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A POSTERIORI ERROR ESTIMATORS FOR NONCONFORMING APPROXIMATION

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Abstract. In this paper, an alternative approach for constructing an a posteriori error estimator for non-conforming approximation of scalar elliptic equation is introduced. The approach is based on the usage of post-processing conforming finite element approximation of the non-conforming solution . Then, the compatible a posteriori error estimator is defined by the local norms of difference between the nonconforming approximation and conforming post-processing approximation on the element plus an additional residual term. We prove in general dimension the efficiency and the reliability of these estimators, without Helmholtz decomposition of the error, nor regularity assumption on the solution or the domain, nor saturation assumption. Finally explicit constants are given, which prove that these estimators are robust in suitable norms

Key Words. Nonconforming finite elements, a posteriori error estimators.

1. 1. Introduction

During the last 15-20 years a big amount of work has been devoted to a posteriori error estimation problem, i.e computing reliable bounds on the error of given numerical approximation to the solution of partial differential equations using only numerical solution and the given data. In order to be operating the a posteriori error estimator should be neither under nor overestimate the error. Most of the work concern the conforming finite element methods [8] and there is no much papers dealing with the nonconforming approximations (see e.g [5][4]). It turned out that in this case some extra terms have to be added to well-know a posteriori error estimator used for conforming case. In [5][4], these extra terms are the jumps across the element edges of the tangential derivatives of the finite element approximation with respect to element edges. In [2], other approach for constructing an a posteriori error estimator is considered which is based on the solution of two local sub-problems.

In this paper, an alternative approach is presented which is based on the usage of post-processing conforming finite elements approximation $\hat{u}_h$ of the nonconforming solution u_h. Then, the compatible a posteriori error estimator is defined as the local norms of $u_h - \hat{u}_h$ on the element plus an additional residual term. We prove in general dimension, without Helmholtz decomposition of the error, nor regularity of the solution or the domain, nor saturation assumption, the efficiency and the reliability of our estimator. Since most known a posteriori error estimates yield two-sided bounds on the error which contain multiplicative constants, an explicit knowledge of such constants is mandatory for a correct calibration of the a posteriori error estimates. The norms of the quasi-interpolation operator have recently been

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estimated explicitly in [7]. In this paper we give explicitly such constants for our estimators.

In the next section, we give some technical lemmas we need in order to estimate the constants in the upper bound of the error. The estimators are introduced in section 3 and the proof of their efficiency and reliability is given.

In order to avoid technical difficulties and to make the underlying ideas as clear as possible, we consider the simple elliptic model problem:

$$\begin{cases} \text{Find } u \text{ such that} \\ -div(A.\nabla u) = f, \quad \text{on } \Omega \\ u = 0, \quad \text{in } \partial\Omega \end{cases},$$

where Ω is an opened bounded polygonal domain in $\mathbb{R}^d$ $(d = 2, 3)$ and A is piecewise constant, elliptic and symmetric matrix.

Let $\mathcal{T}_h$ be a conforming triangulation of Ω by triangles or tetrahedrons but nor regular in the sense of Ciarlet [3], we denoted by E_I the set of interior edges (faces)and by E_f the set of all edges (faces) included in $\Gamma := \partial\Omega$. Let V_h be the lowest order nonconforming finite element space defined in [3] :

$$V_h = \{v_h \in L^2(\Omega); \forall T \in \mathcal{T}_h,\ v_{h|T} \in P_1(T), \forall e \in E_I,\ \textstyle\int_e [v_h]_e d\sigma = 0 \\ \text{and } \ \forall e \in E_f,\ \int_e v_h d\sigma = 0\}$$

where $[.]_e$ denotes the jump of the concerned function across e.

We consider the following discrete problem

$$\begin{cases} \text{Find } u_h \in V_h \ \text{ such that} \\ \forall v_h \in V_h; \ \displaystyle\sum_{T \in \mathcal{T}_h} \int_T A.\nabla u_h.\nabla v_h dx = \int_\Omega \hat{f} v_h dx, \end{cases}$$

where $\hat{f}$ is an approximation of f.

2. Some technical Lemmas

Let us introduce the norms $\|A^{1/2}.\|$ and $\|A^{-1/2}.\|$ defined by :

$$\forall x \in \mathbb{R}^d, \quad \|A^{1/2}x\|^2 =< Ax, x > \quad \text{and} \quad \|A^{-1/2}x\|^2 =< A^{-1}x, x > .$$

For all $T \in \mathcal{T}_h$, we denote by E_T the set of edges (faces) of T and we set :

$$h_{A,T} = \max_{x,y \in T} \|A^{-1/2}(x - y)\|,$$

and

$$\rho_{T,A} = 2 \sup_{x \in T} \inf_{y \in \partial T} \|A^{-1/2}(x - y)\|.$$

In the sequel of this paper, we set

$$\mu = \inf_{0 \le \epsilon < 1/2} \frac{(\int_0^1 (1-t)^{2\epsilon} min(t^{-d}, (1-t)^{-d}) dt)^{1/2}}{(1 - 2\epsilon)^{1/2}}.$$

Let us remark that :

$$\mu \le (\int_0^1 min(t^{-d}, (1-t)^{-d}) dt)^{1/2} = (2\frac{2^{d-1} - 1}{d - 1})^{1/2} = d^{1/2}, \quad d = 2, 3.$$

First, we have the lemma

Lemma 1. *For all $T \in \mathcal{T}_h$ and $v \in H^1(T)$. We have*

$$(\int_T (\int_T \int_0^1 |\nabla v(x+t(y-x)).(y-x)|dtdx)^2 dy)^{1/2} \leq \mu \times meas_d(T) h_{A,T} \|A^{1/2}\nabla v\|_{0,T}.$$

Proof: First, for all $\epsilon \in [0, 1/2[$ we have

$$\begin{aligned}
\int_T \int_0^1 |\nabla v(x+t(y-x)).(y-x)|dtdx &\leq h_{A,T} \int_{]0,1[\times T} \|A^{1/2}\nabla v(x+t(y-x))\|dxdt \\
&\leq h_{A,T} (\int_{]0,1[\times T} (1-t)^{-2\epsilon} dt)^{1/2} (\int_{]0,1[\times T} (1-t)^{2\epsilon} \|A^{1/2}\nabla v(x+t(y-x))\|^2 dtdx)^{1/2} \\
&\leq \frac{h_{A,T}(meas_d(T))^{1/2}}{(1-2\epsilon)^{1/2}} (\int_{]0,1[\times T} (1-t)^{2\epsilon} \|A^{1/2}\nabla v(x+t(y-x))\|^2 dtdx)^{1/2}.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&(\int_T (\int_T \int_0^1 |\nabla v(x+t(y-x)).(y-x)|dtdx)^2 dy)^{1/2} \\
&\quad \leq \frac{h_{A,T}(meas_d(T))^{1/2}}{(1-2\epsilon)^{1/2}} (\int_0^1 (1-t)^{2\epsilon} \{\int_{T\times T} \|A^{1/2}\nabla v(x+t(y-x))\|^2 dxdy\} dt)^{1/2}.
\end{aligned}$$

Using the change of variables:

$$x \longrightarrow z = x+t(y-x) \ \text{ if } t \leq 1/2 \ , \ y \longrightarrow z = x+t(y-x) \ \text{ if } t \geq 1/2,$$

we have

$$\begin{aligned}
&(\int_T (\int_T \int_0^1 |\nabla v(x+t(y-x)).(y-x)|dtdx)^2 dy)^{1/2} \\
&\leq \frac{h_{A,T}(meas_d(T))^{1/2}}{(1-2\epsilon)^{1/2}} (\int_0^1 (1-t)^{2\epsilon} min(t^{-d}, (1-t)^{-d}) dt)^{1/2} (\int_T \|A^{1/2}\nabla v(x))\|^2 dx)^{1/2}.
\end{aligned}$$

Lemma 2. *Let $T \in \mathcal{T}_h$ and $e \in E_T$, for all $v \in H^1(T)$, we have:*

$$\frac{1}{meas_{d-1}(e)} |\int_e v d\sigma| \leq \frac{1}{meas_d(T)^{1/2}} (\|v\|_{0,T} + \frac{h_A}{d} |A^{1/2}\nabla v|_{0,T}).$$

Proof: Let a_e the opposite node at e. We set $p = \dfrac{meas_{d-1}(e)}{dmeas_d(T)}(x - a_e)$, it is easy to verify that

$$\forall f \in E_T, \qquad p.n_T = \delta_e^f \ \text{ in } e,$$

where n_T is usual outward normal to T. Using Green formula we obtain:

$$\begin{aligned}
|\int_e v d\sigma| &= |\int_{\partial T} v p.n_T d\sigma| = |\int_T (\nabla v.p + v div p) dx| \\
&\leq \frac{meas_{d-1}(e)}{meas_d(T)^{1/2}} (\|v\|_{0,T} + \frac{h_A}{d} |A^{1/2}\nabla v|_{0,T}).
\end{aligned}$$

We also have the following

Lemma 3. *For all $T \in \mathcal{T}_h$, $v \in H^1(T)$. We have*

$$\|v - v_T\|_{0,T} \leq \mu h_A \|A^{1/2}.\nabla v\|_{0,T},$$

where $v_T = \dfrac{1}{meas_d(T)} \displaystyle\int_T v dx.$

Proof: Using Taylor formula and lemma 2.1, we have

$$\begin{cases} \|v - v_T\|_{0,T} = (\int_T (v(y) - \frac{1}{meas_d(T)} \int_T v(x)dx)^2 dy)^{1/2} \\ \quad = \frac{1}{meas_d(T)} (\int_T (\int_T \int_0^1 \nabla v(x + t(y-x)).(y-x)dtdx)^2 dy)^{\frac{1}{2}} \leq \mu h_A \|A^{1/2}.\nabla v\|_{0,T}. \end{cases}$$

By the same arguments, we have also

Lemma 4. *For all $T \in \mathcal{T}_h$, $v \in H^1(T)$ and $f \in L^2(T)$. We have:*

$$|\int_T (f - f_T)vdx| \leq \mu \times h_{A,T} \|f - f_T\|_{0,T} \|A^{1/2} \nabla v\|_{0,T},$$

where $f_T = \frac{1}{meas_d(T)} \int_T f(x)dx.$

Proof: Remark that

$$\begin{array}{rcl} \int_T (f(y) - f_T)v(y)dy & = & \int_T (f(y) - f_T)(v(y) - \frac{1}{meas_d(T)} \int_T v(x)dx)dy \\ & \leq & \|f - f_T\|_{0,T} \|v - v_T\|_{0,T}, \end{array}$$

where $v_T = \frac{1}{meas_d(T)} \int_T vdx.$

Using Lemma 2.3, we obtain

$$|\int_T (f - f_T)vdx| \leq \mu \times h_{A,T} \|f - f_T\|_{0,T} \|A^{1/2}.\nabla v\|_{0,T}.$$

Now, let Π_h the interpolation defined from $W^{1,p}(\Omega)$, $p \geq 1$ onto V_h by:

$$\forall e \in E, \int_e (v - \Pi_h v)d\sigma = 0.$$

Recall that [1], for all symmetric matrix B, we have:

$$\forall T \in \mathcal{T}_h; \quad \|B.\nabla \Pi_h v\|_{1,T} \leq \|B.\nabla v\|_{1,T} \quad \text{and } \|v - \Pi_h v\|_{0,T} \leq C_T h_T \|B.\nabla v\|_{1,T}.$$

and

$$\forall T \in \mathcal{T}_h, \forall v_h \in P_1(T), \quad \int_T B.\nabla v_h \nabla(u - \Pi_h u)dx = 0.$$

In the next lemma, we give explicit bound of the constant C_T. First we have

Lemma 5. *For all $T \in \mathcal{T}_h$, for all $v \in H^1(\Omega)$, we have*

$$\|\Pi_h v\|_{0,T} \leq (d+1)(d-1)^{1/2}(\|v\|_{0,T} + \frac{h_A}{d} \|A^{1/2} \nabla v\|_{0,T}).$$

Proof: Since,

$$\Pi_h v = \sum_{e \in E_T} (\frac{1}{meas_{d-1}(e)} \int_e vd\sigma)\phi_e,$$

where $\phi_e \in P_1(T)$ and $\|\phi_e\|_{0,T}^2 \leq meas_d(T)(d-1)$. Using the lemma 2.2, we obtain

$$\|\Pi_h v\|_{0,T} \leq (d+1)(d-1)^{1/2}(\|v\|_{0,T} + \frac{h_A}{d} \|A^{1/2} \nabla v\|_{0,T}),$$

Lemma 6. *For all $T \in \mathcal{T}_h$, and all $v \in H^1(\Omega)$, we have*

$$\|v - \Pi_h v\|_{0,T} \leq (\mu + (d+1)(d-1)^{1/2}(\mu + \frac{1}{d}))h_A \|A^{1/2}.\nabla v\|_{0,T}.$$

Proof: Let $v_T = \frac{1}{meas_d(T)} \int_T v dx$. On one hand, using lemma 2.3 and 2.5, we have:

$$\begin{aligned}\|\Pi_h(v - v_T)\|_{0,T} &\leq (d+1)(d-1)^{1/2}(\|v - v_T\|_{0,T} + \frac{h_A}{d}\|A^{1/2}\nabla v\|_{0,T}) \\ &\leq (d+1)(d-1)^{1/2}(\mu + \frac{1}{d}) \times h_A\|A^{1/2}.\nabla v\|_{0,T}.\end{aligned}$$

On the other hand,

$$\|v - v_T\|_{0,T} \leq \mu h_A \|A^{1/2}.\nabla v\|_{0,T}.$$

By triangular inequality, and using the fact that $\Pi_h v_T = v_T$, we have

$$\begin{aligned}\|v - \Pi_h v\|_{0,T} &= \|v - v_T - \Pi_h(v - v_T)\|_{0,T} \\ &\leq (\mu + (d+1)(d-1)^{1/2}(\mu + \tfrac{1}{d}))h_A\|A^{1/2}.\nabla v\|_{0,T}.\end{aligned}$$

The final lemma is needed to estimate the constants in the proof of the efficiency of the estimator. For this, and following Verfürth [8], we introduce the bubble function Φ_T defined by:

$$\Phi_T = (d+1)^{d+1}\prod_{i=0}^{d}\lambda_i,$$

where λ_i are the barycentric coordinates on T. We have

Lemma 7. *For all $T \in \mathcal{T}_h$, for all $v \in P_0(T)$, we have*

$$\|v\|_{0,T} = \gamma_1 \|\Phi_T^{1/2} v\|_{0,T},$$

and

$$\|A^{1/2}.\nabla(\Phi_T v)\|_{0,T} \leq \frac{\gamma_2}{\rho_{T,A}}\|v\|_{0,T},$$

where

$$\gamma_1 = (\frac{(2d+1)!}{d!(d+1)^{d+1}})^{1/2},$$

and

$$\gamma_2 = (\frac{(d+1)!2^d}{3d!})^{1/2}.$$

Proof: Let us prove the first inequality. Since

$$\forall (\alpha_i)_{i=0}^d \in I\!N^{d+1}, \quad \int_T \prod_{i=0}^{d} \lambda_i^{\alpha_i} dx = \frac{d! \prod_{i=0}^d \alpha_i!}{(\alpha_0 + ... + \alpha_d + d)!} meas_d(T),$$

and $v \in P_0(T)$, we have

$$\|v\|_{0,T} = (\frac{(2d+1)!}{d!(d+1)^{d+1}})^{1/2}\|\Phi_T^{1/2} v\|_{0,T} = \gamma_1\|\Phi_T^{1/2} v\|_{0,T}.$$

On the other hand, since

$$\forall i = 0, .., d, \|A^{1/2}.\nabla\lambda_i\|_{0,T} \leq \frac{1}{\rho_{T,A}} \quad ,$$

we have

$$\begin{aligned}\|A^{1/2}.\nabla(\Phi_T v)\|_{0,T} &\leq \sum_{i=0}^{d}\|A^{-1/2}.(\nabla\lambda_i)\prod_{j\neq i}\lambda_j\|_{0,T}|v| \\ &\leq (\frac{(d+1)!2^d}{3d!\rho_{T,A}^2})^{1/2}\|v\|_{0,T} = \frac{\gamma_2}{\rho_{T,A}}\|v\|_{0,T}.\end{aligned}$$

2.1. The Transfer operator. We introduce a transfer operator defined in [2], that is an application I_h defined from V_h into $V_h \cap H_0^1(\Omega)$, such that

$$\forall u \in H_0^1(\Omega), u_h \in V_h, \quad \|u - I_h(u_h)\|_{1,\Omega} \leq C(\sum_{T\in\mathcal{T}_h} \|u - u_h\|_{1,T}^2)^{1/2}.$$

We denote by $\mathcal{N}$ the set of all nodes of the triangulation $\mathcal{T}_h$. For each $a \in \mathcal{N}$, Γ_a will denote the union of the sides contained in Ω a belongs to; K_a will denote the union of the elements of $\mathcal{T}_h$ having a as node and by M_a card(K_a). Finally $\Delta(T)$ is the union of the elements shearing a node with T and M_T is card$(\Delta(T))$.
We define the operator I_h from V_h onto $V_h \cap H_0^1(\Omega)$ by

$$I_h(u_h)(a) = \begin{cases} 0 & \text{if } a \in \Gamma, \\ \dfrac{1}{M_a} \displaystyle\sum_{T\in K_a} u_{h|T}(a), & \text{otherwise} . \end{cases}$$

Using the same proof as in [2] and the previous lemmas, we have the following

Theorem 1. *For all $u_h \in V_h$, $u \in H_0^1(\Omega)$ and $T \in \mathcal{T}_h$, we have*

$$\|A^{1/2}.\nabla(u_h - I_h(u_h))\|_{0,T} \leq \frac{\beta(meas_d(T))^{1/2}}{\rho_{A,T}} \sum_{K\in\Delta(T)} \frac{h_{A,K}}{(meas_d(K))^{1/2}} \|A^{1/2}.\nabla(u-u_h)\|_{0,K},$$

where

$$\beta = 2^{d-1}d(2 + (\mu + (d+1)(d-1)^{1/2}(\mu + \frac{1}{d}))).$$

3. The a posteriori error estimator.

Let $\hat{u}_h \in H_0^1(\Omega)$ a conforming post-processing of u_h which can be obatined using transfer operator. We assume that $\hat{u}_h$ is obtained by black-box post-processing. It is interesting to give a posteriori error estimator for $u - \hat{u}_h$.

3.1. Error Estimator using Constitutive law. In this subsection, we assume that:

$$\forall T \in \mathcal{T}_h, \quad \hat{f} = f_T.$$

Adapting the proof given in [1], we prove that the vector field defined by

$$\forall T \in \mathcal{T}_h, \quad p_h = A.\nabla u_h - \frac{f_T}{d}(x - x_{G,T}) \text{ on } T,$$

where $x_{G,T}$ is the barycenter of T, belongs to $H(div;\Omega)$ and satisfies

$$\forall T \in \mathcal{T}_h, \quad -div p_h = f_T \ , \quad \text{on } T.$$

Since p_h is of physical interesting, we want to give an a posteriori error estimator for $u - \hat{u}_h$ and $A.\nabla u - p_h$. We have

Theorem 2. *For all $\hat{u}_h \in H_0^1(\Omega) \cap V_h$ and all $\zeta \in]0,1[$,*

$$\forall T \in \mathcal{T}_h, \quad \|A^{-1/2}p_h - A^{1/2}\nabla\hat{u}_h\|_{0,T} \leq \|A^{-1/2}(p - p_h)\|_{0,T} + \|A^{1/2}.\nabla(u - \hat{u}_h)\|_{0,T},$$

and

$$\begin{aligned}(1-\zeta)\|A^{1/2}.\nabla(u - \hat{u}_h)\|_{0,\Omega}^2 + \|A^{-1/2}(p - p_h)\|_{0,\Omega}^2 \quad &\leq \quad \|A^{-1/2}p_h - A^{1/2}\nabla\hat{u}_h\|_{0,\Omega}^2 \\ &+ \quad \tfrac{\mu^2}{2\zeta} \textstyle\sum_{T\in\mathcal{T}_h} h_{A,T}^2 \|f - f_T\|_{0,T}^2.\end{aligned}$$

Proof: The first inequality is clear, let us prove the second one. On one hand, we have:

$$\|A^{-1/2}p_h - A^{1/2}\nabla\hat{u}_h\|_{0,\Omega}^2 = \|A^{1/2}.\nabla(u-\hat{u}_h)\|_{0,\Omega}^2 + \|A^{-1/2}(p-p_h)\|_{0,\Omega}^2$$
$$+2\int_\Omega A^{-1/2}(p_h-p).A^{1/2}(\nabla u - \nabla\hat{u}_h)dx,$$

On the other hand, using the Green formula and Lemma 2.4, yiels

$$|\int_\Omega A^{-1/2}(p_h-p).A^{1/2}(\nabla u-\nabla\hat{u}_h)dx| = |\int_\Omega (p_h-p).(\nabla u-\nabla\hat{u}_h)dx|$$
$$= |-\int_\Omega div(p_h-p)(u-u_h)dx|$$
$$= |\sum_{T\in\mathcal{T}_h}\int_T (f-f_T)(u-\hat{u}_h)dx| \le \mu\sum_{T\in\mathcal{T}_h} h_{A,T}\|f-f_T\|_{0,T}\|A^{1/2}\nabla(u-\hat{u}_h)\|_{0,T}$$
$$\le \frac{\mu^2}{2\zeta}\sum_{T\in\mathcal{T}_h} h_{A,T}\|f-f_T\|_{0,T}^2 + \frac{\zeta}{2}\|A^{1/2}\nabla(u-\hat{u}_h)\|_{0,\Omega}^2.$$

Then we obtain

$$(1-\zeta)\|A^{1/2}.\nabla(u-\hat{u}_h)\|_{0,\Omega}^2 + \|A^{-1/2}(p-p_h)\|_{0,\Omega}^2 \le \|A^{-1/2}p_h - A^{1/2}\nabla\hat{u}_h\|_{0,\Omega}^2$$
$$+\frac{\mu^2}{2\zeta}\sum_{T\in\mathcal{T}_h} h_{A,T}^2\|f-f_T\|_{0,T}^2.$$

Using the same arguments, we can prove the following precise version of the last Theorem:

Theorem 3. *For all $u_h \in H_0^1(\Omega)$ and $p_h \in H(div;\Omega)$ such that*

$$\forall T\in T_h; \quad -div p_h = f_T, \text{ on } T,$$

for all family of reals $\{\zeta_T\}_{T\in T_h}$ such that

$$\zeta_T = 0, \quad \text{if } f = f_T \quad \text{and} \quad \zeta_T \in]0,1[\text{ if } f \ne f_T.$$

We have

$$\|A^{-1/2}p_h - A^{1/2}\nabla\hat{u}_h\|_{0,T} \le \|A^{-1/2}(p-p_h)\|_{0,T} + \|A^{1/2}.\nabla(u-\hat{u}_h)\|_{0,T},$$

and

$$\sum_{T\in\mathcal{T}_h}(1-\zeta_T)\|A^{1/2}.\nabla(u-\hat{u}_h)\|_{0,T}^2 + \|A^{-1/2}(p-p_h)\|_{0,\Omega}^2 \le \|A^{-1/2}p_h - A^{1/2}\nabla u_h\|_{0,\Omega}^2$$
$$+\frac{\mu^2}{2}\sum_{T\in\mathcal{T}_h}\frac{h_{A,T}^2}{\zeta_T}\|f-f_T\|_{0,T}^2.$$

3.2. Estimator in general case. We give an a posteriori error estimator for $u-u_h$ and $u-\hat{u}_h$. First, concerning the upper bound of the error we have

Theorem 4. *Let $\hat{u}_h \in H_0^1(\Omega)\cap V_h$, we have*

$$(\sum_{T\in\mathcal{T}_h}\|A^{1/2}\nabla(u-u_h)\|_{0,T}^2)^{1/2} \le (\sum_{T\in\mathcal{T}_h}\|A^{1/2}\nabla(\hat{u}_h-u_h)\|_{0,T}^2)^{\frac{1}{2}} + \alpha(\sum_{T\in\mathcal{T}_h} h_{A,T}^2\|f\|_{0,T}^2)^{1/2}.$$

where

$$\alpha = \mu + (d+1)(d-1)^{1/2}(\mu + \frac{1}{d}).$$

Proof: We set $V = \{v \in L^2(\Omega), \forall T \in \mathcal{T}_h;\ \ v_{|T} \in H^1(T)\}$, and define the bilinear form on V^2 by

$$\forall u, v \in V;\ a(u, v) = \sum_{T\in\mathcal{T}_h} \int_T A.\nabla u.\nabla v dx.$$

First, we have

$$(\sum_{T\in\mathcal{T}_h} \|A^{1/2}\nabla(u - u_h)\|^2_{0,T} = a(u - u_h, u - u_h)$$

$$= a(u, u - u_h) - \int_\Omega f(u - u_h)dx + \int_\Omega f(u - u_h)dx - a(u_h, u - u_h).$$

On the one hand

$$\begin{aligned}\int_\Omega f(u - u_h)dx - a(u_h, u - u_h) &= \int_\Omega f(u - u_h)dx \\ &- \sum_{T\in\mathcal{T}_h} \int_T A.\nabla u_h.\nabla(\Pi_h(u - u_h))dx \\ &= \int_\Omega f(u - u_h - \Pi_h(u - u_h))dx \\ &\leq \alpha(\sum_{T\in\mathcal{T}_h} h_{A,T}\|f\|^2_{0,T})^{1/2}(\sum_{T\in\mathcal{T}_h} \|A^{1/2}\nabla(u - u_h)\|^2_{0,T})^{1/2},\end{aligned}$$

where

$$\alpha = \mu + (d + 1)(d - 1)^{1/2}(\mu + \frac{1}{d}).$$

On the other hand

$$\begin{aligned}a(u, u - u_h) - \int_\Omega f(u - u_h)dx &= -a(u, u_h) + \int_\Omega f u_h dx \\ &= -a(u, u_h) + a(u_h, u_h) = a(u_h - u, u_h) = a(u_h - \hat{u}_h, u_h - u) \\ &\leq (\sum_{T\in\mathcal{T}_h} \|A^{1/2}\nabla(u - u_h)\|^2_{0,T})^{1/2} \times (\sum_{T\in\mathcal{T}_h} \|A^{1/2}\nabla(\hat{u}_h - u_h)\|^2_{0,T})^{\frac{1}{2}}\end{aligned}$$

The last inequalities give the result.

Concerning the a posteriori error estimator for $u - \hat{u}_h$, by triangular inequality, we have the following

Theorem 5. *Let $\hat{u}_h \in H^1_0(\Omega)$, we have*

$$\begin{aligned}(\sum_{T\in\mathcal{T}_h} \|A^{1/2}\nabla(u - \hat{u}_h)\|^2_{0,T})^{1/2} \quad &\leq \quad 2(\textstyle\sum_{T\in\mathcal{T}_h} \|A^{1/2}\nabla(\hat{u}_h - u_h)\|^2_{0,T} \\ &+ \quad \alpha(\textstyle\sum_{T\in\mathcal{T}_h} h^2_{A,T}\|f\|^2_{0,T})^{1/2}.\end{aligned}$$

where

$$\alpha = \mu + (d + 1)(d - 1)^{1/2}(\mu + \frac{1}{d}).$$

Finally, concerning the lower bound. Using classical arguments [8], we have the following

Theorem 6. *For all $T \in \mathcal{T}_h$, we have the following estimation*

$$\|A^{1/2}\nabla(\hat{u}_h - u_h)\|_{0,T} \leq \|A^{1/2}\nabla(u - u_h)\|_{0,T} + \|A^{1/2}\nabla(u - \hat{u}_h)\|_{0,T},$$

$$\|f\|_{0,T} \leq (1 + \gamma_1^2)\|f - f_T\|_{0,T} + \gamma_1^2 \frac{\gamma_2}{\rho_{A,T}}(\|A^{1/2}\nabla(u - u_h)\|_{0,T},$$

and

$$\|f\|_{0,T} \leq (1 + \gamma_1^2)\|f - f_T\|_{0,T} + \gamma_1^2 \frac{\gamma_2}{\rho_{A,T}}(\|A^{1/2}\nabla(u - \hat{u}_h)\|_{0,T},$$

Proof: The first inequality is obtained by triangular inequality. Let us prove the last inequalities. We set $v_h = u_h$ or $\hat{u}_h$, $f_T = \dfrac{1}{meas_d(T)} \displaystyle\int_T f dx$ and $w_T = \Phi_T \times f_T \in H_0^1(T)$, since f_T belong to $P_0(T)$. Using Lemma 2.7, we have

$$\begin{aligned}
\|f_T\|_{0,T}^2 &\leq \gamma_1^2 \|\Phi_T^{1/2} \times f_T\|_{0,T} = \gamma_1^2 \int_T w_T f_T dx \\
&= \gamma_1^2 \int_T w_T (f_T - f + div(A.\nabla(u - v_h)) dx \\
&= \gamma_1^2 (\int_T w_T (f_T - f) dx + \int_T A.\nabla w_T.\nabla(v_h - u) dx.
\end{aligned}$$

Since $\|\Phi_T\|_{0,\infty,T} = 1$ and using lemma 2.7, we have

$$|\int_T w_T (f_T - f) dx| \leq \|f_T - f\|_{0,T} \|f_T\|_{0,T},$$

and

$$|\int_T A.\nabla w_T.\nabla(v_h - u) dx| \leq \frac{\gamma_2}{\rho_{A,T}} \|A^{1/2}.\nabla(u - v_h)\|_{0,T} \|f_T - \sigma v_h\|_{0,T}.$$

Then

$$\|f_T\|_{0,T} \leq \gamma_1^2 \|f - f_T\|_{0,T} + \frac{\gamma_1^2 \gamma_2}{\rho_{A,T}} \|A^{1/2} \nabla(u - v_h)\|_{0,T}.$$

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