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NUMERICAL SOLUTION OF A TWO-DIMENSIONAL PARABOLIC TRANSMISSION PROBLEM

BOŠKO S. JOVANOVIĆ AND LUBIN G. VULKOV

Abstract. An initial boundary value problem for a two-dimensional parabolic equation in two disconnected rectangles is investigated. A finite difference scheme approximating this problem is proposed and analyzed. An estimate of the convergence rate, compatible with the smoothness of the input data (up to a logarithmic factor of the mesh-size), is obtained.

Key words. transmission problem, initial-boundary value problem, disconnected domains, Sobolev spaces, finite differences, convergence rate.

1. Introduction and formulation of the problem

Layers with material properties which significantly differ from those of the surrounding medium appear in a variety of applications. The layer may have a structural role (as in the case of glue), a thermal role (as in the case of a thin thermal insulator), an electromagnetic or optical role etc. Mathematical models of energy and mass transfer in domains with layers lead to so called interface or transmission problems. For example, in [11] we investigated the heat transfer process in the presence of thin layer with concentrated capacity.

In this paper we focus our attention to transmission problems whose solutions are defined in two (or more) disconnected domains. For example, such a situation occurs when the solution in the intermediate region is known, or can be determined from a simpler equation. The effect of the intermediate region can be modelled (see [4, 5, 6, 7, 16, 19, 22]) by means of nonlocal jump conditions across the intermediate region.

In [12, 14] we considered transmission problem for one-dimensional parabolic equation in two disconnected intervals. After an analysis of the strong and weak solutions in specific Sobolev-like spaces, difference schemes for its approximation are constructed and investigated for convergence. Also, one-dimensional elliptic and hyperbolic problems was studied in [17] and [13], respectively. Analytical properties two-dimensional parabolic problem in two disconnected rectangles are investigated in [15]. Here we propose finite difference schemes for its numerical solution.

As a model example, we consider the following initial-boundary-value problem (IBVP): Find functions $u_1(x,y,t)$ and $u_2(x,y,t)$ that satisfy the system of parabolic equations

$$\frac{\partial u_1}{\partial t}-\frac{\partial}{\partial x}\left(p_1(x,y)\frac{\partial u_1}{\partial x}\right)-\frac{\partial}{\partial y}\left(q_1(x,y)\frac{\partial u_1}{\partial y}\right)+r_1(x,y)u_1=f_1(x,y,t), \tag{1}$$
$$(x,y)\in\Omega_1\equiv(a_1,b_1)\times(c,d),\quad t>0,$$

$$\frac{\partial u_2}{\partial t}-\frac{\partial}{\partial x}\left(p_2(x,y)\frac{\partial u_2}{\partial x}\right)-\frac{\partial}{\partial y}\left(q_2(x,y)\frac{\partial u_2}{\partial y}\right)+r_2(x,y)u_2=f_2(x,y,t), \tag{2}$$
$$(x,y)\in\Omega_2\equiv(a_2,b_2)\times(c,d),\quad t>0,$$

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where $-\infty < a_1 < b_1 < a_2 < b_2 < +\infty$, the internal conjugation conditions of non-local Robin-Dirichlet type

$$p_1(b_1, y)\frac{\partial u_1}{\partial x}(b_1, y, t) + \alpha_1(y)u_1(b_1, y, t) = \int_c^d \beta_1(y, y_*)u_2(a_2, y_*, t)\,\mathrm{d}y_*, \tag{3}$$

$$-p_2(a_2, y)\frac{\partial u_2}{\partial x}(a_2, y, t) + \alpha_2(y)u_2(a_2, y, t) = \int_c^d \beta_2(y, y_*)u_1(b_1, y_*, t)\,\mathrm{d}y_*, \tag{4}$$

$$y \in (c, d), \quad t > 0,$$

the simplest external Dirichlet boundary conditions

$$\begin{aligned} &u_1(a_1, y, t) = 0,\ \ y \in (c, d); \quad u_1(x, c, t) = u_1(x, d, t) = 0,\ \ x \in (a_1, b_1), \\ &u_2(b_2, y, t) = 0,\ \ y \in (c, d); \quad u_2(x, c, t) = u_2(x, d, t) = 0,\ \ x \in (a_2, b_2), \end{aligned} \tag{5}$$

and the initial conditions

$$u_1(x, y, 0) = u_{10}(x, y),\ \ (x, y) \in \Omega_1; \quad u_2(x, y, 0) = u_{20}(x, y),\ \ (x, y) \in \Omega_2. \tag{6}$$

In particular, for a special choice of α_i and β_i such initial-boundary value problem describes radiative heat transfer in a system of absolutely black bodies [1, 2].

Throughout the paper we assume that the input data satisfy the usual regularity and ellipticity conditions

$$p_i(x, y),\ q_i(x, y) \in L_\infty(\Omega_i), \quad r_i(x, y) \in L_p(\Omega_i), \quad p > 1, \quad i = 1, 2, \tag{7}$$

$$0 < p_{i0} \le p_i(x, y), \quad 0 < q_{i0} \le q_i(x, y), \quad \text{a.e. in } \Omega_i, \quad i = 1, 2 \tag{8}$$

and

$$\alpha_i \in L_\infty(c, d), \quad \beta_i \in L_\infty\left((c, d) \times (c, d)\right), \quad i = 1, 2. \tag{9}$$

In real physical problems (see [2]) we also often have

$$\alpha_i > 0\,, \quad \beta_i > 0, \quad i = 1, 2.$$

By C, c_j and C_j we denote positive constants, independent of the solution of the IBVP and the mesh-sizes. C can take different values in the different formulas.

The aim of the present paper is to construct efficient finite difference scheme for the numerical solution of IBVP (1) – (6) and to investigate their convergence.

The layout of the paper is as follows. In Section 2 we briefly expose the properties of IBVP (1) – (6) and give some a priori estimates for its weak solution. In Section 3 we introduce meshes, finite-difference operators and discrete Sobolev-like normd. In Section 4 we define implicit finite difference scheme (FDS) approximating IBVP (1) – (6) and investigate its properties. A convergence rate estimate, compatible with the smoothness of the input data (up to a logarithmic factor of mesh-size), is obtained. In Section 5 we define factorized FDS and investigate its properties and convergence.

2. Weak solutions and a priori estimates

We introduce the product space

$$L = L_2(\Omega_1) \times L_2(\Omega_2) = \{v = (v_1, v_2)\,|\,v_i \in L_2(\Omega_i)\},$$

endowed with the inner product and the associated norm

$$(u, v)_L = (u_1, v_1)_{L_2(\Omega_1)} + (u_2, v_2)_{L_2(\Omega_2)}, \quad \|v\|_L = (v, v)_L^{1/2},$$

where

$$(u_i, v_i)_{L_2(\Omega_i)} = \int_{\Omega_i} u_i v_i \,\mathrm{d}x\,\mathrm{d}y, \quad i = 1, 2.$$

We also define the spaces

$$H^k = \{v = (v_1, v_2)\,|\, v_i \in H^k(\Omega_i)\}, \quad k = 1, 2, \ldots$$

endowed with the inner products and norms

$$(u, v)_{H^k} = (u_1, v_1)_{H^k(\Omega_1)} + (u_2, v_2)_{H^k(\Omega_2)}, \quad \|v\|_{H^k} = (v, v)^{1/2}_{H^k},$$

where

$$(u_i, v_i)_{H^k(\Omega_i)} = \sum_{j=0}^{k} \sum_{l=0}^{j} \Big(\frac{\partial^j u_i}{\partial x^l \partial y^{j-l}}, \frac{\partial^j v_i}{\partial x^l \partial y^{j-l}} \Big)_{L_2(\Omega_i)}, \quad i = 1, 2, \quad k = 1, 2, \ldots.$$

In particular, we set

$$H^1_0 = \{v = (v_1, v_2) \in H^1 \mid v_i = 0 \;\text{ on }\; \Gamma_i, \;\; i = 1, 2\},$$

where $\Gamma_1 = \partial\Omega_1 \setminus \{(b_1, y)\,|\, y \in (c, d)\}$ and $\Gamma_2 = \partial\Omega_2 \setminus \{(a_2, y)\,|\, y \in (c, d)\}$. Finally, with $u = (u_1, u_2)$ and $v = (v_1, v_2)$ we define the bilinear form:

$$\begin{aligned} A(u, v) = &\int_{\Omega_1} \Big(p_1 \frac{\partial u_1}{\partial x} \frac{\partial v_1}{\partial x} + q_1 \frac{\partial u_1}{\partial y} \frac{\partial v_1}{\partial y} + r_1 u_1 v_1\Big) \mathrm{d}x\,\mathrm{d}y \\ &+ \int_{\Omega_2} \Big(p_2 \frac{\partial u_2}{\partial x} \frac{\partial v_2}{\partial x} + q_2 \frac{\partial u_2}{\partial y} \frac{\partial v_2}{\partial y} + r_2 u_2 v_2\Big) \mathrm{d}x\,\mathrm{d}y \\ &+ \int_c^d \alpha_1(y) u_1(b_1, y) v_1(b_1, y)\,\mathrm{d}y + \int_c^d \alpha_2(y) u_2(a_2, y) v_2(a_2, y)\,\mathrm{d}y \\ &- \int_c^d \int_c^d \beta_1(y, y_*) u_2(a_2, y_*) v_1(b_1, y)\,\mathrm{d}y\,\mathrm{d}y_*. \\ &- \int_c^d \int_c^d \beta_2(y_*, y) u_1(b_1, y) v_2(a_2, y_*)\,\mathrm{d}y\,\mathrm{d}y_* \end{aligned} \tag{10}$$

The following assertion holds true (see [15]):

Lemma 1. *Under the conditions (7) and (9) the bilinear form A, defined by (10), is bounded on $H^1 \times H^1$. If in adition the conditions (8) are fulfilled, this form satisfies the Gårding's inequality on H^1_0, i.e. there exist positive constants m and κ such that*

$$A(u, u) + \kappa\, \|u\|^2_L \ge m\, \|u\|^2_{H^1}, \quad \forall\, u \in H^1_0.$$

Let Ω be a domain in R^n and $u(t)$ a function mapping Ω into a Hilbert space H. In a standard manner (see [18]) we define Sobolev space of vector-valued functions $H^k(\Omega, H)$, endowed with the inner product

$$(u, v)_{H^k(\Omega,H)} = \int_\Omega \sum_{|\alpha| \le k} (D^\alpha u(t),\, D^\alpha v(t))_H\, \mathrm{d}t\,, \quad k = 0, 1, 2, \ldots$$

with the usual modification for non-integer k. We set $L_2(\Omega, H) = H^0(\Omega, H)$ and define $H^{1,1/2} = L_2((0, T), H^1) \cap H^{1/2}((0, T), L)$.

Let $H^{-1} = (H^1_0)^*$ be the dual space for H^1_0. The spaces H^1_0, L and H^{-1} form a Gelfand triple $H^1_0 \subset L \subset H^{-1}$ ([23]), with continuous and dense embeddings. We also introduce the space

$$W(0, T) = \Big\{u \,\Big|\, u \in L_2((0, T), H^1_0),\; \frac{\partial u}{\partial t} \in L_2((0, T), H^{-1})\Big\}$$

with inner product

$$(u, v)_{W(0,T)} = \int_0^T \Big[(u(\cdot, t),\, v(\cdot, t))_{H^1} + \Big(\frac{\partial u}{\partial t}(\cdot, t),\, \frac{\partial v}{\partial t}(\cdot, t)\Big)_{H^{-1}}\Big]\, \mathrm{d}t.$$

The weak form of (1) – (5) is:

$$\Big(\frac{\partial u}{\partial t}(\cdot,t),\, v(\cdot,t)\Big)_L + A(u(\cdot,t),\, v(\cdot,t)) = (f(\cdot,t),\, v(\cdot,t))_L, \quad \forall v \in L_2((0,T),H^1). \tag{11}$$

The problem (11) fit in the general theory of parabolic differential operators in Hilbert spaces (see [23]). Applying Theorem 26.1 from [23] to (11) we obtain the following assertion (see [15]):

Theorem 1. *Let the assumptions (7), (8) and (9) hold and suppose that* $u_0 = (u_{10}, u_{20}) \in L,\ f = (f_1, f_2) \in L_2((0,T), H^{-1})$. *Then for* $0 < T < +\infty$ *the IBVP (1) – (6) has a unique weak solution* $u \in W(0,T)$, *and it depends continuously on* f *and* u_0.

Because the norm $\|\cdot\|_{H^{-1}}$ is not computable, in the sequel we restrict our investigations to the IBVPs (1) – (6) with right-hand sides of the special form. In the first case we set

$$\begin{aligned} f_i(x,y,t) = f_{i0}(x,y,t) &+ \frac{\partial(\varrho_i(x)\, f_{i1}(x,y,t))}{\partial x} + \frac{\partial f_{i2}(x,y,t)}{\partial y} \\ &+ \int_0^T \frac{f_{i3}(x,y,t,t') - f_{i2}(x,y,t',t)}{|t-t'|}\,\mathrm{d}t', \qquad i=1,2, \end{aligned} \tag{12}$$

where $f_{i0},\, f_{i1},\, f_{i2} \in L_2((0,T), L_2(\Omega_i)) = L_2(Q_i),\ Q_i = \Omega_i \times (0,T),\ f_{i3} \in L_2((0,T)^2, L_2(\Omega_i)) = L_2(R_i),\ R_i = \Omega_i \times (0,T)^2,\ \varrho_i \in C([a_i,b_i])$ and

$$c_1(b_1 - x) \le \varrho_1(x) \le C_1(b_1 - x), \quad x \in (a_1,b_1), \quad C_1 \ge c_1 > 0,$$

$$c_2(x - a_2) \le \varrho_2(x) \le C_2(x - a_2), \quad x \in (a_2,b_2), \quad C_2 \ge c_2 > 0.$$

Also, we will consider the case

$$f_i(x,t) = \frac{\partial g_i(x,t)}{\partial t}\,, \qquad i=1,2, \tag{13}$$

where $g_i \in H_{00}^{1/2}((0,T), L_2(\Omega_i)),\ i=1,2$. The norm in $H_{00}^{1/2}(0,T)$ is defined in the following way:

$$\|\varphi\|^2_{H_{00}^{1/2}(0,T)} \equiv |\varphi|^2_{H^{1/2}(0,T)} + \int_0^T \Big(\frac{1}{t} + \frac{1}{T-t}\Big)\,\varphi^2(t)\,\mathrm{d}t.$$

The following assertions hold true (see [15]):

Theorem 2. *Let the assumptions (7), (8) and (9) hold and let* $u_{i0} \in L_2(\Omega_i)$, f_{i0}, $f_{i1}\, f_{i2} \in L_2(Q_i)$, $f_{i3} \in L_2(R_i)$, $i = 1, 2$. *Then the IBVP (1) – (6), (12) has a unique weak solution* $u = (u_1, u_2) \in H^{1,1/2}$ *and the a priori estimate*

$$\begin{aligned} \|u\|^2_{H^{1,1/2}} \le C \sum_{i=1}^{2} \Big(\|u_{i0}\|^2_{L_2(\Omega_i)} + \|f_{i0}\|^2_{L_2(Q_i)} \\ + \|f_{i1}\|^2_{L_2(Q_i)} + \|f_{i2}\|^2_{L_2(Q_i)} + \|f_{i3}\|^2_{L_2(R_i)} \Big) \end{aligned} \tag{14}$$

holds.

Theorem 3. *Let the assumptions (7), (8) and (9) hold and let* $u_{i0} \in L_2(\Omega_i)$, $g_i \in H^{1/2}_{00}((0, T), L_2(\Omega_i))$, $i = 1, 2$. *Then the IBVP (1) – (6), (13) has a unique weak solution* $u = (u_1, u_2) \in H^{1,1/2}$ *and the a priori estimate*

$$\|u\|^2_{H^{1,1/2}} \le C \Big(\|u_0\|^2_L + \|g\|^2_{H^{1/2}_{00}((0,T),L)} \Big) \tag{15}$$

holds.

In both cases C is computable constant depending on T, i.e.

$$C \le C_1\, T\, e^{C_2\, T} .$$

3. Meshes, finite differences and discrete norms

Let $\bar\omega_{i,h_i}$ be a uniform mesh in $[a_i, b_i]$ with the step-size $h_i = (b_i - a_i)/n_i$, $i = 1, 2$. We denote $\omega_{i,h_i} := \bar\omega_{i,h_i} \cap (a_i, b_i)$, $\omega^-_{i,h_i} := \omega_{i,h_i} \cup \{a_i\}$, $\omega^+_{i,h_i} := \omega_{i,h_i} \cup \{b_i\}$. Analogously we define a uniform mesh $\bar\omega_k$ in $[c, d]$ with the step-size $k = (d - c)/n_3$ and its submeshes $\omega_k := \bar\omega_k \cap (c, d)$, $\omega^-_k := \omega_k \cup \{c\}$, $\omega^+_k := \omega_k \cup \{d\}$. We assume that $h_1 \asymp h_2 \asymp k$. Finally, we introduce a uniform mesh $\bar\omega_\tau$ in $[0, T]$ with the step-size $\tau = T/n$ and set $\omega_\tau := \bar\omega_\tau \cap (0, T)$, $\omega^-_\tau := \omega_\tau \cup \{0\}$, $\omega^+_\tau := \omega_\tau \cup \{T\}$. We will consider vector-functions of the form $v = (v_1,\, v_2)$ where v_i is a mesh function defined on $\bar\omega_{i,h_i} \times \bar\omega_k \times \bar\omega_\tau$, $i = 1, 2$. We define difference quotients in the usual way (see [20]):

$$\begin{aligned} v_{i,x}(x, y, t) &= \frac{v_i(x + h_i,\, y,\, t) - v_i(x,\, y,\, t)}{h_i} = v_{i,\bar x}(x + h_i,\, y,\, t), \\ v_{i,y}(x, y, t) &= \frac{v_i(x,\, y + k,\, t) - v_i(x,\, y,\, t)}{k} = v_{i,\bar y}(x,\, y + k,\, t), \\ v_{i,t}(x, y, t) &= \frac{v_i(x,\, y,\, t + \tau) - v_i(x,\, y,\, t)}{\tau} = v_{i,\bar t}(x,\, y,\, t + \tau). \end{aligned}$$

We define the Steklov averaging operators

$$T_x f_i(x, y, t) = T^-_x f_i(x + \frac{h_i}{2},\, y, t) = T^+_x f_i(x - \frac{h_i}{2},\, y, t) = \frac{1}{h_i} \int\limits_{x-h_i/2}^{x+h_i/2} f_i(x', y, t)\, \mathrm{d}x',$$

$$T_y f_i(x, y, t) = T^-_y f_i(x, y + \frac{k}{2},\, t) = T^+_y f_i(x, y - \frac{k}{2}, t) = \frac{1}{k} \int\limits_{y-k/2}^{y+k/2} f_i(x, y', t)\, \mathrm{d}y',$$

$$T^-_t f_i(x, y, t) = T^+_t f_i(x, y,\, t - \tau) = \frac{1}{\tau} \int\limits_{t-\tau}^{t} f_i(x, y, t')\, \mathrm{d}t',$$

$$T_x^2 f_i(x,y,t) = T_x\Big(T_x f_i(x,y,t)\Big) = \frac{1}{h_i}\int\limits_{x-h_i}^{x+h_i}\Big(1-\frac{|x-x'|}{h_i}\Big) f_i(x',y,t)\,\mathrm{d}x',$$

$$T_y^2 f_i(x,y,t) = T_y\Big(T_y f_i(x,y,t)\Big) = \frac{1}{k}\int\limits_{y-k}^{y+k}\Big(1-\frac{|y-y'|}{k}\Big) f_i(x,y',t)\,\mathrm{d}y'.$$

For $x=b_1$ and $x=a_2$ we need the following asymmetric averaging operators

$$T_x^{2-} f_1(b_1,y,t) = \frac{2}{h_1}\int_{b_1-h_1}^{b_1}\Big(1-\frac{b_1-x'}{h_1}\Big) f_1(x',y,t)\,\mathrm{d}x',$$

$$T_x^{2+} f_2(a_2,y,t) = \frac{2}{h_2}\int_{a_2}^{a_2+h_2}\Big(1-\frac{x'-a_2}{h_2}\Big) f_2(x',y,t)\,\mathrm{d}x'.$$

With the notational conventions $\hbar_i = h_i$, $x\in\omega_{i,h_i}$, $i=1,2$, $\hbar_1(b_1)=h_1/2$, $\hbar_2(a_2)=h_2/2$, we also introduce the discrete inner products

$$(v,w)_{L_h} = k\sum_{x\in\omega_{1,h_1}^+}\sum_{y\in\omega_k} v_1 w_1 \hbar_1 + k\sum_{x\in\omega_{2,h_2}^-}\sum_{y\in\omega_k} v_2 w_2 \hbar_2,$$

$$(v,w)_{L_{h'}} = h_1 k\sum_{x\in\omega_{1,h_1}^+}\sum_{y\in\omega_k} v_1 w_1 + h_2 k\sum_{x\in\omega_{2,h_2}^+}\sum_{y\in\omega_k} v_2 w_2,$$

$$(v,w)_{L_{h''}} = k\sum_{x\in\omega_{1,h_1}^+}\sum_{y\in\omega_k^+} v_1 w_1 \hbar_1 + k\sum_{x\in\omega_{2,h_2}^-}\sum_{y\in\omega_k^+} v_2 w_2 \hbar_2,$$

and the associated norms

$$\|v\|_{L_h}^2 = (v,v)_{L_h}, \qquad \|v\|_{L_{h'}}^2 = (v,v)_{L_{h'}}, \qquad \|v\|_{L_{h''}}^2 = (v,v)_{L_{h''}}.$$

We also define the following discrete norms

$$\|v_i\|_{L_2(\omega_\tau^\pm)}^2 = \tau\sum_{t\in\omega_\tau^\pm} v_i^2, \qquad \|v\|_{L_2(\omega_k)}^2 = k\sum_{y\in\omega_k}\big(v_1^2+v_2^2\big),$$

$$\|v\|_{L_2(\omega_\tau^\pm,H)}^2 = \tau\sum_{t\in\omega_\tau^\pm}\|v(\cdot,t)\|_H^2, \qquad H = L_h,\, L_{h'},\, L_{h''},\, L_2(\omega_k),$$

$$\|v\|_{\tilde L_2(\bar\omega_\tau,L_h)}^2 = \tau\sum_{t\in\bar\omega_\tau}\Big(\frac{1}{t+\tau/2}+\frac{1}{T-t+\tau/2}\Big)\|v(\cdot,t)\|_{L_h}^2,$$

$$|v|_{H^{1/2}(\bar\omega_\tau,L_h)}^2 = \tau^2\sum_{t\in\bar\omega_\tau}\sum_{t'\in\bar\omega_\tau,\ t'\neq t}\frac{\|v(\cdot,t)-v(\cdot,t')\|_{L_h}^2}{|t-t'|^2},$$

$$\|v\|_{H_{00}^{1/2}(\bar\omega_\tau,L_h)}^2 = |v|_{H^{1/2}(\bar\omega_\tau,L_h)}^2 + \|v\|_{\tilde L_2(\bar\omega_\tau,L_h)}^2,$$

$$\|v\|_{H_{h\tau}^{1,1/2}}^2 = \|v_{\bar x}\|_{L_2(\omega_\tau^+,L_{h'})}^2 + \|v_{\bar y}\|_{L_2(\omega_\tau^+,L_{h''})}^2 + \|v\|_{L_2(\omega_\tau^+,L_h)}^2 + |v|_{H^{1/2}(\bar\omega_\tau,L_h)}^2.$$

4. Explicit finite difference scheme and its convergence

In this and subsequent sections we will assume that u_i belongs to $H^{3,3/2}(Q_i)$, while $p_i,\, q_i\in H^2(\Omega_i)$, $r_i\in H^1(\Omega_i)$, $\alpha_i\in H^2(c,d)$ and $\beta_i\in H^2((c,d)^2)$. Consequently, $f_i\in H^{1,1/2}(Q_i)$ and need not be a continuous function. Therefore, we approximate the IBVP (1) – (6) with the following explicit FDS with averaged input data:

$$v_{1,t} - (\bar{p}_1 v_{1,\bar{x}})_x - (\bar{q}_1 v_{1,\bar{y}})_y + \bar{r}_1 v_1 = \bar{f}_1, \qquad x \in \omega_{1,h_1}, \quad y \in \omega_k, \quad t \in \omega_\tau^-, \tag{16}$$

$$\begin{aligned} &v_{1,t}(b_1,y,t) + \frac{2}{h_1}\Big[\bar{p}_1(b_1,y)v_{1,\bar{x}}(b_1,y,t) + \alpha_1(y)v_1(b_1,y,t) \\ &\qquad -k\sum_{y_*\in\omega_k}\beta_1(y,y_*)v_2(a_2,y_*,t)\Big] - (\bar{q}_1 v_{1,\bar{y}})_y(b_1,y,t) \\ &\qquad +\bar{r}_1(b_1,y)v_1(b_1,y,t) = \bar{f}_1(b_1,y,t), \qquad y \in \omega_k, \quad t \in \omega_\tau^-, \end{aligned} \tag{17}$$

$$v_{2,t} - (\bar{p}_2 v_{2,\bar{x}})_x - (\bar{q}_2 v_{2,\bar{y}})_y + \bar{r}_2 v_2 = \bar{f}_2, \qquad x \in \omega_{2,h_2} \quad y \in \omega_k, \quad t \in \omega_\tau^-, \tag{18}$$

$$\begin{aligned} &v_{2,t}(a_2,y,t) - \frac{2}{h_2}\Big[\bar{p}_2(a_2+h_2,y)v_{2,x}(a_2,y,t) - \alpha_2(y)v_2(a_2,y,t) \\ &\qquad +k\sum_{y_*\in\omega_k}\beta_2(y,y_*)v_1(b_1,y_*,t)\Big] - (\bar{q}_2 v_{2,\bar{y}})_y(a_2,y,t) \\ &\qquad +\bar{r}_2(a_2,y)v_2(a_2,y,t) = \bar{f}_2(a_2,y,t), \qquad y \in \omega_k, \quad t \in \omega_\tau^-, \end{aligned} \tag{19}$$

$$\begin{aligned} &v_1(a_1,y,t) = 0, \qquad v_2(b_2,y,t) = 0, \qquad y \in \omega_k, \qquad t \in \bar{\omega}_\tau, \\ &v_1(x,c,t) = 0, \qquad v_1(x,d,t) = 0, \qquad x \in \bar{\omega}_{1,h_1}, \quad t \in \bar{\omega}_\tau, \\ &v_2(x,c,t) = 0, \qquad v_2(x,d,t) = 0, \qquad x \in \bar{\omega}_{2,h_2}, \quad t \in \bar{\omega}_\tau, \end{aligned} \tag{20}$$

$$v_i(x,y,0) = u_{i0}(x,y), \qquad x \in \omega_{i,h_i}^\pm, \quad y \in \omega_k, \qquad i = 1,2, \tag{21}$$

where

$$\bar{p}_i(x,y) = \frac{1}{2}\,[p_i(x,y) + p_i(x-h_i,y)], \quad x \in \omega_{i,h_i}^+, \quad y \in \omega_k, \quad i = 1,2,$$

$$\bar{q}_i(x,y) = \frac{1}{2}\,[q_i(x,y) + q_i(x,y-k)], \quad x \in \omega_{i,h_i}^\pm, \quad y \in \omega_k^+, \quad i = 1,2,$$

$$\bar{r}_i(x,y) = T_x^2 T_y^2 r_i(x,y), \quad x \in \omega_{i,h_i}, \quad y \in \omega_k, \quad i = 1,2,$$

$$\bar{r}_1(b_1,y) = T_x^{2-} T_y^2 r_1(b_1,y), \qquad \bar{r}_2(a_2,y) = T_x^{2+} T_y^2 r_2(a_2,y), \quad y \in \omega_k,$$

$$\bar{f}_i(x,y,t) = T_x^2 T_y^2 T_t^+ f_i(x,y,t), \quad x \in \omega_{i,h_i}, \quad y \in \omega_k, \quad t \in \omega_\tau^-, \quad i = 1,2,$$

$$\bar{f}_1(b_1,y,t) = T_x^{2-} T_y^2 T_t^+ f_1(b_1,y,t), \qquad \bar{f}_2(a_2,y,t) = T_x^{2+} T_y^2 T_t^+ f_2(a_2,y,t).$$

FDS (16) – (21) is computationally efficient. It follows from the general theory of difference schemes, [21], that the FDS (16) – (21) is stable under condition

$$\tau \le c_3 \, \min\{h_1^2, h_2^2, k^2\}. \tag{22}$$

Here c_3 is computable constant depending on $\max p_i$ and $\max q_i$, $i = 1,2$.

Let $u = (u_1, u_2)$ be the solution of the IBVP (1) – (6) and $v = (v_1, v_2)$ the solution of the FDS (16) – (21). Then the error $z = u - v$ satisfies the following

FDS:

$$z_{1,t} - (\bar p_1 z_{1,\bar x})_x - (\bar q_1 z_{1,\bar y})_y + \bar r_1 z_1 = \xi_{1,t} + \eta_{1,x} + \zeta_{1,y} + \chi_1, \qquad x \in \omega_{1,h_1},\ \ y \in \omega_k,\ \ t \in \omega_\tau^-, \tag{23}$$

$$\begin{aligned} & z_{1,t}(b_1,y,t) + \frac{2}{h_1}\Big[\bar p_1(b_1,y) z_{1,\bar x}(b_1,y,t) + \alpha_1(y) z_1(b_1,y,t) \\ & \quad -k \sum_{y_* \in \omega_k} \beta_1(y,y_*) z_2(a_2,y_*,t)\Big] - (\bar q_1 z_{1,\bar y})_y(b_1,y,t) + \bar r_1(b_1,y) z_1(b_1,y,t) \\ & \quad = \xi_{1,t}(b_1,y,t) + \zeta_{1,y}(b_1,y,t) + \chi_1(b_1,y,t) - \frac{2}{h_1}\,\eta_1(b_1,y,t) + \frac{2}{h_1}\,\mu_1(y,t), \qquad y \in \omega_k,\ \ t \in \omega_\tau^-, \end{aligned} \tag{24}$$

$$z_{2,t} - (\bar p_2 z_{2,\bar x})_x - (\bar q_2 z_{2,\bar y})_y + \bar r_2 z_2 = \xi_{2,t} + \eta_{2,x} + \zeta_{2,y} + \chi_2, \qquad x \in \omega_{2,h_2},\ \ y \in \omega_k,\ \ t \in \omega_\tau^-, \tag{25}$$

$$\begin{aligned} & z_{2,t}(a_2,y,t) + \frac{2}{h_2}\Big[\bar p_2(a_2+h_2,y) z_{2,x}(a_2,y,t) + \alpha_2(y) z_2(a_2,y,t) \\ & \quad +k \sum_{y_* \in \omega_k} \beta_2(y,y_*) z_1(b_1,y_*,t)\Big] - (\bar q_2 z_{2,\bar y})_y(a_2,y,t) + \bar r_2(a_2,y) z_2(a_2,y,t) \\ & \quad = \xi_{2,t}(a_2,y,t) + \zeta_{2,y}(a_2,y,t) + \chi_2(a_2,y,t) \\ & \quad + \frac{2}{h_2}\eta_2(a_2+h_2,y,t) + \frac{2}{h_2}\mu_2(y,t), \qquad y \in \omega_k,\ \ t \in \omega_\tau^-, \end{aligned} \tag{26}$$

$$\begin{aligned} & z_1(a_1,y,t) = 0, && z_2(b_2,y,t) = 0, && y \in \omega_k, && t \in \bar\omega_\tau, \\ & z_1(x,c,t) = 0, && z_1(x,d,t) = 0, && x \in \bar\omega_{1,h_1}, && t \in \bar\omega_\tau, \\ & z_2(x,c,t) = 0, && z_2(x,d,t) = 0, && x \in \bar\omega_{2,h_2}, && t \in \bar\omega_\tau, \end{aligned} \tag{27}$$

$$z_i(x,y,0) = 0, \qquad x \in \omega_{i,h_i}^\pm,\ \ y \in \omega_k,\ \ i = 1,2, \tag{28}$$

where

$$\begin{aligned} \xi_i &= u_i - T_x^2 T_y^2 u_i, && x \in \omega_{i,h_i},\ \ y \in \omega_k,\ \ t \in \bar\omega_\tau,\ \ i = 1,2, \\ \xi_1 &= u_1 - T_x^{2-} T_y^2 u_1 - \frac{h_1}{3}\, T_y^2\, \frac{\partial u_1}{\partial x}, && x = b_1,\ \ y \in \omega_k,\ \ t \in \bar\omega_\tau \\ \xi_2 &= u_2 - T_x^{2+} T_y^2 u_2 + \frac{h_2}{3}\, T_y^2\, \frac{\partial u_2}{\partial x}, && x = a_2,\ \ y \in \omega_k,\ \ t \in \bar\omega_\tau \end{aligned}$$

$$\eta_i = T_x^- T_y^2 T_t^+\Big(p_i\, \frac{\partial u_i}{\partial x}\Big) - \bar p_i\, u_{i,\bar x}, \qquad x \in \omega_{i,h_i}^+,\ \ y \in \omega_k,\ \ t \in \omega_\tau^-,\ \ i = 1,2,$$

$$\begin{aligned} \zeta_i &= T_x^2 T_y^- T_t^+\Big(q_i\, \frac{\partial u_i}{\partial y}\Big) - \bar q_i\, u_{i,\bar y}, \qquad x \in \omega_{i,h_i},\ \ y \in \omega_k^+,\ \ t \in \omega_\tau^-,\ \ i = 1,2, \\ \zeta_1 &= T_x^{2-} T_y^- T_t^+\Big(q_1\, \frac{\partial u_1}{\partial y}\Big) - \frac{h_1}{3}\, T_y^- T_t^+\, \frac{\partial}{\partial x}\Big(q_1\, \frac{\partial u_1}{\partial y}\Big) - \bar q_1\, u_{1,\bar y}, \quad x = b_1, \\ \zeta_2 &= T_x^{2+} T_y^- T_t^+\Big(q_2\, \frac{\partial u_2}{\partial y}\Big) + \frac{h_2}{3}\, T_y^- T_t^+\, \frac{\partial}{\partial x}\Big(q_2\, \frac{\partial u_2}{\partial y}\Big) - \bar q_2\, u_{2,\bar y}, \quad x = a_2, \end{aligned}$$

$$\begin{aligned} \chi_i &= (T_x^2 T_y^2 r_i)\, u_i - T_x^2 T_y^2 (r_i\, T_t^+ u_i), \qquad x \in \omega_{i,h_i},\ \ y \in \omega_k,\ \ t \in \omega_\tau^-,\ \ i = 1,2, \\ \chi_1 &= \big(T_x^{2-} T_y^2 r_1\big)\, u_1 - T_x^{2-} T_y^2 \big(r_1\, T_t^+ u_1\big) - \frac{h_1}{3}\, (T_x^{2-} T_y^2 r_1)\Big(T_y^2 T_t^+\, \frac{\partial u_1}{\partial x}\Big), \ \ x = b_1, \\ \chi_2 &= \big(T_x^{2+} T_y^2 r_2\big)\, u_2 - T_x^{2+} T_y^2 \big(r_2\, T_t^+ u_2\big) + \frac{h_2}{3}\, (T_x^{2+} T_y^2 r_2)\Big(T_y^2 T_t^+\, \frac{\partial u_2}{\partial x}\Big), \ \ x = a_2, \end{aligned}$$

$$
\begin{aligned}
\mu_1(y,t) = &\big[\alpha_1(y)u_1(b_1,y,t) - T_y^2T_t^+\big(\alpha_1(y)u_1(b_1,y,t)\big)\big] \\
&-\Big[k\sum_{y_*\in\omega_k}\beta_1(y,y_*)u_2(a_2,y_*,t) - \int_c^d T_y^2T_t^+\big(\beta_1(y,y_*)u_2(a_2,y_*,t)\big)\,\mathrm{d}y_*\Big] \\
&+\frac{h_1^2}{6}\,T_y^2T_t^+\frac{\partial^2u_1}{\partial x\partial t}(b_1,y,t) + \frac{h_1^2}{6}\,T_y^-T_t^+\,\frac{\partial}{\partial x}\Big(q_1\,\frac{\partial u_1}{\partial y}\Big)(b_1,y,t) \\
&+\frac{h_1^2}{6}\,(T_x^{2-}T_y^2r_1(b_1,y))\Big(T_y^2T_t^+\frac{\partial u_1}{\partial x}(b_1,y,t)\Big), \qquad y\in\omega_k, \quad t\in\omega_\tau^-, \\
\mu_2(y,t) = &\big[\alpha_2(y)u_2(a_2,y,t) - T_y^2T_t^+\big(\alpha_2(y)u_2(a_2,y,t)\big)\big] \\
&-\Big[k\sum_{y_*\in\omega_k}\beta_2(y,y_*)u_1(b_1,y_*,t) - \int_c^d T_y^2T_t^+\big(\beta_2(y,y_*)u_1(b_1,y_*,t)\big)\,\mathrm{d}y_*\Big] \\
&-\frac{h_2^2}{6}\,T_y^2T_t^+\frac{\partial^2u_2}{\partial x\partial t}(a_2,y,t) - \frac{h_2^2}{6}\,T_y^-T_t^+\,\frac{\partial}{\partial x}\Big(q_2\,\frac{\partial u_2}{\partial y}\Big)(a_2,y,t) \\
&-\frac{h_2^2}{6}\,(T_x^{2+}T_y^2r_2(a_2,y))\Big(T_y^2T_t^+\frac{\partial u_2}{\partial x}(a_2,y,t)\Big), \qquad y\in\omega_k, \quad t\in\omega_\tau^-.
\end{aligned}
$$

The a priori estimate for the solution of FDS (23) – (28) is given in the following lemma.

Lemma 2. *Let the assumptions (8) and (22) hold and let the coefficients of FDS (23) – (28) are well defined in the mesh nodes. Then the solution z of FDS (23) – (28) satisfies a priori estimate*

$$
\begin{aligned}
\|z\|^2_{H^{1,1/2}_{h\tau}} \le C\Big(&\|\xi\|^2_{H^{1/2}_{00}(\bar\omega_\tau,L_h)} + \|\eta\|^2_{L_2(\omega_\tau^-,L_{h'})} + \|\zeta\|^2_{L_2(\omega_\tau^-,L_{h''})} \\
&+\|\chi\|^2_{L_2(\omega_\tau^-,L_h)} + \|\mu\|^2_{L_2(\omega_\tau^-,L_2(\omega_k))}\Big),
\end{aligned}
\tag{29}
$$

where C is computable constant depending on T.

The proof is analogous to the proof of Theorems 2 and 3 (see [15] and also [8, 9]).

Therefore, in order to determine the convergence rate of the FDS (16) – (21), it is sufficient to estimate the right-hand side terms in the inequality (29).

From obvious inequality

$$
\begin{aligned}
&\sum_{x\in\omega_{1,h_1}^+}\hbar_1k\sum_{y\in\omega_k}\tau^2\sum_{t\in\bar\omega_\tau}\sum_{t'\in\bar\omega_\tau,\ t'\ne t}\frac{|\xi_1(x,y,t)-\xi_1(x,y,t')|^2}{|t-t'|^2} \\
\le 4&\sum_{x\in\omega_{1,h_1}^+}\hbar_1k\sum_{y\in\omega_k}\tau^2\sum_{t\in\omega_\tau^+}\sum_{t'\in\omega_\tau^-,\ t'<t}\frac{|T_t^-\xi_1(x,y,t)-T_t^+\xi_1(x,y,t')|^2}{|t-t'|^2} \\
&+\frac{4}{3}\,\pi^2\sum_{x\in\omega_{1,h_1}^+}\hbar_1k\sum_{y\in\omega_k}\sum_{t\in\omega_\tau^+}|\xi_1(x,y,t)-T_t^-\xi_1(x,y,t)|^2 \\
&+\frac{4}{3}\,\pi^2\sum_{x\in\omega_{1,h_1}^+}\hbar_1k\sum_{y\in\omega_k}\sum_{t\in\omega_\tau^-}|\xi_1(x,y,t)-T_t^+\xi_1(x,y,t)|^2,
\end{aligned}
$$

using integral representations

$$\xi_1(x,y,t)=\frac{1}{h_1k}\int\limits_{x-h_1}^{x+h_1}\int\limits_{y-k}^{y+k}\Big(1-\frac{|x'-x|}{h_1}\Big)\Big(1-\frac{|y'-y|}{k}\Big)\Bigg(\int\limits_{x'}^{x}\int\limits_{y'}^{y}\frac{\partial^2u_1}{\partial x\partial y}(x'',y'',t)\,\mathrm{d}y''\mathrm{d}x''$$
$$-\int\limits_{x'}^{x}\int\limits_{x''}^{x}\frac{\partial^2u_1}{\partial x^2}(x''',y',t)\,\mathrm{d}x'''\,\mathrm{d}x''-\int\limits_{y'}^{y}\int\limits_{y''}^{y}\frac{\partial^2u_1}{\partial y^2}(x',y''',t)\,\mathrm{d}y'''\,\mathrm{d}y''\Bigg)\mathrm{d}y'\,\mathrm{d}x',\quad x\in\omega_{1,h_1},$$

$$\xi_1(b_1,y,t)=\frac{2}{h_1k}\int\limits_{b_1-h_1}^{b_1}\int\limits_{y-k}^{y+k}\Big(1+\frac{x'-b_1}{h_1}\Big)\Big(1-\frac{|y'-y|}{k}\Big)\Bigg(\int\limits_{x'}^{b_1}\int\limits_{y'}^{y}\frac{\partial^2u_1}{\partial x\partial y}(x'',y'',t)\,\mathrm{d}y''\mathrm{d}x''$$
$$-\int\limits_{x'}^{b_1}\int\limits_{x''}^{b_1}\frac{\partial^2u_1}{\partial x^2}(x''',y',t)\,\mathrm{d}x'''\,\mathrm{d}x''-\int\limits_{y'}^{y}\int\limits_{y''}^{y}\frac{\partial^2u_1}{\partial y^2}(x',y''',t)\,\mathrm{d}y'''\,\mathrm{d}y''\Bigg)\mathrm{d}y'\,\mathrm{d}x',$$

$$\Big(\xi_1-T_t^+\xi_1\Big)(x,y,t)=\frac{1}{h_1k\tau}\int\limits_{x-h_1}^{x+h_1}\int\limits_{y-k}^{y+k}\int\limits_{t}^{t+\tau}\Big(1-\frac{|x'-x|}{h_1}\Big)\Big(1-\frac{|y'-y|}{k}\Big)$$
$$\times\Bigg\{\int\limits_{x'}^{x}\Big[\frac{\partial u_1}{\partial x}(x'',y,t)-\frac{\partial u_1}{\partial x}(x'',y',t)-\frac{\partial u_1}{\partial x}(x'',y,t')+\frac{\partial u_1}{\partial x}(x'',y',t')\Big]\mathrm{d}x''$$
$$+\int\limits_{x'}^{x}\int\limits_{t'}^{t}\frac{\partial^2u_1}{\partial x\partial t}(x'',y',t'')\,\mathrm{d}t''\mathrm{d}x''+\int\limits_{y'}^{y}\int\limits_{t'}^{t}\frac{\partial^2u_1}{\partial y\partial t}(x',y'',t'')\,\mathrm{d}t''\mathrm{d}y''\Bigg\}\mathrm{d}t'\mathrm{d}y'\mathrm{d}x',\quad x\in\omega_{1,h_1},$$

$$\Big(\xi_1-T_t^+\xi_1\Big)(b_1,y,t)=\frac{2}{h_1k\tau}\int\limits_{b_1-h_1}^{b_1}\int\limits_{y-k}^{y+k}\int\limits_{t}^{t+\tau}\Big(1+\frac{x'-b_1}{h_1}\Big)\Big(1-\frac{|y'-y|}{k}\Big)$$
$$\times\Bigg\{\int\limits_{x'}^{b_1}\Big[\frac{\partial u_1}{\partial x}(x'',y,t)-\frac{\partial u_1}{\partial x}(x'',y',t)-\frac{\partial u_1}{\partial x}(x'',y,t')+\frac{\partial u_1}{\partial x}(x'',y',t')\Big]\mathrm{d}x''$$
$$+\int\limits_{x'}^{b_1}\Big[\frac{\partial u_1}{\partial x}(x'',y',t)-\frac{\partial u_1}{\partial x}(b_1,y',t)-\frac{\partial u_1}{\partial x}(x'',y',t')+\frac{\partial u_1}{\partial x}(b_1,y',t')\Big]\mathrm{d}x''$$
$$+\int\limits_{y'}^{y}\int\limits_{t'}^{t}\frac{\partial^2u_1}{\partial y\partial t}(x',y'',t'')\,\mathrm{d}t''\mathrm{d}y''\Bigg\}\mathrm{d}t'\,\mathrm{d}y'\,\mathrm{d}x'$$

and Lemmas 2–4 from [10], in the same manner as in [3], [8], we obtain the estimates

$$\sum_{x\in\omega_{1,h_1}^+}\hbar_1k\sum_{y\in\omega_k}\tau^2\sum_{t\in\bar\omega_\tau}\sum_{t'\in\bar\omega_\tau,\ t'\neq t}\frac{|\xi_1(x,y,t)-\xi_1(x,y,t')|^2}{|t-t'|^2}$$
$$\le C(h_1^4+k^4+\tau^2)\,\|u_1\|^2_{H^{3,3/2}(Q_1)}$$

and

$$\sum_{x\in\omega_{1,h_1}^+}\hbar_1k\sum_{y\in\omega_k}\tau\sum_{t\in\bar\omega_\tau}\Big(\frac{1}{t+\tau/2}+\frac{1}{T-t+\tau/2}\Big)\xi_1^2(x,y,t)$$
$$\le C(h_1^4+k^4)\,\log\frac{1}{\tau}\,\|u_1\|^2_{H^{3,3/2}(Q_1)}.$$

Analogous results hold for ξ_2. In such a way we obtain

$$\begin{aligned} |\xi|^2_{H^{1/2}(\bar\omega_\tau, L_h)} &\le C(h^4+\tau^2)\,\|u\|^2_{H^{3,3/2}}, \\ \|\xi\|^2_{\widetilde{L}_2(\bar\omega_\tau, L_h)} &\le Ch^4\log\frac{1}{\tau}\,\|u\|^2_{H^{3,3/2}}, \end{aligned}$$

and

$$\|\xi\|^2_{H^{1/2}_{00}(\bar\omega_\tau, L_h)} \le C\Big(h^4\log\frac{1}{\tau}+\tau^2\Big)\,\|u\|^2_{H^{3,3/2}}, \tag{30}$$

where $h=\max\{h_1,h_2,k\}$ and $\|u\|^2_{H^{3,3/2}}=\|u_1\|^2_{H^{3,3/2}(Q_1)}+\|u_2\|^2_{H^{3,3/2}(Q_2)}$.

Term η_1 can be decomposed in the following way:

$$\begin{aligned} &\eta_1=\eta_{11}+\eta_{12}+\eta_{13}, \quad \text{where} \\ &\eta_{11}=T_x^-T_y^2\Big(p_1\Big(T_t^+\frac{\partial u_1}{\partial x}\Big)\Big)-\big(T_x^-T_y^2p_1\big)\Big(T_x^-T_y^2\Big(T_t^+\frac{\partial u_1}{\partial x}\Big)\Big), \\ &\eta_{12}=\Big[\big(T_x^-T_y^2p_1\big)-\bar p_1\Big]\Big(T_x^-T_y^2\Big(T_t^+\frac{\partial u_1}{\partial x}\Big)\Big), \\ &\eta_{13}=\bar p_1\Big[T_x^-T_y^2\Big(T_t^+\frac{\partial u_1}{\partial x}\Big)-\,u_{1,\bar x}\Big]. \end{aligned}$$

From the estimates obtained in [8] it immediately follows that

$$h_1k\tau\sum_{x\in\omega^+_{1,h_1}}\sum_{y\in\omega_k}\sum_{t\in\omega^-_\tau}\eta^2_{1j}(x,y,t)\le C(h_1^4+k^4)\,\|p_1\|^2_{H^2(\Omega_1)}\,\|u_1\|^2_{H^{3,3/2}(Q_1)}, \quad j=1,2.$$

Further, we have

$$\begin{aligned} T_x^-T_y^2\Big(T_t^+\frac{\partial u_1}{\partial x}\Big)-\,u_{1,\bar x}=\frac{1}{k\tau}\int_{y-k}^{y+k}\int_t^{t+\tau}\Big(1-\frac{|y'-y|}{k}\Big)\Bigg\{\int_y^{y'}\int_y^{y''}\frac{\partial^2U_1}{\partial y^2}(y'',t')\,\mathrm{d}y'''\mathrm{d}y'' \\ +\int_t^{t'}\frac{\partial U_1}{\partial t}(y',t'')\,\mathrm{d}t''+\Big[U_1(y',t)-U_1(y,t)-U_1(y',t')+U_1(y,t')\Big]\Bigg\}\,\mathrm{d}t'\,\mathrm{d}y', \end{aligned}$$

where we denoted $U_1=T_x^-\frac{\partial u_1}{\partial x}$. Estimating this integral using Lemmas 2–4 from [10] and performing summation over the mesh we obtain

$$\begin{aligned} h_1k\tau\sum_{x\in\omega^+_{1,h_1}}\sum_{y\in\omega_k}\sum_{t\in\omega^-_\tau}\eta^2_{13}(x,y,t) &\le C(k^4+\tau^2)\,\|p_1\|^2_{C(\bar\Omega_1)}\,\|u_1\|^2_{H^{3,3/2}(Q_1)} \\ &\le C(k^4+\tau^2)\,\|p_1\|^2_{H^2(\Omega_1)}\,\|u_1\|^2_{H^{3,3/2}(Q_1)}. \end{aligned}$$

From previous inequalities and analogous inequalities for η_2 we get

$$\|\eta\|^2_{L_2(\omega^-_\tau, L_{h'})}\le C(h^4+\tau^2)\,\max_i\|p_i\|^2_{H^2(\Omega_i)}\,\|u\|^2_{H^{3,3/2}}. \tag{31}$$

Term ζ_1 for $x\in\omega_{1,h_1}$ is fully analogous to η_1. For $x=b_1$ we set

$$\zeta_1(b_1,y,t)=\zeta_{11}(b_1,y,t)+\zeta_{12}(b_1,y,t)+\zeta_{13}(b_1,y,t)+\zeta_{14}(b_1,y,t)+\zeta_{15}(b_1,y,t),$$

where

$$\begin{aligned}
\zeta_{11}(b_1,y,t) &= T_x^{2-}T_y^-\Big(q_1\Big(T_t^+\frac{\partial u_1}{\partial y}\Big)\Big) - \big(T_x^{2-}T_y^-q_1\big)\Big(T_x^{2-}T_y^-\Big(T_t^+\frac{\partial u_1}{\partial y}\Big)\Big),\\
\zeta_{12}(b_1,y,t) &= \Big[\big(T_x^{2-}T_y^-q_1\big) - \bar q_1 - \frac{h_1}{3}\,T_y^-\frac{\partial q_1}{\partial x}\Big]\Big(T_x^{2-}T_y^-\Big(T_t^+\frac{\partial u_1}{\partial y}\Big)\Big),\\
\zeta_{13}(b_1,y,t) &= \bar q_1\Big[T_x^{2-}T_y^-\Big(T_t^+\frac{\partial u_1}{\partial y}\Big) - u_{1,\bar y} - \frac{h_1}{3}\,T_y^-T_t^+\frac{\partial^2 u_1}{\partial x\partial y}\Big],\\
\zeta_{14}(b_1,y,t) &= \frac{h_1}{3}\Big[\Big(T_y^-\frac{\partial q_1}{\partial x}\Big)\Big(T_x^{2-}T_y^-\Big(T_t^+\frac{\partial u_1}{\partial y}\Big)\Big) - T_y^-\Big(\frac{\partial q_1}{\partial x}\Big(T_t^+\frac{\partial u_1}{\partial y}\Big)\Big)\Big],\\
\zeta_{15}(b_1,y,t) &= \frac{h_1}{3}\Big[\bar q_1\,T_y^-\Big(T_t^+\frac{\partial^2 u_1}{\partial x\partial y}\Big) - T_y^-\Big(q_1T_t^+\frac{\partial^2 u_1}{\partial x\partial y}\Big)\Big].
\end{aligned}$$

Terms $\zeta_{11}(b_1,y,t)$, $\zeta_{12}(b_1,y,t)$ and $\zeta_{12}(b_1,y,t)$ can be estimated in the same manner as the corresponding terms for $x\neq b_1$, while $\zeta_{14}(b_1,y,t)$ and $\zeta_{15}(b_1,y,t)$ satisfy the same estimate as ζ_{11}. Analogous results hold for ζ_2, whereby

$$\|\zeta\|^2_{L_2(\omega_\tau^-,L_{h''})} \le C(h^4+\tau^2)\,\max_i\|q_i\|^2_{H^2(\Omega_i)}\,\|u\|^2_{H^{3,3/2}}. \tag{32}$$

For $x\in\omega_{1,h_1}$ the term χ_1 can be decomposed in the following way:

$$\begin{aligned}
\chi_1 &= \chi_{11}+\chi_{12}+\chi_{13}+\chi_{14}, \quad \text{where}\\
\chi_{11} &= \big(T_x^2T_y^2r_1\big)\big(\xi_1 - T_t^+\xi_1\big),\\
\chi_{12} &= \big(T_x^2T_y^2r_1\big)\big(T_t^+\xi_1\big),\\
\chi_{13} &= \big(T_x^2T_y^2r_1\big)\Big(T_x^2T_y^2\big(u_1 - T_t^+u_1\big)\Big),\\
\chi_{14} &= \big(T_x^2T_y^2r_1\big)\Big(T_x^2T_y^2\big(T_t^+u_1\big)\Big) - T_x^2T_y^2\Big(r_1\big(T_t^+u_1\big)\Big).
\end{aligned}$$

For $x=b_1$ we set

$$\begin{aligned}
\chi_1 &= \chi_{11}+\chi_{12}+\chi_{13}+\chi_{14}+\chi_{15}, \quad \text{where}\\
\chi_{11}(b_1,y,t) &= \big(T_x^{2-}T_y^2r_1\big)\big(\xi_1 - T_t^+\xi_1\big),\\
\chi_{12}(b_1,y,t) &= \big(T_x^{2-}T_y^2r_1\big)\big(T_t^+\xi_1\big),\\
\chi_{13}(b_1,y,t) &= \big(T_x^{2-}T_y^2r_1\big)\Big(T_x^{2-}T_y^2\big(u_1 - T_t^+u_1\big)\Big),\\
\chi_{14}(b_1,y,t) &= \big(T_x^{2-}T_y^2r_1\big)\Big(T_x^{2-}T_y^2\big(T_t^+u_1\big)\Big) - T_x^{2-}T_y^2\Big(r_1\big(T_t^+u_1\big)\Big),\\
\chi_{15}(b_1,y,t) &= \frac{h_1}{3}\,\big(T_x^{2-}T_y^2r_1\big)\Big(T_y^2\frac{\partial u_1}{\partial x} - T_t^+T_y^2\frac{\partial u_1}{\partial x}\Big).
\end{aligned}$$

From the integral representations of ξ_1 we easily obtain

$$k\tau\sum_{x\in\omega_{1,h_1}^+}\hbar_1\sum_{y\in\omega_k}\sum_{t\in\omega_\tau^-}\chi_{11}^2(x,y,t) \le C(h_1^4+k^4+\tau^2)\,\|r_1\|^2_{L_2(\Omega_1)}\,\|u_1\|^2_{H^{3,3/2}(Q_1)}$$

and

$$k\tau\sum_{x\in\omega_{1,h_1}^+}\hbar_1\sum_{y\in\omega_k}\sum_{t\in\omega_\tau^-}\chi_{12}^2(x,y,t) \le C(h_1^4+k^4)\,\|r_1\|^2_{H^1(\Omega_1)}\,\|u_1\|^2_{H^{3,3/2}(Q_1)}.$$

Similarly,

$$k\tau \sum_{x\in\omega_{1,h_1}^+} \hbar_1 \sum_{y\in\omega_k} \sum_{t\in\omega_\tau^-} \chi_{13}^2(x,y,t) \le C\tau^2 \|r_1\|^2_{H^1(\Omega_1)} \|u_1\|^2_{H^{3,3/2}(Q_1)}$$

and

$$k\tau \sum_{x\in\omega_{1,h_1}^+} \hbar_1 \sum_{y\in\omega_k} \sum_{t\in\omega_\tau^-} \chi_{14}^2(x,y,t) \le C(h_1^4+k^4) \|r_1\|^2_{H^1(\Omega_1)} \|u_1\|^2_{H^{3,3/2}(Q_1)}.$$

Finally, using Lemmas 2–4 from [10] and the trace theorem for anisotropic Sobolev spaces [18], we obtain

$$\frac{h_1}{2}\, k\tau \sum_{y\in\omega_k} \sum_{t\in\omega_\tau^-} \chi_{15}^2(b_1,y,t) \le C(h_1^4+\tau^2) \|r_1\|^2_{H^1(\Omega_1)} \|u_1\|^2_{H^{3,3/2}(Q_1)}.$$

From these estimates, and analogous ones for χ_2, we get

$$\|\chi\|^2_{L_2(\omega_\tau^-,L_h)} \le C(h^4+\tau^2)\, \max_i \|r_i\|^2_{H^1(\Omega_i)} \|u\|^2_{H^{3,3/2}}. \tag{33}$$

The term μ_1 can be decomposed as

$$\begin{aligned}
\mu_1 &= \mu_{11}+\mu_{12}+\mu_{13}+\mu_{14}+\mu_{15}+\mu_{16}+\mu_{17}, \quad \text{where}\\
\mu_{11} &= \alpha_1(y)u_1(b_1,y,t) - T_y^2T_t^+\big(\alpha_1(y)u_1(b_1,y,t)\big),\\
\mu_{12} &= k\sum_{y_*\in\omega_k}\Big\{\beta_1(y,y_*)\big[T_{y_*}^2T_t^+u_2(a_2,y_*,t) - T_{y_*}^2u_2(a_2,y_*,t)\big]\\
&\qquad + \big[T_y^2T_{y_*}^2\beta_1(y,y_*) - T_{y_*}^2u_2(a_2,y_*,t)\big]T_t^+u_2(a_2,y_*,t)\Big\},\\
\mu_{13} &= k\sum_{y_*\in\omega_k}\Big\{\beta_1(y,y_*)\big[-T_{y_*}^2T_t^+u_2(a_2,y_*,t) + T_{y_*}^2u_2(a_2,y_*,t)\\
&\qquad + T_t^+u_2(a_2,y_*,t) - u_2(a_2,y_*,t)\big] + \big[-T_y^2T_{y_*}^2\beta_1(y,y_*) + T_{y_*}^2\beta_1(y,y_*)\\
&\qquad + T_y^2\beta_1(y,y_*) - \beta_1(y,y_*)\big]T_t^+u_2(a_2,y_*,t)\Big\},\\
\mu_{14} &= \int_c^d \big(T_y^2\beta_1(y,y_*)\big)\big(T_t^+u_2(a_2,y_*,t)\big)\,\mathrm{d}y_* - k\sum_{y_*\in\omega_k}\big(T_y^2\beta_1(y,y_*)\big)\big(T_t^+u_2(a_2,y_*,t)\big),\\
\mu_{15} &= \frac{h_1^2}{6}\,T_y^2T_t^+\frac{\partial^2u_1}{\partial x\partial t}(b_1,y,t),\\
\mu_{16} &= \frac{h_1^2}{6}\,T_y^-T_t^+\frac{\partial}{\partial x}\Big(q_1\frac{\partial u_1}{\partial y}\Big)(b_1,y,t),\\
\mu_{17} &= \frac{h_1^2}{6}\,\big(T_x^{2-}T_y^2r_1(b_1,y)\big)\Big(T_y^2T_t^+\frac{\partial u_1}{\partial x}(b_1,y,t)\Big).
\end{aligned}$$

Further, we have

$$\begin{aligned}
\mu_{11}(y,t) = \frac{1}{k\tau}\int_{y-k}^{y+k}\int_t^{t+\tau}\Big(1-\frac{|y'-y|}{k}\Big)\Big\{&U_1(y,t)-U_1(y',t)-U_1(y,t')+U_1(y',t')\\
&-\int_t^{t'}\frac{\partial U_1}{\partial t}(y',t'')\,\mathrm{d}t'' - \int_y^{y'}\int_y^{y''}\frac{\partial^2U_1}{\partial y^2}(y'',t')\,\mathrm{d}y'''\mathrm{d}y''\Big\}\,\mathrm{d}t'\,\mathrm{d}y',
\end{aligned}$$

where we denoted $U_1(y,t) = \alpha_1(y)u_1(b_1,y,t)$. Using Lemmas 2–4 from [10] we obtain

$$k\tau \sum_{y\in\omega_k} \sum_{t\in\omega_\tau^-} \mu_{11}^2(y,t) \le C(h_1^4+\tau^2)\,\|U_1\|^2_{H^{2,1}((c,d)\times(0,T))}$$
$$\le C(h_1^4+\tau^2)\,\|\alpha_1\|^2_{H^2(c,d)}\,\|u_1\|^2_{H^{3,3/2}(Q_1)}.$$

Similarly, one obtains

$$k\tau \sum_{y\in\omega_k} \sum_{t\in\omega_\tau^-} \mu_{12}^2(y,t) \le C\tau^2\,\|\beta_1\|^2_{C([c,d]^2)}\Big\|\frac{\partial u_2}{\partial t}(a_2,\cdot,\cdot)\Big\|^2_{L_2((c,d)\times(0,T))}$$
$$+Ck^4\Big\|\frac{\partial^2\beta_1}{\partial y^2}\Big\|^2_{L_2((c,d)^2)}\|u_2\|^2_{C(\bar Q_2)} \le C(k^4+\tau^2)\,\|\beta_1\|^2_{H^2((c,d)^2)}\,\|u_2\|^2_{H^{3,3/2}(Q_2)}$$

and

$$k\tau \sum_{y\in\omega_k} \sum_{t\in\omega_\tau^-} \mu_{13}^2(y,t) \le Ck^2\tau^{3/2}\,\|\beta_1\|^2_{C([c,d]^2)}\,\|u_2\|^2_{H^{5/2,5/4}((c,d)\times(0,T))}$$
$$+Ck^4\Big\|\frac{\partial^2\beta_1}{\partial y\partial y_*}\Big\|^2_{L_2((c,d)^2)}\|u_2\|^2_{C(\bar Q_2)} \le C(k^4+\tau^2)\,\|\beta_1\|^2_{H^2((c,d)^2)}\,\|u_2\|^2_{H^{3,3/2}(Q_2)}.$$

Using the error formula for trapezoidal rule we get

$$\mu_{14}^2(y,t) \le Ck^4 \int_c^d \Big|\frac{\partial^2}{\partial y_*^2}\Big(\big(T_y^2\beta_1(y,y_*)\big)\big(T_t^+u_2(a_2,y_*,t)\big)\Big)\Big|^2 \mathrm{d}y_*,$$

whereby

$$k\tau \sum_{y\in\omega_k} \sum_{t\in\omega_\tau^-} \mu_{14}^2(y,t) \le Ck^4\,\|\beta_1\|^2_{H^2((c,d)^2)}\,\|u_2\|^2_{H^{3,3/2}(Q_2)}.$$

Using boundary condition (3) we derive the inequality

$$k\tau \sum_{y\in\omega_k} \sum_{t\in\omega_\tau^-} \mu_{15}^2(y,t) \le Ch_1^4\Big(\|\alpha_1\|^2_{H^2(c,d)}\,\|u_1\|^2_{H^{3,3/2}(Q_1)}$$
$$+\|\beta_1\|^2_{L_2((c,d)^2)}\,\|u_2\|^2_{H^{3,3/2}(Q_2)}\Big).$$

Terms μ_{16} and μ_{17} can be estimated directly:

$$k\tau \sum_{y\in\omega_k} \sum_{t\in\omega_\tau^-} \mu_{16}^2(y,t) \le Ch_1^4\,\|q_1\|^2_{H^2(\Omega_1)}\,\|u_1\|^2_{H^{3,3/2}(Q_1)},$$

$$k\tau \sum_{y\in\omega_k} \sum_{t\in\omega_\tau^-} \mu_{17}^2(y,t) \le Ch_1^4\,\|r_1\|^2_{H^1(\Omega_1)}\,\|u_1\|^2_{H^{3,3/2}(Q_1)}.$$

From obtained inequalities, and analogous ones for μ_2, we have that

$$\|\mu\|^2_{L_2(\omega_\tau^-,L_2(\omega_k))} \le C(h^4+\tau^2)\Big(\max_i\|q_i\|^2_{H^2(\Omega_i)}+\max_i\|r_i\|^2_{H^1(\Omega_i)}$$
$$+\max_i\|\alpha_i\|^2_{H^2(c,d)}+\max_i\|\beta_i\|^2_{H^2((c,d)^2)}\Big)\|u\|^2_{H^{3,3/2}}. \tag{34}$$

Finally, from (29) – (34) one obtains the next assertion.

Theorem 4. *Let p_i, $q_i \in H^2(\Omega_i)$, $r_i \in H^1(\Omega_i)$, $\alpha_i \in H^2(c,d)$, $\beta_i \in H^2((c,d)^2)$, $i=1,2$, and let the assumptions (8) and (22) hold. Further, let the solution of IBVP (1) – (6) belong to the space $H^{3,3/2}$. Then the solution v of FDS (16) – (21)*

converges to the solution u of IBVP (1) – (6) and the following convergence rate estimate holds:

$$\|u-v\|_{H^{1,1/2}_{h\tau}} \le C\big(h^2\sqrt{\log 1/\tau}+\tau\big)\Big(1+\max_i \|p_i\|_{H^2}+\max_i \|q_i\|_{H^2} + \max_i \|r_i\|_{H^1}+\max_i \|\alpha_i\|_{H^2(c,d)}+\max_i \|\beta_i\|_{H^2((c,d)^2)}\Big)\|u\|_{H^{3,3/2}}. \tag{35}$$

5. Factorized finite difference scheme

Substituting in (16) – (19) the upward finite differences $v_{i,t}$ with backward differences $v_{i,\bar t}$, $i=1,2$, one obtains implicit FDS for the IBVP (1) – (6). Implicit FDS is unconditionally stable (i.e. without the assumption (22)) and also satisfies the convergence rate estimate (35). Unfortunately, implicit FDS is numerically inefficient, because on each time level we need to solve an elliptic difference problem.

Let us now consider the following factorized finite difference scheme

$$v_{1,t}-\sigma_1\tau v_{1,t\bar x x}-\theta_1\tau v_{1,t\bar y y}+\sigma_1\theta_1\tau^2 v_{1,t\bar x x\bar y y}-(\bar p_1 v_{1,\bar x})_x -(\bar q_1 v_{1,\bar y})_y+\bar r_1 v_1=\bar f_1, \qquad x\in\omega_{1,h_1},\quad y\in\omega_k,\quad t\in\omega_\tau^-, \tag{36}$$

$$\begin{aligned} &v_{1,t}(b_1,y,t)+\frac{2\sigma_1\tau}{h_1}\,v_{1,t\bar x}(b_1,y,t)-\theta_1\tau v_{1,t\bar y y}(b_1,y,t)-\frac{2\sigma_1\theta_1\tau^2}{h_1}\,v_{1,t\bar x\bar y y}(b_1,y,t)\\ &\qquad +\frac{2}{h_1}\Big[\bar p_1(b_1,y)v_{1,\bar x}(b_1,y,t)+\alpha_1(y)v_1(b_1,y,t)\\ &\qquad -k\sum_{y_*\in\omega_k}\beta_1(y,y_*)v_2(a_2,y_*,t)\Big]-(\bar q_1 v_{1,\bar y})_y(b_1,y,t)\\ &\qquad +\bar r_1(b_1,y)v_1(b_1,y,t)=\bar f_1(b_1,y,t), \qquad y\in\omega_k,\quad t\in\omega_\tau^-, \end{aligned} \tag{37}$$

$$v_{2,t}-\sigma_2\tau v_{2,t\bar x x}-\theta_2\tau v_{2,t\bar y y}+\sigma_2\theta_2\tau^2 v_{2,t\bar x x\bar y y}-(\bar p_2 v_{2,\bar x})_x -(\bar q_2 v_{2,\bar y})_y+\bar r_2 v_2=\bar f_2, \qquad x\in\omega_{2,h_2}\quad y\in\omega_k,\quad t\in\omega_\tau^-, \tag{38}$$

$$\begin{aligned} &v_{2,t}(a_2,y,t)-\frac{2\sigma_2\tau}{h_2}\,v_{2,tx}(a_2,y,t)-\theta_2\tau v_{2,t\bar y y}(a_2,y,t)+\frac{2\sigma_2\theta_2\tau^2}{h_2}\,v_{2,tx\bar y y}(a_2,y,t)\\ &\qquad -\frac{2}{h_2}\Big[\bar p_2(a_2+h_2,y)v_{2,x}(a_2,y,t)-\alpha_2(y)v_2(a_2,y,t)\\ &\qquad +k\sum_{y_*\in\omega_k}\beta_2(y,y_*)v_1(b_1,y_*,t)\Big]-(\bar q_2 v_{2,\bar y})_y(a_2,y,t)\\ &\qquad +\bar r_2(a_2,y)v_2(a_2,y,t)=\bar f_2(a_2,y,t), \qquad y\in\omega_k,\quad t\in\omega_\tau^-, \end{aligned} \tag{39}$$

subject to initial and boundary conditions (20) – (21).

For sufficiently large σ_i and θ_i, i.e.

$$\sigma_i=\frac{1+\varepsilon}{2}\max_{(x,y)\in\Omega_i} p_i(x,y)\,,\quad \theta_i=\frac{1+\varepsilon}{2}\max_{(x,y)\in\Omega_i} q_i(x,y)\,,\quad \varepsilon>0,\quad i=1,2, \tag{40}$$

from the general theory of difference schemes [21] follows that the FDS (36) – (39), (20), (21) is unconditionally stable. This FDS is computationally efficient, because on each time level it can be resolved by two application of Thomas algorithm. In such a manner, factorized finite difference scheme combine the good properties of explicit and implicit schemes.

Factorized FDS satisfies an a priori estimate analogous to (29):

$$\begin{aligned} \|z\|^2_{H^{1,1/2}_{h\tau}}\le C\Big(&\|\xi\|^2_{H^{1/2}_{00}(\bar\omega_\tau,L_h)}+\|\eta\|^2_{L_2(\omega_\tau^-,L_{h'})}+\|\zeta\|^2_{L_2(\omega_\tau^-,L_{h''})}+\|\chi\|^2_{L_2(\omega_\tau^-,L_h)}\\ &+\|\lambda\|^2_{L_2(\omega_\tau^-,L_{h'})}+\|\nu\|^2_{L_2(\omega_\tau^-,L_{h''})}+\|\upsilon\|^2_{L_2(\omega_\tau^-,L_{h'})}+\|\mu\|^2_{L_2(\omega_\tau^-,L_2(\omega_k))}\Big), \end{aligned} \tag{41}$$

where ξ, η, ζ, χ and μ are as in the previous section and

$$\lambda_i = \sigma_i \tau \, u_{i,t\bar{x}}, \qquad \nu_i = \theta_i \tau \, u_{i,t\bar{y}}, \qquad \upsilon_i = \sigma_i \theta_i \tau^2 \, u_{i,t\bar{x}y\bar{y}}, \qquad i = 1, 2.$$

From integral representation

$$\lambda_1(x,y,t) = \frac{\sigma}{h_1 k \tau} \int\limits_{x-h_1}^{x} \int\limits_{y-k}^{y+k} \int\limits_{t}^{t+\tau} \Big(1 - \frac{|y'-y|}{k}\Big) \Big\{ \tau \, \frac{\partial^2 u_1}{\partial x \partial t}(x'', y', t'')$$
$$+ \Big[\frac{\partial u_1}{\partial x}(x', y, t+\tau) - \frac{\partial u_1}{\partial x}(x', y, t') - \frac{\partial u_1}{\partial x}(x', y', t+\tau) + \frac{\partial u_1}{\partial x}(x', y', t') \Big]$$
$$- \Big[\frac{\partial u_1}{\partial x}(x', y, t) - \frac{\partial u_1}{\partial x}(x', y, t') - \frac{\partial u_1}{\partial x}(x', y', t) + \frac{\partial u_1}{\partial x}(x', y', t') \Big] \Big\} \mathrm{d}t' \mathrm{d}y' \mathrm{d}x',$$

using Lemmas 2–4 from [10] we obtain

$$h_1 k \tau \sum_{x \in \omega_{1,h_1}^{+}} \sum_{y \in \omega_k} \sum_{t \in \omega_\tau^{-}} \lambda_1^2(x,y,t) \le C(\tau^2 + k^4) \, \|u_1\|^2_{H^{3,3/2}(Q_1)}.$$

An analogous inequality holds for λ_2, whereby

$$\|\lambda\|^2_{L_2(\omega_\tau^-, L_{h'})} \le C(h^4 + \tau^2) \, \|u\|^2_{H^{3,3/2}}. \tag{42}$$

Term ν can be estimated in the same manner:

$$\|\nu\|^2_{L_2(\omega_\tau^-, L_{h''})} \le C(h^4 + \tau^2) \, \|u\|^2_{H^{3,3/2}}. \tag{43}$$

From

$$\upsilon_i(x,y,t) = \frac{\theta_i \tau}{k^2} \, \big[\lambda_i(x, y+k, t) - 2\lambda_i(x,y,t) + \lambda_i(x, y+k, t) \big]$$

immediately follows

$$\|\upsilon\|^2_{L_2(\omega_\tau^-, L_{h'})} \le C\Big(\tau^2 + \frac{\tau^4}{h^4}\Big) \|u\|^2_{H^{3,3/2}}. \tag{44}$$

Finally, from (41), (30) – (34) and (42) – (44) one obtains the next assertion.

Theorem 5. *Let p_i, $q_i \in H^2(\Omega_i)$, $r_i \in H^1(\Omega_i)$, $\alpha_i \in H^2(c,d)$, $\beta_i \in H^2((c,d)^2)$, $i = 1, 2$, and let the assumptions (8) hold. Further, let the solution of IBVP (1) – (6) belong to the space $H^{3,3/2}$ and let $\tau \asymp h_1^2 \asymp h_2^2 \asymp k^2$. Then the solution v of FDS (36) – (40), (20), (21) converges to the solution u of IBVP (1) – (6) and the following convergence rate estimate holds:*

$$\|u - v\|_{H^{1,1/2}_{h\tau}} \le C \, h^2 \sqrt{\log 1/h} \, \Big(1 + \max_i \|p_i\|_{H^2} + \max_i \|q_i\|_{H^2}$$
$$+ \max_i \|r_i\|_{H^1} + \max_i \|\alpha_i\|_{H^2(c,d)} + \max_i \|\beta_i\|_{H^2((c,d)^2)} \Big) \|u\|_{H^{3,3/2}}.$$

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University of Belgrade, Faculty of Mathematics, 11000 Belgrade, Serbia
E-mail: bosko@matf.bg.ac.yu

University of Rousse, Department of Mathematics, 7017 Rousse, Bulgaria
E-mail: vulkov@ami.ru.acad.bg