Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **5**, No. 2, pp. 222-234

DOI: 10.4208/aamm.11-m1155
April 2013

Toward a New Algorithm for Nonlinear Fractional Differential Equations

Fadi Awawdeh[1] and S. Abbasbandy[2,*]

[1] *Department of Mathematics, Hashemite University, Jordan*
[2] *Department of Mathematics, Imam Khomeini International University, Ghazvin 34149-16818, Iran*

Received 14 March 2011; Accepted (in revised version) 16 October 2012

Available online 8 March 2013

Abstract. This paper is concerned with the development of an efficient algorithm for the analytic solutions of nonlinear fractional differential equations. The proposed algorithm Laplace homotopy analysis method (LHAM) is a combined form of the Laplace transform method with the homotopy analysis method. The biggest advantage the LHAM has over the existing standard analytical techniques is that it overcomes the difficulty arising in calculating complicated terms. Moreover, the solution procedure is easier, more effective and straightforward. Numerical examples are examined to demonstrate the accuracy and efficiency of the proposed algorithm.

AMS subject classifications: 34A08, 74G10, 35C10, 44A10

Key words: Homotopy analysis method, Laplace transform, fractional differential equations.

---

1 Introduction

Fractional calculus is an emerging field and over the last decades it has represented an alternative tool to solve several problems from various fields. Interest in the differentiation and integration of non-integer orders dates back to the nineteenth century. Nowadays, fractional calculus is used to model various phenomena in physics, materials science, control theory, polymer modelling and engineering, such as the rheological behavior of viscoelastic materials, heat transfer and diffusion [1,2].

The increasing use of fractional differential equations in mathematical models motivates the desire to develop good quality numerical and analytical methods for their solution. Most nonlinear fractional differential equations do not have exact analytic solutions; therefore, approximation and analytical techniques must be used.

*Corresponding author.
Email: awawdeh@hu.edu.jo (F. Awawdeh), abbasbandy@yahoo.com (S. Abbasbandy)

The variational iteration method (VIM) [3, 4], Adomian decomposition method (ADM) [5], homotopy perturbation method (HPM) [6] and homotopy analysis method (HAM) [7–16] are relatively new efficient approaches to provide an analytical approximation to linear and nonlinear problems. In recent years, the application of these methods has been extended to obtain an analytical approximate solution to differential equations of fractional order [3, 4, 17–21].

The ADM and VIM are limited in that the former has complicated algorithms in calculating Adomian polynomials for nonlinear problems, and the latter has an inherent inaccuracy in identifying the Lagrange multiplier for fractional operators, which is necessary for constructing variational iteration formula. The HPM is indeed a special case of the homotopy analysis method [22]. However, mostly, the results given by HPM converge to the corresponding numerical solutions in a rather small region. Although the HAM provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameter $\hbar$, we face the difficulty in calculating complicated integrals that arise when dealing with strongly nonlinear problems.

Therefore, in this work we will introduce a new alternative procedure to eliminate these disadvantages in solving nonlinear fractional differential equations. The newly developed technique by no means depends on complicated tools from any field. This can be the most important advantage over other methods. It is worth mentioning that the proposed algorithm is an elegant combination of the Laplace transform method and the homotopy analysis method. Some nonlinear fractional differential equations are examined to illustrate the effectiveness, accuracy and convenience of this method, and in all cases, the presented technique performed excellently.

2 Analysis of the new algorithm

The homotopy analysis method (HAM) is a general analytic approach to get series solutions of various types of nonlinear equations. The validity of the HAM is that it provides a simple way to adjust and control the convergence of solution series and provides great freedom to choose proper base functions to approximate a nonlinear problem. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques so that it provides a possibility to analyze strongly nonlinear problems.

In this section, we present a modified algorithm of the homotopy analysis method with the help of Laplace transform. This algorithm can be implemented to handle, in a realistic and efficient way, nonlinear fractional differential equations. This new modification improves the accuracy of applying the HAM directly and facilitates the computational work.

To illustrate the basic ideas of the new algorithm, we consider the following nonlinear differential equation of fractional order (more general form can be considered without loss of generality):

$$D^{\alpha}u(t)=\mathcal{N}(u(t))+g(t), \quad t>0, \tag{2.1}$$

with the initial conditions

$$u^{(k)}(0)=c_k, \quad k=0,1,2,\cdots,n-1, \tag{2.2}$$

where n is an integer that satisfies $n-1<\alpha\le n$, $\mathcal{N}$ is a nonlinear operator which might include other fractional derivatives of order less than α, $g(t)$ is a known analytic function and D^{α} is the Caputo fractional derivative of order α. For details about fractional derivatives in the Caputo sense see [2].

To solve the nonlinear fractional differential Eq. (2.1) by using the Laplace transform method, we recall that Laplace transforms of the derivatives of $u(t)$ are defined by

$$\mathcal{L}[D^{\alpha}u(t)]=s^{\alpha}U(s)-\sum_{k=0}^{n-1}s^{\alpha-k-1}u^{(k)}(0+).$$

Taking the Laplace transform of both sides of Eq. (2.1), we get:

$$\mathcal{L}[D^{\alpha}u(t)]=\mathcal{L}[\mathcal{N}(u(t))]+\mathcal{L}[g(t)].$$

This can be written as

$$s^{\alpha}U(s)-\sum_{k=0}^{n-1}s^{\alpha-k-1}u^{(k)}(0+)=\mathcal{L}[\mathcal{N}(u(t))]+\mathcal{L}[g(t)]$$

or equivalently

$$U(s)=\sum_{k=0}^{n-1}s^{-(k+1)}u^{(k)}(0+)+\frac{1}{s^{\alpha}}\mathcal{L}[\mathcal{N}(u(t))]+\frac{1}{s^{\alpha}}\mathcal{L}[g(t)], \tag{2.3}$$

where $U(s)=\mathcal{L}[u(t)]$. Applying the inverse Laplace transform to both sides of Eq. (2.3) gives

$$u(t)=\sum_{k=0}^{n-1}\frac{u^{(k)}(0+)}{k!}t^k+\mathcal{L}^{-1}\Big[\frac{1}{s^{\alpha}}\mathcal{L}[g(t)]\Big]+\mathcal{L}^{-1}\Big[\frac{1}{s^{\alpha}}\mathcal{L}[\mathcal{N}(u(t))]\Big]. \tag{2.4}$$

The homotopy analysis method can be used to handle Eq. (2.4) and to address the nonlinear term $\mathcal{N}(u(t))$. Our method based on representing the solution $u(t)$ by an infinite series whose components will be determined recursively.

It is reasonable to express the solution using the set of base functions

$$\Big\{t^{k_1\alpha+k_2}:\ k_1,k_2\in\mathbb{N}\Big\}. \tag{2.5}$$

In view of the homotopy technique, we can construct the following homotopy for Eq. (2.1):

$$(1-q)L[\phi(t;q)-u_0(t)]=q\hbar H(t)(D^{\alpha}\phi(t;q)-\mathcal{N}(\phi(t;q))-g(t)), \tag{2.6}$$

where $q\in[0,1]$ is the embedding parameter, $\hbar$ is a non zero auxiliary parameter, $H(t)\neq 0$ is an auxiliary function, $u_0(t)$ is an initial guess of $u(t)$ and L is an auxiliary linear operator. When $q=0$, Eq. (2.6) becomes

$$L[\phi(t;0)-u_0(t)]=0$$

and when $q=1$, Eq. (2.6) becomes the original nonlinear Eq. (2.1). Thus as q varies from 0 to 1, the solution $\phi(t;q)$ varies from the initial guess $u_0(t)$ to the solution $u(t)$.

Assume that $\phi(t;q)$ is analytic in $q\in[0,1]$ so that $\phi(t;q)$ can be expanded in Maclaurin's series of q as follows

$$\phi(t;q)=u_0(t)+\sum_{m=1}^{\infty}q^m u_m(t), \tag{2.7}$$

where

$$u_m(t)=\frac{1}{m!}\frac{\partial^m\phi(t;q)}{\partial q^m}\Big|_{q=0}.$$

If the auxiliary linear operator L, the nonzero auxiliary parameter $\hbar$ and the auxiliary function $H(t)$ are properly chosen so that the power series (2.7) of $\phi(t;q)$ converges at $q=1$. Then, we have under these assumptions the so-called homotopy series solution

$$u(t)=\sum_{m=0}^{\infty}u_m(t) \tag{2.8}$$

and the solution at nth-order approximation is given by

$$u(t)\approx\sum_{m=0}^{n}u_m(t).$$

Define the vector

$$\vec{u}_n=\{u_0(t),u_1(t),\cdots,u_n(t)\}.$$

Differentiating Eq. (2.6) m times with respect to the embedding parameter q, then setting $q=0$ and dividing them by $m!$, we obtain the high-order deformation equations

$$L[u_m(t)-\chi_m u_{m-1}(t)]=\hbar H(t)\Re_m(\vec{u}_{m-1}),\quad m\geq 1,$$

where

$$\Re_m(\vec{u}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(t;q)]}{\partial q^{m-1}}\Big|_{q=0} \tag{2.9}$$

and

$$\chi_m=\begin{cases}0, & m\leq 1,\\ 1, & m>1.\end{cases}$$

The initial guess $u_0(t)$ will be chosen so that it satisfies the ICs (2.2). Therefore, it is reasonable that

$$u_m(0)=0\quad\text{for}\ \ m\geq 1.$$

Taking $H(t)=1$. The auxiliary linear operator L may be defined as the identity operator and the initial guess will be chosen as:

$$u_0(t)=\sum_{k=0}^{n-1}\frac{u^{(k)}(0+)}{k!}t^k+\mathcal{L}^{-1}\Big[\frac{1}{s^\alpha}\mathcal{L}[g(t)]\Big]. \tag{2.10}$$

In Eq. (2.10), it is possible that the term $\mathcal{L}^{-1}[\mathcal{L}[g(t)]/s^\alpha]$ disobey the solution expression (2.5). In this case, since $g(t)$ is assumed to be analytic, we expanded $g(t)$ in Maclaurin's series.

Then

$$u_1(t)=-\hbar\Big(\mathcal{L}^{-1}\Big[\frac{1}{s^\alpha}\mathcal{L}[\mathcal{N}(u_0(t))]\Big]\Big) \tag{2.11}$$

and for $m\geq 2$,

$$u_m(t)=(1+\hbar)u_{m-1}(t)-\hbar\mathcal{L}^{-1}\Big[\frac{1}{s^\alpha}\mathcal{L}[\Re_m(\mathcal{N}(\phi(t;q)))]\Big]. \tag{2.12}$$

We can summarize our LHAM algorithm as follows:

- Determine $u_0(t)$ and $u_1(t)$ using (2.10) and (2.11) respectively.
- Use $u_0(t)$ and $u_1(t)$ to determine $\Re_2$ according to (2.9) and then obtain $u_2(t)$ by (2.12).
- Determine $\Re_m$, $m\geq 3$, using (2.9) and then evaluate the components $u_m(t)$, $m\geq 1$, by (2.12).
- The series solution follows immediately after using (2.8).

The obtained series solution may converge to an exact solution if such a solution exists. Otherwise, the series solution can be used for numerical purposes.

The first goal of the LHAM approach is employing the powerful homotopy analysis method to investigate nonlinear fractional differential equations. The other goals are to overcome the difficulty arising in calculating complicated integrals.

3 Implementation of the method

The combined Laplace transform HAM method (LHAM) for solving nonlinear fractional differential equations is illustrated by studying the following examples:

Example 3.1 (see [3]). Consider the following nonlinear differential equation of the fractional order:

$$D^\alpha u(t)+e^{u(t)}=0, \quad 0<\alpha\leq 1, \tag{3.1}$$

with initial condition

$$u(0)=0. \tag{3.2}$$

In accordance with the proposed algorithm, we get the following recursive relations:

$$u_0(t)=0, \quad u_1(t)=-h\mathcal{L}^{-1}\Big[\frac{1}{s^\alpha}\mathcal{L}[-e^{u_0(t)}]\Big]=h\frac{t^\alpha}{\Gamma(\alpha+1)},$$

and for $m\geq 2$, we obtain

$$u_m(t)=(1+\hbar)u_{m-1}(t)+\hbar\mathcal{L}^{-1}\Big[\frac{1}{s^\alpha}\mathcal{L}[\Re_m(e^\phi)]\Big].$$

Note that $\Re_m(e^\phi)$ can be obtained recursively as follows:

$$\Re_0(e^\phi)=e^{u_0(t)}, \tag{3.3a}$$

$$\Re_m(e^\phi)=\sum_{k=0}^{m-1}\Big(1-\frac{k}{m}\Big)\Re_k(e^{\phi(t;q)})\Re_{m-k}(\phi(t;q)). \tag{3.3b}$$

For more details about the derivation of (3.3a), one can see [11]. Now, we can easily obtain the components $u_m(t)$. As a result, we can verify that

$$\begin{aligned}u_2(t)=&(1+\hbar)u_1(t)+\hbar\mathcal{L}^{-1}\Big[\frac{1}{s^\alpha}\mathcal{L}\Big[\Big(\frac{u_0(t)^2}{2}+u_1(t)\Big)e^{u_0(t)}\Big]\Big]\\ =&\hbar(1+\hbar)\frac{t^\alpha}{\Gamma(\alpha+1)}+\hbar^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\end{aligned}$$

and that the solution of (3.1)-(3.2) can be given as:

$$u(t)=\hbar(2+\hbar)\frac{t^\alpha}{\Gamma(\alpha+1)}+\hbar^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots. \tag{3.4}$$

We can investigate the influence of $\hbar$ on the series solution by means of the so-called $\hbar$-curve. As pointed by Liao [13], the valid region of $\hbar$ is a horizontal line segment. According to the $\hbar$-curve shown in Fig. 1, we conclude that all values $-0.3\leq\hbar\leq-0.1$ are acceptable.

At the N^{th}-order approximation, we have the analytic solution of (3.1)-(3.2), namely

$$u(t)\approx\tilde{u}_N(t)=\sum_{m=0}^{N}u_m(t).$$

Fig. 2 shows the residual error for N^{th}-order approximation as

$$\text{Residual Error}\approx D^\alpha\tilde{u}_N(t)+e^{\tilde{u}_N(t)}$$

and clearly indicates that the LHAM gives rapid convergence. Moreover the best value for $\hbar$ is not -1.

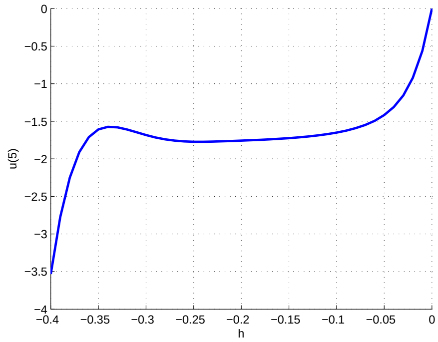

Figure 1: Plots for the $\hbar$-curve for Example 3.1.

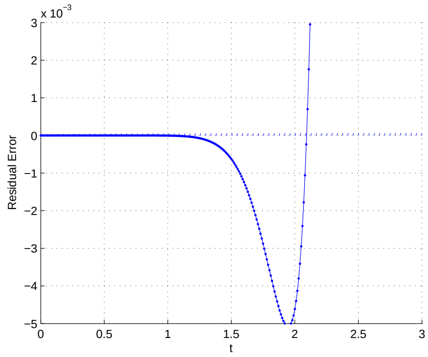

Figure 2: The residual error for Eqs. (3.1)-(3.2) when $N=15$; $\hbar=-0.2$ $(\cdots)$, $\hbar=-1$ $(-\cdot-)$.

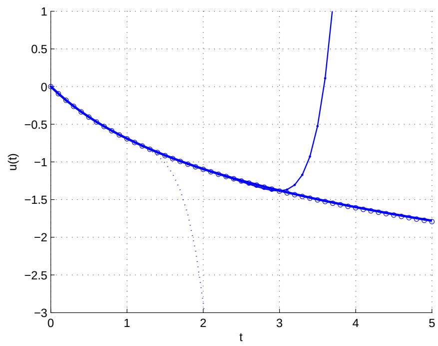

Figure 3: Plots of 10 terms of the LHAM approximate solutions $\alpha=1$: $\hbar=-1$ $(\cdots)$, $\hbar=-0.5$ $(-\cdot-)$, $\hbar=-0.2$ $(\circ\circ)$ and the exact solution $(---)$ for Example 3.1.

Fig. 3 shows the approximate solutions obtained in (3.4) using the LHAM when $\alpha=1$ versus the exact solution, $u(t)=-\ln(1+t)$. Clearly, $\hbar=-1$ is not a good choice since it gives a solution agrees with the exact solution in a small range. You can see that $\hbar=-0.2$ is optimal and agrees with the exact solution in a large range.

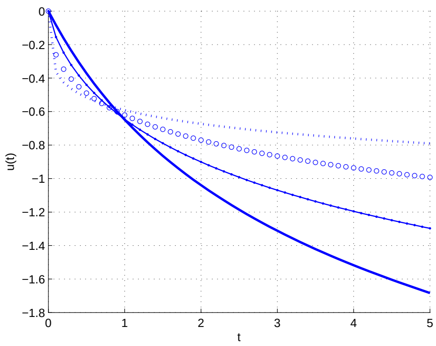

Figure 4: Plots of 10 terms of the LHAM approximate solutions using $\hbar=-0.2$: $\alpha=1$ $(---)$, $\alpha=0.75$ $(-\cdot-)$, $\alpha=0.5$, $(\circ\circ)$ and $\alpha=0.3$ $(\cdots)$ for Example 3.1.

Fig. 4 shows the approximate solutions for (3.1)-(3.2) obtained for different values of $\alpha=1,0.75,0.5,0.3$. A comparison between the results presented in Figs. 3 and 4 indicates that the algorithm presented in this paper can handle nonlinear differential equations of fractional order. It is to be noted that only 10 terms of the series solution were used in evaluating the approximate solutions given in Figs. 3 and 4.

Solution using the VIM: By applying the VIM, we get the following iteration procedure:

$$u_{n+1}(t)=u_n(t)+D^{-\alpha}[D^\alpha u_n(t)+e^{u_n(t)}], \tag{3.5}$$

where $u_0(t)=0$ is assumed. Since the integration of the nonlinear term in Eq. (3.5) is not easily evaluated, we replace the nonlinear term with a series of finite components. Under this assumption, we have the following variational iteration formulation:

$$u_{n+1}(t)=u_n(t)-\int_0^t \Big(D^\alpha u_n(x)+1+u_n(x)+\frac{1}{2}u_n^2(x)+\frac{1}{6}u_n^3(x)\Big)dx.$$

Solution using the ADM: The ADM suggests the solution be decomposed into the infinite series of components

$$u(t)=\sum_{n=0}^{\infty} u_n(t)$$

with the following recursive relation:

$$u_0(t)=0, \quad u_{n+1}(t)=-D^{-\alpha}(A_n(t)), \quad N\geq 0,$$

where

$$A_0=e^{u_0}, \quad A_1=u_1e^{u_0}, \quad A_2=\Big(\frac{u_1^2}{2}+u_2\Big)e^{u_0}, \quad \cdots.$$

It is to be noted that only the ten-order term of the VIM and only ten terms of the decomposition series were used in evaluating the approximate solutions for Fig. 5. From

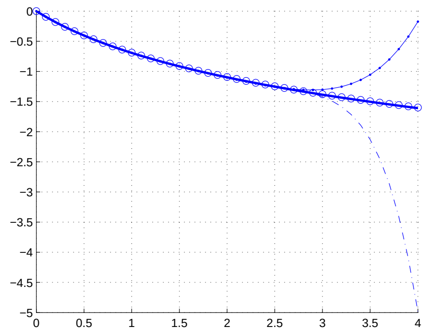

Figure 5: Approximate solutions for Example 3.1, $\alpha=1$, using ADM: dashes, VIM: dotted dashes, LHAM when $\hbar=-0.2$: circles and exact solution: solid line.

the numerical results in Fig. 5, it is easy to conclude that our approximate solution using the LHAM is more accurate than the approximate solutions obtained using the VIM and ADM.

Example 3.2 (see [20])**.** Consider the fractional Riccati equation

$$D^{\alpha}u(t)+u^2(t)=1, \quad t>0, \tag{3.6}$$

where $0<\alpha\le 1$, subject to the initial condition

$$u(0)=0. \tag{3.7}$$

In view of (2.10), we choose the initial guess

$$u_0(t)=u(0+)+\mathcal{L}^{-1}\Big[\frac{1}{s^{\alpha}}\mathcal{L}[1]\Big]=\frac{t^{\alpha}}{\Gamma(\alpha+1)}.$$

Then by (2.11) we obtain

$$u_1(t)=-h\Big(\mathcal{L}^{-1}\Big[\frac{1}{s^{\alpha}}\mathcal{L}[-u_0(t)^2]\Big]\Big)=h\Big(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}\Big)t^{3\alpha}$$

and for $m\ge 2$, we have by (2.12) that

$$\begin{aligned} u_m(t)&=(1+\hbar)u_{m-1}(t)-\hbar\mathcal{L}^{-1}\Big[\frac{1}{s^{\alpha}}\mathcal{L}\Big[\Re_m(-\phi(t;q)^2)\Big]\Big] \\ &=(1+\hbar)u_{m-1}(t)+\hbar\mathcal{L}^{-1}\Big[\frac{1}{s^{\alpha}}\mathcal{L}\Big[\sum_{k=0}^{m-1}u_k(t)u_{m-1-k}(t)\Big]\Big]. \end{aligned} \tag{3.8}$$

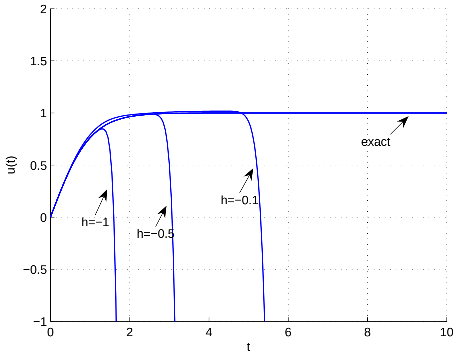

Figure 6: Plots of 10 terms of the LHAM approximate solutions and the exact solution for Example 3.2 when $\alpha=1$.

According to (3.8), it is easy to derive the components of the homotopy series solution by symbolic software such as Mathematica. For example $u_2(t)$ is derived as follows:

$$\begin{aligned} u_2(t)=&(1+\hbar)u_1(t)+\hbar\mathcal{L}^{-1}\Big[\frac{1}{s^\alpha}\mathcal{L}\Big[\sum_{k=0}^{1}u_k(t)u_{m-1-k}(t)\Big]\Big]\\ =&(\hbar+\hbar^2)\Big(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}\Big)t^{3\alpha}+2\hbar^2\Big(\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}\Big)t^{5\alpha}. \end{aligned}$$

Therefore, the solution of problem (3.6)-(3.7) is given by:

$$\begin{aligned} u(t)=&\frac{t^\alpha}{\Gamma(\alpha+1)}+(2\hbar+\hbar^2)\Big(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}\Big)t^{3\alpha}\\ &+2\hbar^2\Big(\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}\Big)t^{5\alpha}+\cdots. \end{aligned} \tag{3.9}$$

Fig. 6 shows the approximate solutions obtained for problem (3.6)-(3.7) using the LHAM when $\alpha=1$ versus the exact solution, $u(t)=\tanh(t)$. The value of $\alpha=1$ is the only case for which we know the exact solution and our approximate solutions are in good agreement with the exact values. We can see that when $\hbar=-1$, the solution (3.9) is exactly the same as that given by the Adomian decomposition method and homotopy perturbation method. This illustrates that the two methods are indeed special cases of the LHAM. However, the results given by the Adomian decomposition method and homotopy perturbation method converge to the corresponding numerical solutions in a rather small region, as shown in Fig. 6. But, different from those two methods, the LHAM provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameter $\hbar$. Using the so-called $\hbar-$curve, all values $-0.3\le\hbar\le-0.1$ are considered optimal and lead to solutions converge in large regions. Fig. 7 shows the approximate solutions (3.9) for different values of $\hbar$ when $\alpha=0.75$.

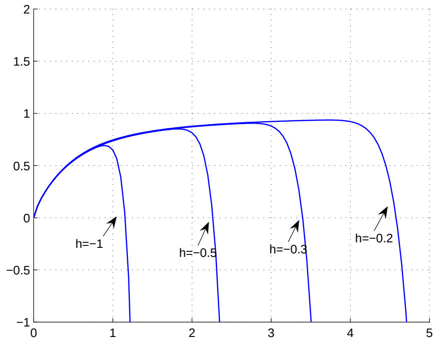

Figure 7: Plots of 10 terms of the LHAM approximate solutions for Example 3.2 when $\alpha=0.75$.

Example 3.3 (see [23])**.** Consider the fractional oscillation equation:

$$D^{3/2}u(t)+u(t)=te^{-t}, \quad t>0, \tag{3.10}$$

subject to the initial conditions

$$u^{(k)}(0)=0, \quad k=0,1.$$

The exact solution is

$$u(t)=\int_0^t G(t-x)xe^{-x}dx, \qquad G(t)=t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}),$$

where $E_{\alpha,\beta}(z)$ is the so-called *Mittag-Leffler* function:

$$E_{\alpha,\beta}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+\beta)}, \qquad \alpha>0, \quad \beta>0.$$

In view of the LHAM algorithm presented in the previous section, we have that:

$$u_0(t)=\mathcal{L}^{-1}\Big[\frac{1}{s^{1.5}}\mathcal{L}[te^{-t}]\Big], \quad u_1(t)=h\mathcal{L}^{-1}\Big[\frac{1}{s^{1.5}}\mathcal{L}[u_0(t)]\Big],$$

and for $m\geq 2$, we obtain

$$u_m(t)=(1+\hbar)u_{m-1}(t)+\hbar\mathcal{L}^{-1}\Big[\frac{1}{s^{1.5}}\mathcal{L}[u_{m-1}(t)]\Big]. \tag{3.11}$$

With the help of the $\hbar$-curve, we can see that $\hbar=-0.1$ is an optimal value and the solution corresponding to that value of $\hbar$ is given by:

$$\begin{aligned}u(t)=&\frac{1}{\sqrt{\pi}}\Big(\frac{8}{15}t^{5/2}-\frac{32}{105}t^{7/2}+\frac{32}{315}t^{9/2}-\frac{304}{12375}t^{11/2}+\cdots\Big)\\&+(-0.0360t^4+0.0102t^5-0.0036t^6+\cdots).\end{aligned}$$

A comparison of the proposed algorithm and the exact solution is shown in Fig. 8. It can be seen that our method is in excellent agreement with the exact solution, and the error is $\mathcal{O}(h^3)$ when we use 20 terms of the series solution.

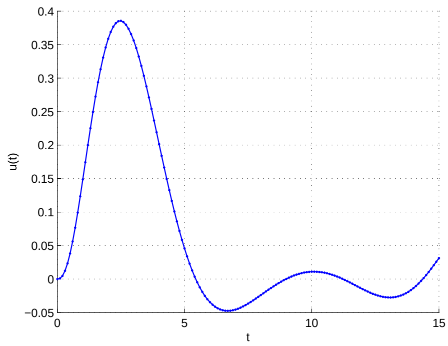

Figure 8: Plots of 20 terms of the LHAM approximate solution when $\hbar = -0.1$ $(---)$ and the exact solution $(\cdots)$ for Example 3.3.

4 Conclusions

In this paper, we proposed a very effective and convenient algorithm called the LHAM to solve nonlinear fractional differential equations. The main advantage of the method is its fast convergence to the solution. Moreover, it avoids the volume of calculations required by the other existing analytical methods. In practice, the utilization of the method is straightforward if some symbolic software as Mathematica is used to implement the calculations. We presented various examples to numerically determine whether the new method leads to higher accuracy and simplicity, and in all cases the solutions obtained are easily programmable approximants to the analytic solutions of the original problems with the accuracy required.

Acknowledgments

The authors would like to express their thankfulness to anonymous referees for their helpful comments. This work is supported by the Imam Khomeini International University of Iran, under the grant 751166-92 (for second author).

References

[1] R. L. Bagley and P. J. Torvik, *On the appearance of the fractional derivative in the behavior of real materials*, J. Appl. Mech., 51 (1984), pp. 294–298.
[2] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.
[3] A. Ghorbani, *Toward a new analytical method for solving nonlinear fractional differential equations*, Comput. Methods Appl. Mech. Eng., 197 (2008), pp. 4173–4179.
[4] Z. M. Odibat and S. Momani, *Application of variational iteration method to nonlinear differential equations of fractional order*, Int. J. Nonlinear Sci. Numer. Simul., 7 (2006), pp. 27–34.

[5] G. ADOMIAN, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston and London, 1994.

[6] S. ABBASBANDY, *Application of He's homotopy perturbation method to functional integral equations*, Chaos Soliton. Fract., 31 (2007), pp. 1243–1247.

[7] S. ABBASBANDY, *Solitary wave solutions to the Kuramoto-Sivashinsky equation by means of the homotopy analysis method*, Nonlinear Dyn., 52 (2008), pp. 35–40.

[8] S. ABBASBANDY, *Approximate solution for the nonlinear model of diffusion and reaction in porous catalysts by means of the homotopy analysis method*, Chem. Eng. J., 136(2-3) (2008), pp. 144–150.

[9] F. AWAWDEH, H. M. JARADAT AND O. ALSAYYED, *Solving system of DAEs by homotopy analysis method*, Chaos Soliton. Fract., 42(3) (2009), pp. 1422–1427.

[10] F. AWAWDEH, A. ADAWI AND Z. MUSTAFA, *Solutions of the SIR models of epidemics using HAM*, Chaos Soliton. Fract., 42(5) (2009), pp. 3047–3052.

[11] F. AWAWDEH, M. KHANDAQJI AND Z. MUSTAFA, *A new approach for the solution of the electrostatic potential differential equations*, Math Probl. Eng., 2009 (2009), doi: 10.1155/2009/980987.

[12] I. HASHIM, O. ABDULAZIZ AND S. MOMANI, *Homotopy analysis method for fractional IVPs*, Commun. Nonlinear. Sci. Numer. Simul., 14(3) (2009), pp. 674–684.

[13] S. J. LIAO, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman & Hall/CRC Press, Boca Raton, 2003.

[14] S. J. LIAO, *On the homotopy analysis method for nonlinear problems*, Appl. Math. Comput., 147 (2004), pp. 499–513.

[15] S. J. LIAO, *Notes on the homotopy analysis method: Some definitions and theorems*, Commun. Nonlinear Sci. Numer. Simul., 14 (2009), pp. 983–997.

[16] S. J. LIAO, *An optimal homotopy-analysis approach for strongly nonlinear differential equations*, Commun. Nonlinear Sci. Numer. Simul., 15 (2010), pp. 2003–2016.

[17] S. MOMANI AND Z. ODIBAT, *Analytical solution of a time-fractional Navier-Stokes equation by Adomian decomposition method*, Appl. Math. Comput., 177(2) (2006), pp. 488–494.

[18] S. MOMANI AND Z. ODIBAT, *Analytical approach to linear fractional partial differential equations arising in fluid mechanics*, Phys. Lett. A, 355 (2006), pp. 271–279.

[19] Z. ODIBAT, *On Legendre polynomial approximation with the VIM or HAM for numerical treatment of nonlinear fractional differential equations*, J. Comput. Appl. Math., (2011), doi: 10.1016/j.cam.2010.12.013.

[20] Z. ODIBAT, S. MOMANI AND H. XU, *A reliable algorithm of homotopy analysis method for solving nonlinear fractional differential equations*, Appl. Math. Model., 34 (2010), pp. 593–600.

[21] S. SAHA RAY AND R. K. BERA, *An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method*, Appl. Math. Comput., 167 (2005), pp. 561–571.

[22] S. J. LIAO, *Comparison between the homotopy analysis method and homotopy perturbation method*, Appl. Math. Comput., 169 (2005), pp. 1186–1194.

[23] R. LIN AND F. LIU, *Fractional high order methods for the nonlinear fractional ordinary differential equation*, Nonlinear Anal., 66 (2007), pp. 856–869.