\boldsymbol{D}(\boldsymbol{U})\nabla\psi\cdot\boldsymbol{n}\}[C]-\sum_{E\in\mathcal{T}_h}\int_E(\boldsymbol{U}\cdot\nabla C)\psi-\sum_{\gamma\in\Gamma_h}\frac{r^2\sigma_\gamma}{h_\gamma}\int_\gamma[C][\psi]\\&=\int_\Omega\big((\widehat{c}-c)-(\widehat{C}-C)\big)q\psi,\qquad\forall\psi\in D_r(\mathcal{T}_h),\quad t\in J.\end{aligned}\tag{3.15}$$

Let ζ satisfy the duality problem

$$\phi\frac{\partial\zeta}{\partial t}+\nabla\cdot(\boldsymbol{u}\zeta)+\nabla\cdot(\boldsymbol{D}(\boldsymbol{u})^T\nabla\zeta)-\Big(g\frac{\partial P}{\partial t}+q^+\Big)\zeta=E_c,\quad(x,t)\in\Omega\times J,\tag{3.16a}$$

$$\boldsymbol{D}(\boldsymbol{u})^T\nabla\zeta\cdot\boldsymbol{n}=0,\quad(x,t)\in\partial\Omega\times J,\tag{3.16b}$$

$$\zeta(x,T)=0,\quad x\in\Omega,\tag{3.16c}$$

where the function g is defined by

$$g(x,t)=\begin{cases}\dfrac{b(c)-b(C)}{c-C}, & \text{if } c-C\neq0,\\ 0, & \text{if } c-C=0.\end{cases}$$

Assume that the dual problem (3.16a)-(3.16c) satisfies the stability estimate

$$\max_{0\le t\le T}\|\zeta(\cdot,t)\|^2+\int_0^T\|\zeta\|^2_{H^2(\Omega)}\le K\int_0^T\|E_c\|^2. \tag{3.17}$$

We derive the error estimation for the transport problem as follows.

Theorem 3.3. *Let* ζ *be the solution of* (3.16a)-(3.16c). *Under Assumptions* 3.1 *and* 3.2, *there exists a positive constant* K *independent of the mesh size* h *and the order* r *of the discontinuous finite element space, such that*

$$\|E_c\|^2_{L^2(L^2(\Omega))}\le K\sum_{E\in\mathcal{T}_h}\breve{\eta}_E^2+\frac{K_1}{K_0+1}\sum_{E\in\mathcal{T}_h}\|E\boldsymbol{u}\|^2_{L^2(L^2(E))}+\frac{K_1}{K_0+1}\sum_{E\in\mathcal{T}_h}\|E_p\|^2_{L^2(L^2(E))}, \tag{3.18}$$

where

$$\begin{aligned}\breve{\eta}_E^2=&\frac{h^4}{r^4}\|R_{I3}\|^2_{L^2(L^2(E))}+\frac{h^4}{r^4}\|R_{B2}\|^2_E+\frac{h^3}{r^3}\sum_{\gamma\in\partial E}\|R_{B3}\|^2_{L^2(L^2(\gamma))}\\&+\frac{h^2}{r^2}\|\boldsymbol{D}(\boldsymbol{U})\|^2_{L^\infty(L^\infty(\Omega))}\sum_{\gamma\in\partial E}\|R_{B2}\|^2_{L^2(L^2(\gamma))}+\|R_{B0}\|^2_E.\end{aligned}$$

Proof. By using the Eqs. (3.16a)-(3.16c) and the integration by parts, we get

$$\begin{aligned}\|E_c\|^2_{L^2(L^2(\Omega))}=&\int_0^T(E_c,E_c)\\=&\int_0^T\Big(\phi\frac{\partial\zeta}{\partial t}+\nabla\cdot(\boldsymbol{u}\zeta)+\nabla\cdot(\boldsymbol{D}(\boldsymbol{u})^T\nabla\zeta)-\Big(g\frac{\partial P}{\partial t}+q^+\Big)\zeta,E_c\Big)\\=&-\int_0^T\Big(\phi\frac{\partial E_c}{\partial t},\zeta\Big)-(\phi R_{B2},\zeta(\cdot,0))+\int_0^T(E_c,\nabla\cdot(\boldsymbol{u}\zeta))\\&-\int_0^T\Big(\Big(g\frac{\partial P}{\partial t}+q^+\Big)\zeta,E_c\Big)+\int_0^T\sum_{\gamma\in\Gamma_h}(\{\boldsymbol{D}(\boldsymbol{u})^T\nabla\zeta\cdot\boldsymbol{n}\},[E_c])_\gamma\\&-\int_0^T(\nabla\zeta,\boldsymbol{D}(\boldsymbol{u})\nabla E_c).\end{aligned} \tag{3.19}$$

Note that [8]

$$(\hat{c}-c)-(\hat{C}-C)=\begin{cases}-E_c, & \text{if } q>0,\\ 0, & \text{if } q<0.\end{cases}$$

Since $[c]=0$, taking $\psi=\tilde{\zeta}\in D_r(\mathcal{T}_h)\cap C^0(\Omega)$ in (3.15) and adding it to (3.19), we obtain

$$\begin{aligned}\|E_c\|^2_{L^2(L^2(\Omega))}=&-\int_0^T\Big(\phi\frac{\partial E_c}{\partial t},\zeta-\tilde{\zeta}\Big)-(\phi R_{B2},(\zeta-\tilde{\zeta})(\cdot,0))-\int_0^T(E_c,q^+(\zeta-\tilde{\zeta}))\\&+\int_0^T\sum_{E\in\mathcal{T}_h}((\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{U}))\nabla C,\nabla\zeta)_E-\int_0^T\Big(b(c)\frac{\partial p}{\partial t}-b(C)\frac{\partial P}{\partial t},(\zeta-\tilde{\zeta})\Big)\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T \sum_{E\in\mathcal{T}_h} \big(\boldsymbol{D}(\boldsymbol{u})\nabla c-\boldsymbol{D}(\boldsymbol{U})\nabla C,\nabla(\zeta-\tilde{\zeta})\big)_E+\int_0^T \Big(b(C)\frac{\partial E_p}{\partial t},\zeta\Big)\\
&+\int_0^T \sum_{\gamma\in\Gamma_h} (\{\boldsymbol{D}(\boldsymbol{u})^T\nabla\zeta\cdot\boldsymbol{n}-\boldsymbol{D}(\boldsymbol{U})\nabla\tilde{\zeta}\cdot\boldsymbol{n}\},[C])_\gamma\\
&+\int_0^T \sum_{E\in\mathcal{T}_h}\int_E(\boldsymbol{u}\cdot\nabla c-\boldsymbol{U}\cdot\nabla C,\tilde{\zeta})+\int_0^T \big(E_c,\nabla\cdot(\boldsymbol{u}\zeta)\big),
\end{aligned}
$$

where the identical relation $(c_0-C_0,\tilde{\zeta}(\cdot,0))=0$, $\forall\tilde{\zeta}\in D_r(\mathcal{T}_h)$ according to (2.1d) is used. Note that $\boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega}=0$. Using the integration by parts, we have

$$
\begin{aligned}
&\int_0^T \big(E_c,\nabla\cdot(\boldsymbol{u}\zeta)\big)=-\int_0^T\big(\nabla E_c,\boldsymbol{u}\zeta\big)+\int_0^T\sum_{\gamma\in\Gamma_h}(\{\zeta\boldsymbol{u}\cdot\boldsymbol{n}\},[E_c])_\gamma\\
=&-\int_0^T\big(\boldsymbol{u}\cdot\nabla c-\boldsymbol{U}\cdot\nabla C,\zeta\big)+\int_0^T\big((\boldsymbol{u}-\boldsymbol{U})\nabla C,\zeta\big)-\int_0^T\sum_{\gamma\in\Gamma_h}(\{\zeta\boldsymbol{u}\cdot\boldsymbol{n}\},R_{B2})_\gamma,\\
&-\int_0^T\sum_{E\in\mathcal{T}_h}\big(\boldsymbol{D}(\boldsymbol{u})\nabla c-\boldsymbol{D}(\boldsymbol{U})\nabla C,\nabla(\zeta-\tilde{\zeta})\big)_E\\
=&\int_0^T\sum_{E\in\mathcal{T}_h}\big(\nabla\cdot\big(\boldsymbol{D}(\boldsymbol{u})\nabla c-\boldsymbol{D}(\boldsymbol{U})\nabla C\big),\zeta-\tilde{\zeta}\big)_E+\int_0^T\sum_{\gamma\in\Gamma_h\cup\partial\Omega}\big(\boldsymbol{D}(\boldsymbol{U})\nabla C\cdot\boldsymbol{n},\zeta-\tilde{\zeta}\big)_\gamma,
\end{aligned}
$$

where for the last step, we have used the boundary condition (1.1d) and the fact that $[\boldsymbol{D}(\boldsymbol{u})\nabla c\cdot\boldsymbol{n}]=0$ because of the regularity of c. So,

$$
\begin{aligned}
\|E_c\|^2_{L^2(L^2(\Omega))}=&-\int_0^T(\check{R}_{I3},\zeta-\tilde{\zeta})-\big(\phi R_{B2},(\zeta-\tilde{\zeta})(\cdot,0)\big)+\int_0^T\sum_{\gamma\in\Gamma_h\cup\partial\Omega}\big(R_{B3},\zeta-\tilde{\zeta}\big)_\gamma\\
&+\int_0^T\big((\boldsymbol{u}-\boldsymbol{U})\nabla C,\zeta\big)+\int_0^T\sum_{E\in\mathcal{T}_h}\big(\big(\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{U})\big)\nabla C,\nabla\zeta)_E\\
&+\int_0^T\Big(b(C)\frac{\partial E_p}{\partial t},\zeta\Big)-\int_0^T\sum_{\gamma\in\Gamma_h}(\zeta\{\boldsymbol{u}\cdot\boldsymbol{n}\},R_{B2})_\gamma\\
&+\int_0^T\sum_{\gamma\in\Gamma_h}\big(\{\boldsymbol{D}(\boldsymbol{u})^T\nabla\zeta\cdot\boldsymbol{n}-\boldsymbol{D}(\boldsymbol{U})\nabla\tilde{\zeta}\cdot\boldsymbol{n}\},R_{B2}\big)_\gamma.
\end{aligned}
$$

Next, we bound the items on the right side of the above equality. By virtue of the approximation properties (3.3a)-(3.3b), the stability estimate (3.17) and the Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
-\int_0^T(R_{I3},\zeta-\tilde{\zeta})\le& K\int_0^T\Big(\frac{h^2}{r^2}\sum_{E\in\mathcal{T}_h}\|\zeta\|_{2,E}\cdot\sum_{E\in\mathcal{T}_h}\|R_{I3}\|_E\Big)\\
\le& K\Big(\sum_{E\in\mathcal{T}_h}\|E_c\|^2_{L^2(L^2(E))}\Big)^{\frac12}\cdot\Big(\frac{h^4}{r^4}\sum_{E\in\mathcal{T}_h}\|R_{I3}\|^2_{L^2(L^2(E))}\Big)^{\frac12},
\end{aligned}
$$

$$
\begin{aligned}
-\big(\phi R_{B2},(\zeta-\tilde{\zeta})(\cdot,0)\big) &\le K\frac{h^2}{r^2}\sum_{E\in\mathcal{T}_h}\|\zeta(\cdot,0)\|_{2,E}\cdot\sum_{E\in\mathcal{T}_h}\|R_{B2}\|_E \\
&\le K\Big(\sum_{E\in\mathcal{T}_h}\|E_c\|^2_{L^2(L^2(E))}\Big)^{\frac{1}{2}}\Big(\frac{h^4}{r^4}\sum_{E\in\mathcal{T}_h}\|R_{B2}\|^2_E\Big)^{\frac{1}{2}}, \\
\int_0^T\sum_{\gamma\in\Gamma_h\cup\partial\Omega}(R_{B3},\zeta-\tilde{\zeta})_\gamma &\le K\int_0^T\Big(\frac{h^{3/2}}{r^{3/2}}\sum_{E\in\mathcal{T}_h}\|\zeta\|_{2,E}\cdot\sum_{\gamma\in\Gamma_h\cup\partial\Omega}\|R_{B3}\|_\gamma\Big) \\
&\le K\|E_c\|_{L^2(L^2(\Omega))}\cdot\Big(\frac{h^3}{r^3}\sum_{\gamma\in\Gamma_h\cup\partial\Omega}\|R_{B3}\|^2_{L^2(L^2(\gamma))}\Big)^{\frac{1}{2}}.
\end{aligned}
$$

We bound the fourth term and the fifth term based on the definition of $\boldsymbol{D}(\boldsymbol{u})$ and the stability estimate (3.17) as follows.

$$
\begin{aligned}
&\int_0^T\big((\boldsymbol{u}-\boldsymbol{U})\nabla C,\zeta\big) \\
\le&K\|\nabla C\|_{L^\infty(L^\infty(\Omega))}\int_0^T\sum_{E\in\mathcal{T}_h}\|\boldsymbol{u}-\boldsymbol{U}\|_E\cdot\sum_{E\in\mathcal{T}_h}\|\zeta\|_E \\
\le&\frac{K_1}{3(K_0+1)}\Big(\sum_{E\in\mathcal{T}_h}\|E_{\boldsymbol{u}}\|^2_{L^2(L^2(E))}\Big)^{\frac{1}{2}}\cdot\Big(\sum_{E\in\mathcal{T}_h}\|E_c\|^2_{L^2(L^2(E))}\Big)^{\frac{1}{2}}, \\
&\int_0^T\sum_{E\in\mathcal{T}_h}\big(\big(\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{U})\big)\nabla C,\nabla\zeta\big)_E \\
\le&K\|\nabla C\|_{L^\infty(L^\infty(\Omega))}\int_0^T\sum_{E\in\mathcal{T}_h}\|\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{U})\|_E\cdot\sum_{E\in\mathcal{T}_h}\|\nabla\zeta\|_E \\
\le&\frac{K_1}{3(K_0+1)}\Big(\sum_{E\in\mathcal{T}_h}\|E_{\boldsymbol{u}}\|^2_{L^2(L^2(E))}\Big)^{\frac{1}{2}}\cdot\Big(\sum_{E\in\mathcal{T}_h}\|E_c\|^2_{L^2(L^2(E))}\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Using the stability estimate (3.17), we proceed to bound the sixth item and the seventh item

$$
\begin{aligned}
&\int_0^T\Big(b(C)\frac{\partial E_p}{\partial t},\zeta\Big) \\
=&-\big(b(C_0)E_p(\cdot,0)\zeta(\cdot,0)\big)-\int_0^T\Big(\frac{\partial\big(b(C)\zeta\big)}{\partial t},E_p\Big) \\
=&-\big(b(C_0)(p_0-P_0)\zeta(\cdot,0)\big)-\int_0^T\Big(\frac{\partial b(C)}{\partial C}\frac{\partial C}{\partial t}\zeta,E_p\Big)-\int_0^T\Big(b(C)\frac{\partial\zeta}{\partial t},E_p\Big) \\
\le&K\sum_{E\in\mathcal{T}_h}\|R_{B0}\|_E\cdot\sum_{E\in\mathcal{T}_h}\|\zeta(\cdot,0)\|_E+K\int_0^T\sum_{E\in\mathcal{T}_h}\|E_p\|_E\cdot\Big(\sum_{E\in\mathcal{T}_h}\|\zeta\|_E+\sum_{E\in\mathcal{T}_h}\Big\|\frac{\partial\zeta}{\partial t}\Big\|_E\Big) \\
\le&\bigg(K\Big(\sum_{E\in\mathcal{T}_h}\|R_{B0}\|^2_E\Big)^{\frac{1}{2}}+\frac{K_1}{K_0+1}\Big(\sum_{E\in\mathcal{T}_h}\|E_p\|^2_{L^2(L^2(E))}\Big)^{\frac{1}{2}}\bigg)\cdot\Big(\sum_{E\in\mathcal{T}_h}\|E_c\|^2_{L^2(L^2(E))}\Big)^{\frac{1}{2}},
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^T \sum_{\gamma\in\Gamma_h} (\zeta\{\boldsymbol{u}\cdot\boldsymbol{n}\},R_{B2})_\gamma \\
\le& K\|\boldsymbol{u}\|_{L^\infty(L^\infty(\Omega))}\int_0^T \sum_{\gamma\in\Gamma_h}\|R_{B2}\|_\gamma\cdot\sum_{E\in\mathcal{T}_h}\|\zeta\|_{1,E} \\
\le& K\Big(\sum_{\gamma\in\Gamma_h}\|R_{B2}\|^2_{L^2(L^2(\gamma))}\Big)^{\frac{1}{2}}\cdot\Big(\sum_{E\in\mathcal{T}_h}\|E_c\|^2_{L^2(L^2(E))}\Big)^{\frac{1}{2}}.
\end{aligned}
$$

For the eighth item, due to the approximation properties (3.3a)-(3.3b), the stability property (3.17), the inverse inequality and the definition of $\boldsymbol{D}(\boldsymbol{u})$, we have

$$
\begin{aligned}
&\int_0^T \sum_{\gamma\in\Gamma_h}\big(\{\boldsymbol{D}(\boldsymbol{u})^T\nabla\zeta\cdot\boldsymbol{n}-\boldsymbol{D}(\boldsymbol{U})\nabla\tilde{\zeta}\cdot\boldsymbol{n}\},R_{B2}\big)_\gamma \\
=&\int_0^T \sum_{\gamma\in\Gamma_h}\big(R_{B2},\{\big(\boldsymbol{D}(\boldsymbol{u})^T-\boldsymbol{D}(\boldsymbol{U})\big)\nabla\zeta\cdot\boldsymbol{n}\}+\{\boldsymbol{D}(\boldsymbol{U})\nabla(\zeta-\tilde{\zeta})\cdot\boldsymbol{n}\}\big)_\gamma \\
\le& K\|C\|_{L^\infty(L^\infty(\Omega))}\int_0^T \sum_{\gamma\in\Gamma_h}\big\|\boldsymbol{u}-\boldsymbol{U}\big\|_\gamma\cdot\sum_{\gamma\in\Gamma_h}\big\|\nabla\zeta\cdot\boldsymbol{n}\big\|_\gamma \\
&\quad+K\|\boldsymbol{D}(\boldsymbol{U})\|_{L^\infty(L^\infty(\Omega))}\int_0^T \sum_{\gamma\in\Gamma_h}\big\|R_{B2}\big\|_\gamma\cdot\sum_{\gamma\in\Gamma_h}\big\|\nabla(\zeta-\tilde{\zeta})\cdot\boldsymbol{n}\big\|_\gamma \\
\le& K\|C\|_{L^\infty(L^\infty(\Omega))}\int_0^T \sum_{E\in\mathcal{T}_h}\big\|\boldsymbol{E_u}\big\|_E\cdot\sum_{E\in\mathcal{T}_h}\big\|\zeta\big\|_{2,E} \\
&\quad+K\|\boldsymbol{D}(\boldsymbol{U})\|_{L^\infty(L^\infty(\Omega))}\int_0^T \sum_{\gamma\in\Gamma_h}\big\|R_{B2}\big\|_\gamma\cdot\frac{h}{r}\sum_{E\in\mathcal{T}_h}\big\|\zeta\big\|_{2,E} \\
\le&\frac{K_1}{3(K_0+1)}\Big(\sum_{E\in\mathcal{T}_h}\|\boldsymbol{E_u}\|^2_{L^2(L^2(E))}\Big)^{\frac{1}{2}}\Big(\sum_{E\in\mathcal{T}_h}\|E_c\|^2_{L^2(L^2(E))}\Big)^{\frac{1}{2}} \\
&\quad+K\frac{h}{r}\|\boldsymbol{D}(\boldsymbol{U})\|_{L^\infty(L^\infty(\Omega))}\Big(\sum_{\gamma\in\Gamma_h}\|R_{B2}\|^2_{L^2(L^2(\gamma))}\Big)^{\frac{1}{2}}\Big(\sum_{E\in\mathcal{T}_h}\|E_c\|^2_{L^2(L^2(E))}\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Collecting all the above estimates, we see that (3.18) is satisfied. □

3.5 A posteriori error estimate for the coupled system

Finally, the a posteriori error estimate for the coupled system is achieved.

Theorem 3.4. *Under Assumptions* 3.1 *and* 3.2, *there exists a positive constant K independent of the mesh size h, the order k of the RT space and the order r of the discontinuous finite element space, such that*

$$
\|E_c\|^2_{L^2(L^2(\Omega))}+\|E_p\|^2_{L^2(L^2(\Omega))}+\|\boldsymbol{E_u}\|^2_{L^2(L^2(\Omega))}\le K\sum_{E\in\mathcal{T}_h}\dot{\eta}^2_E,
$$

where

$$\begin{aligned}\dot{\eta}_E^2 =& \frac{h^4}{r^4}\|R_{I3}\|^2_{L^2(L^2(E))} + \frac{h^4}{r^4}\|R_{B2}\|^2_E + \frac{h^3}{r^3}\sum_{\gamma\in\partial E}\|R_{B3}\|^2_{L^2(L^2(\gamma))} \\ &+ \frac{h^2}{r^2}\|\boldsymbol{D}(\boldsymbol{U})\|^2_{L^\infty(L^\infty(\Omega))}\sum_{\gamma\in\partial E}\|R_{B1}\|^2_{L^2(L^2(\gamma))} + \frac{h^{\min(4,2k+2)}}{k^4}\|R_{I2}\|^2_{L^2(L^2(E))} \\ &+ \frac{h^{\min(4,2k+2)}}{k^4}\|R_{B0}\|^2_E + \frac{h^{\min(2,2k+2)}}{k^2}\|R_{I1}\|^2_{L^2(L^2(E))}.\end{aligned}$$

Proof. Multiply (3.8), (3.14) and (3.18) with K_1+1, K_1+1, and K_0+1, respectively. Then, put them together to yield the desired inequality, which completes the proof. □

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