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Finite Element Analysis of Maxwell's Equations in Dispersive Lossy Bi-Isotropic Media

Yunqing Huang[1], Jichun Li[2,*] and Yanping Lin[3]

[1] *Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Xiangtan University, Hunan 411105, China*
[2] *Department of Mathematical Sciences, University of Nevada Las Vegas, Las Vegas, Nevada 89154-4020, USA*
[3] *Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Hong Kong, and Department of Mathematical and Statistics Science, University of Alberta, Edmonton AB, Alberta T6G 2G1, Canada*

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Dedicated to Graeme Fairweather on the occasion of his 70^{th} birthday.

Abstract. In this paper, the time-dependent Maxwell's equations used to modeling wave propagation in dispersive lossy bi-isotropic media are investigated. Existence and uniqueness of the modeling equations are proved. Two fully discrete finite element schemes are proposed, and their practical implementation and stability are discussed.

AMS subject classifications: 65N30, 35L15, 78M10

Key words: Maxwell's equations, bi-isotropic media, finite element method.

1 Introduction

The research on numerical analysis and modeling of electromagnetic wave propagation in dispersive media (especially metamaterials) has been a subject of increasing interest over the recent years (cf. [1,6,10–14,16,19–21] and references cited therein). In this paper, we consider the wave propagation problem in dispersive lossy bi-isotropic (BI) media, which are characterized by more complicated constitutive relations than those classical dispersive media models such as Debye and Lorentz models [10]. In BI media, the magnetic and electric fields are coupled. Electromagnetic waves in such media have some interesting characteristics such as optical rotatory dispersion [15].

*Corresponding author.
Email: huangyq@xtu.edu.cn (Y. Huang), jichun@unlv.nevada.edu (J. Li), ylin@math.ualberta.ca (Y. Lin)

Though some FDTD schemes (cf. [7]) have been developed for solving BI media, to our best knowledge, there is no rigorous mathematical analysis (such as the existence and uniqueness) of this model. Furthermore, to overcome the disadvantage of FDTD schemes for complex geometric problems, it is interesting to develop some finite element method for modeling wave propagation in BI media. Our major goal of this paper is to initiate the analysis of these new modeling equations and develop some efficient finite element methods to solve them.

In this paper, we denote C (sometimes with a sub-index) a generic constant independent of the mesh size h and the time step size Δt. We also use some common notations [17]:

$$
\begin{aligned}
&H(div;\Omega)=\{\boldsymbol{v}\in(L^2(\Omega))^3:\ \nabla\cdot\boldsymbol{v}\in(L^2(\Omega))^3\},\\
&H(curl;\Omega)=\{\boldsymbol{v}\in(L^2(\Omega))^3:\ \nabla\times\boldsymbol{v}\in(L^2(\Omega))^3\},\\
&H_0(curl;\Omega)=\{\boldsymbol{v}\in H(curl;\Omega):\ \boldsymbol{n}\times\boldsymbol{v}=\boldsymbol{0}\ \text{ on }\ \partial\Omega\},
\end{aligned}
$$

for any bounded Lipschitz polyhedral domain Ω in $\mathcal{R}^3$ with connected boundary $\partial\Omega$. Moreover, we let $(H^\alpha(\Omega))^3$ be the standard Sobolev space equipped with norm $\|\cdot\|_\alpha$. When $\alpha=0$, we just denote $\|\cdot\|_0$ for the $(L^2(\Omega))^3$ norm.

The rest of the paper is organized as follows. In Section 2, we first present the time-dependent governing equations for modeling wave propagation in BI media. Then we prove the existence and uniqueness of the modeling equations. We also present a stability result. In Section 3, we develop two fully-discrete finite element schemes for solving the BI media model equations. Solvability, stability of these schemes are discussed. Finally, we conclude the paper in Section 4.

2 The governing equations

The description of the dispersive lossy BI media is given by the constitutive relations [15]:

$$\boldsymbol{D}=\epsilon(\omega)\boldsymbol{E}+\sqrt{\epsilon_0\mu_0}(\chi-i\kappa(\omega))\boldsymbol{H}, \tag{2.1a}$$

$$\boldsymbol{B}=\mu(\omega)\boldsymbol{H}+\sqrt{\epsilon_0\mu_0}(\chi+i\kappa(\omega))\boldsymbol{E}, \tag{2.1b}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ denote the electric field and magnetic field, $\boldsymbol{D}$ and $\boldsymbol{B}$ denote the electric and magnetic flux densities respectively, ϵ_0 and μ_0 are the vacuum permittivity and permeability respectively, the number $i=\sqrt{-1}$, $\chi\geq 0$ is the nonreciprocity parameter, and $\kappa(\omega)$ is the chirality parameter. Furthermore, the permittivity $\epsilon(\omega)$ and permeability $\mu(\omega)$ depend on the wave frequency ω. Experiments found that a Condon model can be used to describe the frequency of the chirality $\kappa(\omega)$, and both $\epsilon(\omega)$ and $\mu(\omega)$ follow a second-order Lorentz model. Since the resonance frequencies of $\kappa(\omega)$, $\epsilon(\omega)$ and $\mu(\omega)$ are found to be very close in experiments, in practice they are assumed to be the same, in which case, the frequency domain constitutive relations (2.1a)-(2.1b) are expressed as (we corrected some typos of (3)-(4) in [7]):

$$\boldsymbol{D}=\epsilon_0\epsilon_\infty\boldsymbol{E}+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)\omega_0^2}{\omega_0^2-\omega^2+i2\omega_0\xi\omega}\boldsymbol{E}+\frac{\chi}{c_v}\boldsymbol{H}-\frac{i}{c_v}\frac{\omega\tau\omega_0^2}{\omega_0^2-\omega^2+i2\omega_0\xi\omega}\boldsymbol{H}, \tag{2.2a}$$

$$\boldsymbol{B}=\mu_0\mu_\infty\boldsymbol{H}+\frac{\mu_0(\mu_s-\mu_\infty)\omega_0^2}{\omega_0^2-\omega^2+i2\omega_0\xi\omega}\boldsymbol{H}+\frac{\chi}{c_v}\boldsymbol{E}+\frac{i}{c_v}\frac{\omega\tau\omega_0^2}{\omega_0^2-\omega^2+i2\omega_0\xi\omega}\boldsymbol{E}, \tag{2.2b}$$

where $c_v=1/\sqrt{\epsilon_0\mu_0}$ represents the light speed in vacuum, $\xi\in[0,1)$ is the loss parameter, ϵ_s and ϵ_∞ are the permittivities at zero and infinity frequencies, respectively, μ_s and μ_∞ are the permeabilities at zero and infinity frequencies, respectively, and $\tau>0$ is a time constant.

Using the following rules

$$i\omega\rightarrow\frac{\partial}{\partial t},\quad \omega^2\rightarrow-\frac{\partial^2}{\partial t^2},$$

the constitutive equations (2.2a)-(2.2b) can be written in time domain as

$$\begin{aligned}\frac{\partial^2\boldsymbol{D}}{\partial t^2}+2\omega_0\xi\frac{\partial\boldsymbol{D}}{\partial t}+\omega_0^2\boldsymbol{D}=&\,\epsilon_0\epsilon_\infty\frac{\partial^2\boldsymbol{E}}{\partial t^2}+2\epsilon_0\epsilon_\infty\omega_0\xi\frac{\partial\boldsymbol{E}}{\partial t}+\epsilon_0\epsilon_s\omega_0^2\boldsymbol{E}\\&+\frac{\chi}{c_v}\Big(\frac{\partial^2\boldsymbol{H}}{\partial t^2}+2\omega_0\xi\frac{\partial\boldsymbol{H}}{\partial t}+\omega_0^2\boldsymbol{H}\Big)-\frac{\tau}{c_v}\omega_0^2\frac{\partial\boldsymbol{H}}{\partial t},\end{aligned} \tag{2.3a}$$

$$\begin{aligned}\frac{\partial^2\boldsymbol{B}}{\partial t^2}+2\omega_0\xi\frac{\partial\boldsymbol{B}}{\partial t}+\omega_0^2\boldsymbol{B}=&\,\mu_0\mu_\infty\frac{\partial^2\boldsymbol{H}}{\partial t^2}+2\mu_0\mu_\infty\omega_0\xi\frac{\partial\boldsymbol{H}}{\partial t}+\mu_0\mu_s\omega_0^2\boldsymbol{H}\\&+\frac{\chi}{c_v}\Big(\frac{\partial^2\boldsymbol{E}}{\partial t^2}+2\omega_0\xi\frac{\partial\boldsymbol{E}}{\partial t}+\omega_0^2\boldsymbol{E}\Big)+\frac{\tau}{c_v}\omega_0^2\frac{\partial\boldsymbol{E}}{\partial t}.\end{aligned} \tag{2.3b}$$

To make the problem complete, (2.3a)-(2.3b) need to be coupled with the Ampere's law and Faraday's law written as follows:

$$\frac{\partial\boldsymbol{D}}{\partial t}=\nabla\times\boldsymbol{H}, \tag{2.4a}$$

$$\frac{\partial\boldsymbol{B}}{\partial t}=-\nabla\times\boldsymbol{E}. \tag{2.4b}$$

Furthermore, we assume that the governing equations (2.2a)-(2.3b) are subject to the perfectly conducting (PEC) boundary condition

$$\boldsymbol{n}\times\boldsymbol{E}=\boldsymbol{0}\quad\text{on }\partial\Omega, \tag{2.5}$$

and initial conditions

$$\boldsymbol{E}(\boldsymbol{x},0)=\boldsymbol{E}_0(\boldsymbol{x}),\qquad \boldsymbol{H}(\boldsymbol{x},0)=\boldsymbol{H}_0(\boldsymbol{x}),\qquad \boldsymbol{D}(\boldsymbol{x},0)=\boldsymbol{D}_0(\boldsymbol{x}),\qquad \boldsymbol{B}(\boldsymbol{x},0)=\boldsymbol{B}_0(\boldsymbol{x}), \tag{2.6a}$$

$$\boldsymbol{E}_t(\boldsymbol{x},0)=\boldsymbol{E}_1(\boldsymbol{x}),\qquad \boldsymbol{H}_t(\boldsymbol{x},0)=\boldsymbol{H}_1(\boldsymbol{x}),\qquad \boldsymbol{D}_t(\boldsymbol{x},0)=\boldsymbol{D}_1(\boldsymbol{x}),\qquad \boldsymbol{B}_t(\boldsymbol{x},0)=\boldsymbol{B}_1(\boldsymbol{x}), \tag{2.6b}$$

where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$, and $\boldsymbol{E}_i$, $\boldsymbol{H}_i$, $\boldsymbol{B}_i$ and $\boldsymbol{D}_i$, $i=0,1$, are some given functions.

In the rest of this section, we shall show that the problem (2.3a)-(2.6b) is well-posed.

For a function $u(t)$ defined for $t\geq 0$, let us denote its Laplace transform by $\hat{u}(s)=\mathcal{L}(u)=\int_0^\infty e^{-st}u(t)dt$. Taking the Laplace transform of (2.3a)-(2.4b), we have

$$
\begin{aligned}
&(i)\ s^2\hat{\boldsymbol{D}}+2\omega_0\xi s\hat{\boldsymbol{D}}+\omega_0^2\hat{\boldsymbol{D}}=\epsilon_0\epsilon_\infty s^2\hat{\boldsymbol{E}}+2\epsilon_0\epsilon_\infty\omega_0\xi s\hat{\boldsymbol{E}}+\epsilon_0\epsilon_s\omega_0^2\hat{\boldsymbol{E}}\\
&\qquad\qquad+\frac{\chi}{c_v}(s^2\hat{\boldsymbol{H}}+2\omega_0\xi s\hat{\boldsymbol{H}}+\omega_0^2\hat{\boldsymbol{H}})-\frac{\tau}{c_v}\omega_0^2 s\hat{\boldsymbol{H}}+\tilde{F}_0(s), &(2.7a)\\
&(ii)\ s^2\hat{\boldsymbol{B}}+2\omega_0\xi s\hat{\boldsymbol{B}}+\omega_0^2\hat{\boldsymbol{B}}=\mu_0\mu_\infty s^2\hat{\boldsymbol{H}}+2\mu_0\mu_\infty\omega_0\xi s\hat{\boldsymbol{H}}+\mu_0\mu_s\omega_0^2\hat{\boldsymbol{H}}\\
&\qquad\qquad+\frac{\chi}{c_v}(s^2\hat{\boldsymbol{E}}+2\omega_0\xi s\hat{\boldsymbol{E}}+\omega_0^2\hat{\boldsymbol{E}})+\frac{\tau}{c_v}\omega_0^2 s\hat{\boldsymbol{E}}+\tilde{G}_0(s), &(2.7b)\\
&(iii)\ s\hat{\boldsymbol{D}}-\boldsymbol{D}_0=\nabla\times\hat{\boldsymbol{H}}, &(2.7c)\\
&(iv)\ s\hat{\boldsymbol{B}}-\boldsymbol{B}_0=-\nabla\times\hat{\boldsymbol{E}}, &(2.7d)
\end{aligned}
$$

where we have absorbed all related initial conditions into $\tilde{F}_0(s)$ and $\tilde{G}_0(s)$, i.e.,

$$
\begin{aligned}
\tilde{F}_0(s)=&s\boldsymbol{D}_0+\boldsymbol{D}_1+2\omega_0\xi\boldsymbol{D}_0-\epsilon_0\epsilon_\infty(s\boldsymbol{E}_0-\boldsymbol{E}_1)-2\epsilon_0\epsilon_\infty\omega_0\xi\boldsymbol{E}_0\\
&-\frac{\chi}{c_v}(s\boldsymbol{H}_0+\boldsymbol{H}_1+2\omega_0\xi\boldsymbol{H}_0)+\frac{\tau\omega_0^2}{c_v}\boldsymbol{H}_0,\\
\tilde{G}_0(s)=&s\boldsymbol{B}_0+\boldsymbol{B}_1+2\omega_0\xi\boldsymbol{B}_0-\mu_0\mu_\infty(s\boldsymbol{H}_0+\boldsymbol{H}_1)-2\mu_0\mu_\infty\omega_0\xi\boldsymbol{H}_0\\
&-\frac{\chi}{c_v}(s\boldsymbol{E}_0+\boldsymbol{E}_1+2\omega_0\xi\boldsymbol{E}_0)-\frac{\tau\omega_0^2}{c_v}\boldsymbol{E}_0.
\end{aligned}
$$

Multiplying (2.7a) and (2.7b) by s, and using (2.7c) and (2.7d) respectively, we obtain

$$
\begin{aligned}
&-p(s)\nabla\times\hat{\boldsymbol{E}}=q(s)\hat{\boldsymbol{H}}+\Big(\frac{\chi}{c_v}sp(s)+\frac{\tau}{c_v}\omega_0^2s^2\Big)\hat{\boldsymbol{E}}+G_0(s), &(2.8a)\\
&p(s)\nabla\times\hat{\boldsymbol{H}}=r(s)\hat{\boldsymbol{E}}+\Big(\frac{\chi}{c_v}sp(s)-\frac{\tau}{c_v}\omega_0^2s^2\Big)\hat{\boldsymbol{H}}+F_0(s), &(2.8b)
\end{aligned}
$$

where we denote

$$
\begin{aligned}
&p(s)=s^2+2\omega_0\xi s+\omega_0^2, &(2.9a)\\
&q(s)=(\mu_0\mu_\infty s^2+2\mu_0\mu_\infty\omega_0\xi s+\mu_0\mu_s\omega_0^2)s, &(2.9b)\\
&r(s)=(\epsilon_0\epsilon_\infty s^2+2\epsilon_0\epsilon_\infty\omega_0\xi s+\epsilon_0\epsilon_s\omega_0^2)s, &(2.9c)\\
&G_0(s)=s\tilde{G}_0(s)-p(s)\boldsymbol{B}_0, &(2.9d)\\
&F_0(s)=s\tilde{F}_0(s)-p(s)\boldsymbol{D}_0. &(2.9e)
\end{aligned}
$$

Multiplying (2.8b) by $q(s)$ and using (2.8a) to eliminate $\hat{\boldsymbol{H}}$, we have

$$
\begin{aligned}
&r(s)q(s)\hat{\boldsymbol{E}}+p(s)\nabla\times\Big[p(s)\nabla\times\hat{\boldsymbol{E}}+\Big(\frac{\chi}{c_v}sp(s)+\frac{\tau}{c_v}\omega_0^2s^2\Big)\hat{\boldsymbol{E}}+G_0(s)\Big]\\
&\quad-\Big(\frac{\chi}{c_v}sp(s)-\frac{\tau}{c_v}\omega_0^2s^2\Big)\Big[p(s)\nabla\times\hat{\boldsymbol{E}}+\Big(\frac{\chi}{c_v}sp(s)+\frac{\tau}{c_v}\omega_0^2s^2\Big)\hat{\boldsymbol{E}}+G_0(s)\Big]+q(s)F_0(s)=0,
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
&p^2(s)\nabla\times\nabla\times\hat{\boldsymbol{E}}+2\frac{\tau}{c_v}\omega_0^2s^2p(s)\nabla\times\hat{\boldsymbol{E}}+\Big[\Big(\frac{\tau}{c_v}\omega_0^2s^2\Big)^2-\Big(\frac{\chi}{c_v}sp(s)\Big)^2+r(s)q(s)\Big]\hat{\boldsymbol{E}}\\
=&\Big(\frac{\chi}{c_v}sp(s)-\frac{\tau}{c_v}\omega_0^2s^2\Big)G_0(s)-q(s)F_0(s)-p(s)\nabla\times G_0(s)\equiv FG(s). \qquad (2.10)
\end{aligned}
$$

A weak formulation of (2.10) can be formed as: Find $\hat{\boldsymbol{E}}\in H_0(\text{curl};\Omega)$ such that

$$
\mathcal{A}(\hat{\boldsymbol{E}},\boldsymbol{u})=(FG(s),\boldsymbol{u}), \quad \forall \boldsymbol{u}\in H_0(\text{curl};\Omega), \qquad (2.11)
$$

where the bilinear form $\mathcal{A}(\cdot,\cdot)$ is given by

$$
\begin{aligned}
\mathcal{A}(\hat{\boldsymbol{E}},\boldsymbol{u})=&p^2(s)(\nabla\times\hat{\boldsymbol{E}},\nabla\times\boldsymbol{u})+2\frac{\tau}{c_v}\omega_0^2s^2p(s)(\nabla\times\hat{\boldsymbol{E}},\boldsymbol{u})\\
&+\Big[\Big(\frac{\tau}{c_v}\omega_0^2s^2\Big)^2-\Big(\frac{\chi}{c_v}sp(s)\Big)^2+r(s)q(s)\Big](\hat{\boldsymbol{E}},\boldsymbol{u}). \qquad (2.12)
\end{aligned}
$$

Theorem 2.1. *Under the conditions*

$$
\sqrt{\mu_\infty\epsilon_\infty}\geq\chi, \quad \epsilon_s\geq\epsilon_\infty, \quad \mu_s\geq\mu_\infty, \qquad (2.13)
$$

where equal signs cannot be true at the same time, there exists a unique solution $\hat{\boldsymbol{E}}\in H_0(curl;\Omega)$ *for the problem* (2.11).

Proof. First, it is easy to see that $\mathcal{A}(\hat{\boldsymbol{E}},\boldsymbol{u})$ is bounded in $H_0(\text{curl};\Omega)$ norm, i.e.,

$$
\mathcal{A}(\hat{\boldsymbol{E}},\boldsymbol{u})\leq C\|\hat{\boldsymbol{E}}\|_{H_0(\text{curl};\Omega)}\|\boldsymbol{u}\|_{H_0(\text{curl};\Omega)}. \qquad (2.14)
$$

To prove the existence and uniqueness, we shall further confirm the coercivity of the bilinear form $\mathcal{A}(\cdot,\cdot)$. Note that $q(s)$ and $r(s)$ defined in (2.9b) and (2.9c) can be written as

$$
q(s)=\mu_0\mu_\infty sp(s)+\mu_0(\mu_s-\mu_\infty)\omega_0^2s
$$

and

$$
r(s)=\epsilon_0\epsilon_\infty sp(s)+\epsilon_0(\epsilon_s-\epsilon_\infty)\omega_0^2s,
$$

from which we obtain

$$
\begin{aligned}
q(s)r(s)=&\mu_0\epsilon_0\mu_\infty\epsilon_\infty(sp(s))^2+\mu_0\epsilon_0\omega_0^2s^2p(s)[\mu_\infty(\epsilon_s-\epsilon_\infty)+\epsilon_\infty(\mu_s-\mu_\infty)]\\
&+\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)(\mu_s-\mu_\infty)\omega_0^4s^2.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\Big(\frac{\tau}{c_v}\omega_0^2s^2\Big)^2-\Big(\frac{\chi}{c_v}sp(s)\Big)^2+r(s)q(s)=&\Big(\frac{\tau}{c_v}\omega_0^2s^2\Big)^2+\mu_0\epsilon_0(\mu_\infty\epsilon_\infty-\chi^2)(sp(s))^2\\
&+\mu_0\epsilon_0\omega_0^2s^2p(s)[\mu_\infty(\epsilon_s-\epsilon_\infty)+\epsilon_\infty(\mu_s-\mu_\infty)]\\
&+\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)(\mu_s-\mu_\infty)\omega_0^4s^2. \qquad (2.15)
\end{aligned}
$$

On the other hand, by the arithmetic-geometric mean inequality, we have

$$2\frac{\tau}{c_v}\omega_0^2 s^2 p(s)(\nabla\times\hat{\boldsymbol{E}},\hat{\boldsymbol{E}})\geq -\delta p^2(s)\|\nabla\times\hat{\boldsymbol{E}}\|_0^2-\frac{1}{\delta}\Big(\frac{\tau}{c_v}\omega_0^2 s^2\Big)^2\|\hat{\boldsymbol{E}}\|_0^2, \tag{2.16}$$

where the arbitrary constant $\delta>0$.

Using (2.15) and (2.16), and the property $p(s)>s^2$, we have

$$\begin{aligned}\mathcal{A}(\hat{\boldsymbol{E}},\hat{\boldsymbol{E}})\geq&(1-\delta)p^2(s)\|\nabla\times\hat{\boldsymbol{E}}\|_0^2+\Big(1+Ind-\frac{1}{\delta}\Big)\epsilon_0\mu_0(\tau\omega_0)^2\omega_0^2 s^4\|\hat{\boldsymbol{E}}\|_0^2\\&+[\mu_0\epsilon_0(\mu_\infty\epsilon_\infty-\chi^2)(sp(s))^2+\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)(\mu_s-\mu_\infty)\omega_0^4 s^2]\|\hat{\boldsymbol{E}}\|_0^2,\end{aligned} \tag{2.17}$$

where we denote $Ind=[\mu_\infty(\epsilon_s-\epsilon_\infty)+\epsilon_\infty(\mu_s-\mu_\infty)]/(\tau\omega_0)^2$.

From (2.13) and (2.17), we can see that $Ind>0$. Hence choosing $1>\delta\geq 1/(1+Ind)$ guarantees that

$$\mathcal{A}(\hat{\boldsymbol{E}},\hat{\boldsymbol{E}})\geq C\|\hat{\boldsymbol{E}}\|^2_{H_0(\mathrm{curl};\Omega)},$$

which, along with the boundness (2.14), guarantees the existence and uniqueness of a solution $\hat{\boldsymbol{E}}\in H_0(\mathrm{curl};\Omega)$ by the Lax-Milgram lemma. □

The existence and uniqueness of a solution $\hat{\boldsymbol{H}}$ is implied from (2.8a). From (2.7c) and (2.7d) and the existence of solutions $\hat{\boldsymbol{H}}$ and $\hat{\boldsymbol{E}}$, we see that solutions $\hat{\boldsymbol{D}}$ and $\hat{\boldsymbol{B}}$ exist and are unique. The inverse Laplace transforms of functions $\hat{\boldsymbol{H}},\hat{\boldsymbol{E}},\hat{\boldsymbol{D}}$ and $\hat{\boldsymbol{B}}$ are the solutions of the original time-dependent problem (2.3a)-(2.6b).

Remark 2.1. In [7], two examples of BI media are considered. The first one chooses the parameters

$$\mu_s=\mu_\infty=1,\quad \epsilon_s=6,\quad \epsilon_\infty=4,\quad \tau=20ps,\quad \omega_0=4\pi GHz,\quad \xi=0,\quad \chi=0,$$

which satisfy the assumption (2.13). In this case, $Ind\approx 31.66$.

The second example chooses the parameters

$$\mu_s=1.5,\quad \mu_\infty=1,\quad \epsilon_s=6,\quad \epsilon_\infty=4,\quad \tau=15ps,\quad \omega_0=6\pi GHz,\quad \xi=0.2,\quad \chi=0.1,$$

which also satisfy the assumption (2.13). In this case, $Ind\approx 50.04$.

By the ordinary differential equation theory, we can solve (2.3a) for $\boldsymbol{D}$, and (2.3b) for $\boldsymbol{B}$ analytically.

Lemma 2.1. *The solution of* (2.3a) *can be written as*

$$\boldsymbol{D}(t)=e^{-\delta t}(C_1\cos\alpha t+C_2\sin\alpha t)+\boldsymbol{D}_p(t), \tag{2.18}$$

where the particular solution (cf. [10,12])

$$\begin{aligned}\boldsymbol{D}_p(t)=\int_0^t g(t-s)\Big[&\epsilon_0\epsilon_\infty\boldsymbol{E}_{tt}+2\epsilon_0\epsilon_\infty\omega_0\xi\boldsymbol{E}_t+\epsilon_0\epsilon_s\omega_0^2\boldsymbol{E}\\&+\frac{\chi}{c_v}(\boldsymbol{H}_{tt}+2\omega_0\xi\boldsymbol{H}_t+\omega_0^2\boldsymbol{H})-\frac{\tau\omega_0^2}{c_v}\boldsymbol{H}_t\Big]ds.\end{aligned} \tag{2.19}$$

Here the kernel $g(t)=\alpha^{-1}e^{-\delta t}\sin\alpha t$*, and the parameters are*

$$\delta=\omega_0\xi,\quad \alpha=\sqrt{\omega_0^2-\delta^2}=\omega_0\sqrt{1-\xi^2},\quad C_1=\boldsymbol{D}(\boldsymbol{x},0),\quad C_2=(\boldsymbol{D}_t(\boldsymbol{x},0)+\delta C_1)/\alpha.$$

Similarly, the solution of (2.3b) *can be written as*

$$\boldsymbol{B}(t)=e^{-\delta t}(\tilde{C}_1\cos\alpha t+\tilde{C}_2\sin\alpha t)+\boldsymbol{B}_p(t), \tag{2.20}$$

where the particular solution

$$\begin{aligned}\boldsymbol{B}_p(t)=\int_0^t g(t-s)\Big[&\mu_0\mu_\infty\boldsymbol{H}_{tt}+2\mu_0\mu_\infty\omega_0\xi\boldsymbol{H}_t+\mu_0\mu_s\omega_0^2\boldsymbol{H}\\&+\frac{\chi}{c_v}(\boldsymbol{E}_{tt}+2\omega_0\xi\boldsymbol{E}_t+\omega_0^2\boldsymbol{E})+\frac{\tau\omega_0^2}{c_v}\boldsymbol{E}_t\Big]ds.\end{aligned} \tag{2.21}$$

Here the kernel $g(t)$ *has the same form as that for* $\boldsymbol{D}_p$*, and the parameters are*

$$\tilde{C}_1=\boldsymbol{B}(\boldsymbol{x},0),\;\; \tilde{C}_2=(\boldsymbol{B}_t(\boldsymbol{x},0)+\delta\tilde{C}_1)/\alpha.$$

Finally, for the problem (2.3a)-(2.6b) we have the following stability.

Theorem 2.2. *The solution* $(\boldsymbol{E},\boldsymbol{H},\boldsymbol{D},\boldsymbol{B})$ *of* (2.3a)-(2.6b) *satisfies the following stability: for any* $t\in[0,T]$,

$$(\|\boldsymbol{E}\|_0^2+\|\boldsymbol{E}_t\|_0^2+\|\boldsymbol{H}\|_0^2+\|\boldsymbol{H}_t\|_0^2+\|\boldsymbol{B}\|_0^2+\|\boldsymbol{B}_t\|_0^2+\|\boldsymbol{D}\|_0^2+\|\boldsymbol{D}_t\|_0^2)(t)\le C, \tag{2.22}$$

where the constant $C>0$ *depends on* T *and the initial condition functions*

$$\|\boldsymbol{E}(0)\|_0,\ \|\boldsymbol{H}(0)\|_0,\ \|\boldsymbol{B}(0)\|_0,\ \|\boldsymbol{D}(0)\|_0,\ \|\boldsymbol{E}_t(0)\|_0,\ \|\boldsymbol{H}_t(0)\|_0,\ \|\boldsymbol{B}_t(0)\|_0,\ \|\boldsymbol{D}_t(0)\|_0. \tag{2.23}$$

Proof. Multiplying (2.3a) by $\boldsymbol{E}_t$ and integrating the resultant over Ω, we have

$$\begin{aligned}&\frac{1}{2}\epsilon_0\epsilon_\infty\frac{d}{dt}\|\boldsymbol{E}_t\|_0^2+2\epsilon_0\epsilon_\infty\omega_0\xi\|\boldsymbol{E}_t\|_0^2+\frac{1}{2}\epsilon_0\epsilon_s\omega_0^2\frac{d}{dt}\|\boldsymbol{E}\|_0^2\\&\quad+\frac{\chi}{c_v}(\boldsymbol{H}_{tt}+2\omega_0\xi\boldsymbol{H}_t+\omega_0^2\boldsymbol{H},\boldsymbol{E}_t)-\frac{\tau\omega_0^2}{c_v}(\boldsymbol{H}_t,\boldsymbol{E}_t)-(\boldsymbol{D}_{tt}+2\omega_0\xi\boldsymbol{D}_t+\omega_0^2\boldsymbol{D},\boldsymbol{E}_t)=0.\end{aligned} \tag{2.24}$$

Similarly, multiplying (2.3b) by $\boldsymbol{H}_t$ and integrating the resultant over Ω, we have

$$\begin{aligned}&\frac{1}{2}\mu_0\mu_\infty\frac{d}{dt}\|\boldsymbol{H}_t\|_0^2+2\mu_0\mu_\infty\omega_0\xi\|\boldsymbol{H}_t\|_0^2+\frac{1}{2}\mu_0\mu_s\omega_0^2\frac{d}{dt}\|\boldsymbol{H}\|_0^2\\&\quad+\frac{\chi}{c_v}(\boldsymbol{E}_{tt}+2\omega_0\xi\boldsymbol{E}_t+\omega_0^2\boldsymbol{E},\boldsymbol{H}_t)+\frac{\tau\omega_0^2}{c_v}(\boldsymbol{E}_t,\boldsymbol{H}_t)-(\boldsymbol{B}_{tt}+2\omega_0\xi\boldsymbol{B}_t+\omega_0^2\boldsymbol{B},\boldsymbol{H}_t)=0.\end{aligned} \tag{2.25}$$

Summing up (2.24) and (2.25), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\epsilon_0\epsilon_\infty\|\boldsymbol{E}_t\|_0^2+\epsilon_0\epsilon_s\omega_0^2\|\boldsymbol{E}\|_0^2+\mu_0\mu_\infty\|\boldsymbol{H}_t\|_0^2+\mu_0\mu_s\omega_0^2\|\boldsymbol{H}\|_0^2\right]\\&\quad+\frac{\chi}{c_v}\left[(\boldsymbol{H}_{tt}+2\omega_0\xi\boldsymbol{H}_t+\omega_0^2\boldsymbol{H},\boldsymbol{E}_t)+(\boldsymbol{E}_{tt}+2\omega_0\xi\boldsymbol{E}_t+\omega_0^2\boldsymbol{E},\boldsymbol{H}_t)\right]\\&\quad-(\boldsymbol{D}_{tt}+2\omega_0\xi\boldsymbol{D}_t+\omega_0^2\boldsymbol{D},\boldsymbol{E}_t)-(\boldsymbol{B}_{tt}+2\omega_0\xi\boldsymbol{B}_t+\omega_0^2\boldsymbol{B},\boldsymbol{H}_t)\le 0.\end{aligned}\tag{2.26}$$

Using (2.4a), (2.4b), and the PEC boundary condition (2.5), we have

$$(\boldsymbol{D}_{tt},\boldsymbol{E}_t)+(\boldsymbol{B}_{tt},\boldsymbol{H}_t)=(\nabla\times\boldsymbol{H}_t,\boldsymbol{E}_t)-(\nabla\times\boldsymbol{E}_t,\boldsymbol{H}_t)=0.\tag{2.27}$$

Integrating (2.26) from 0 to t, and using (2.27) and the following identity

$$\int_0^t\left[(\boldsymbol{H}_{tt},\boldsymbol{E}_t)+(\boldsymbol{E}_{tt},\boldsymbol{H}_t)\right]dt=(\boldsymbol{H}_t,\boldsymbol{E}_t)(t)-(\boldsymbol{H}_t,\boldsymbol{E}_t)(0),\tag{2.28}$$

we have

$$\begin{aligned}&\frac{1}{2}\left[\epsilon_0\epsilon_\infty\|\boldsymbol{E}_t\|_0^2+\epsilon_0\epsilon_s\omega_0^2\|\boldsymbol{E}\|_0^2+\mu_0\mu_\infty\|\boldsymbol{H}_t\|_0^2+\mu_0\mu_s\omega_0^2\|\boldsymbol{H}\|_0^2\right](t)\\\le&\frac{1}{2}\left[\epsilon_0\epsilon_\infty\|\boldsymbol{E}_t\|_0^2+\epsilon_0\epsilon_s\omega_0^2\|\boldsymbol{E}\|_0^2+\mu_0\mu_\infty\|\boldsymbol{H}_t\|_0^2+\mu_0\mu_s\omega_0^2\|\boldsymbol{H}\|_0^2\right](0)\\&-\frac{\chi}{c_v}\left[(\boldsymbol{H}_t,\boldsymbol{E}_t)(t)-(\boldsymbol{H}_t,\boldsymbol{E}_t)(0)\right]-\int_0^t\frac{\chi}{c_v}\left[(2\omega_0\xi\boldsymbol{H}_t+\omega_0^2\boldsymbol{H},\boldsymbol{E}_t)+(2\omega_0\xi\boldsymbol{E}_t+\omega_0^2\boldsymbol{E},\boldsymbol{H}_t)\right]dt\\&+\int_0^t\left[(2\omega_0\xi\boldsymbol{D}_t+\omega_0^2\boldsymbol{D},\boldsymbol{E}_t)+(2\omega_0\xi\boldsymbol{B}_t+\omega_0^2\boldsymbol{B},\boldsymbol{H}_t)\right]dt.\end{aligned}\tag{2.29}$$

It is easy to see that all the right hand side terms can be bounded by the left hand side terms except those involving $\boldsymbol{D}$ and $\boldsymbol{B}$, which can be bounded by Lemma 2.1 as shown below.

From (2.18) and (2.19), we can see that $\|\boldsymbol{D}(t)\|_0$ can be bounded by a function of

$$\|\boldsymbol{E}_t(t)\|_0,\ \|\boldsymbol{E}(t)\|_0,\ \|\boldsymbol{H}_t(t)\|_0,\ \|\boldsymbol{H}(t)\|_0,\ \|\boldsymbol{D}(0)\|_0,\ \|\boldsymbol{D}_t(0)\|_0,\ \|\boldsymbol{E}_t(0)\|_0,\ \|\boldsymbol{H}_t(0)\|_0,$$

where we used integration by parts for terms $\boldsymbol{E}_{tt}$ and $\boldsymbol{H}_{tt}$ in (2.19).

Similarly, from (2.20) and (2.21), we can see that $\|\boldsymbol{B}(t)\|_0$ can be bounded by a function of

$$\|\boldsymbol{E}_t(t)\|_0,\ \|\boldsymbol{E}(t)\|_0,\ \|\boldsymbol{H}_t(t)\|_0,\ \|\boldsymbol{H}(t)\|_0,\ \|\boldsymbol{B}(0)\|_0,\ \|\boldsymbol{B}_t(0)\|_0,\ \|\boldsymbol{E}_t(0)\|_0,\ \|\boldsymbol{H}_t(0)\|_0.$$

Differentiating (2.18), we obtain

$$\boldsymbol{D}_t(t)=e^{-\delta t}[(C_2\alpha-C_1\delta)\cos\alpha t-(C_2\delta+C_1\alpha)\sin\alpha t)]+\boldsymbol{D}'_p(t),\tag{2.30}$$

where the derivative

$$\boldsymbol{D}'_p(t)=\int_0^t g_t(t-s)\Big[\epsilon_0\epsilon_\infty \boldsymbol{E}_{tt}+2\epsilon_0\epsilon_\infty\omega_0\xi\boldsymbol{E}_t+\epsilon_0\epsilon_s\omega_0^2\boldsymbol{E}$$
$$+\frac{\chi}{c_v}(\boldsymbol{H}_{tt}+2\omega_0\xi\boldsymbol{H}_t+\omega_0^2\boldsymbol{H})-\frac{\tau\omega_0^2}{c_v}\boldsymbol{H}_t\Big]ds. \tag{2.31}$$

Here kernel

$$g_t(t)=\frac{\omega_0}{\alpha}e^{-\delta t}\cos(\theta+\alpha t),\quad \text{where }\ \theta=\cos^{-1}\Big(\frac{\alpha}{\sqrt{\alpha^2+\delta^2}}\Big)=\cos^{-1}\sqrt{1-\xi^2}.$$

From (2.30) and (2.31), we can see that $\|\boldsymbol{D}_t(t)\|_0$ can be bounded by a function of

$$\|\boldsymbol{E}_t(t)\|_0,\ \|\boldsymbol{E}(t)\|_0,\ \|\boldsymbol{H}_t(t)\|_0,\ \|\boldsymbol{H}(t)\|_0,\ \|\boldsymbol{D}(0)\|_0,\ \|\boldsymbol{D}_t(0)\|_0,\ \|\boldsymbol{E}_t(0)\|_0,\ \|\boldsymbol{H}_t(0)\|_0.$$

By the same arguments, we can prove that $\|\boldsymbol{B}_t(t)\|_0$ can be bounded by a function of

$$\|\boldsymbol{E}_t(t)\|_0,\ \|\boldsymbol{E}(t)\|_0,\ \|\boldsymbol{H}_t(t)\|_0,\ \|\boldsymbol{H}(t)\|_0,\ \|\boldsymbol{B}(0)\|_0,\ \|\boldsymbol{B}_t(0)\|_0,\ \|\boldsymbol{E}_t(0)\|_0,\ \|\boldsymbol{H}_t(0)\|_0.$$

Substituting estimates of $\|\boldsymbol{D}(t)\|_0$, $\|\boldsymbol{B}(t)\|_0$, $\|\boldsymbol{D}_t(t)\|_0$, $\|\boldsymbol{B}_t(t)\|_0$ into (2.29), and using the Gronwall inequality [5], we obtain

$$\big[\epsilon_0\epsilon_\infty\|\boldsymbol{E}_t\|_0^2+\epsilon_0\epsilon_s\omega_0^2\|\boldsymbol{E}\|_0^2+\mu_0\mu_\infty\|\boldsymbol{H}_t\|_0^2+\mu_0\mu_s\omega_0^2\|\boldsymbol{H}\|_0^2\big](t)\le C,\quad \forall t\in(0,T], \tag{2.32}$$

where the constant $C>0$ depends on T and initial condition functions (2.23).

Eq. (2.32), along with the fact that $\|\boldsymbol{B}(t)\|_0$, $\|\boldsymbol{B}_t(t)\|_0$, $\|\boldsymbol{D}(t)\|_0$ and $\|\boldsymbol{D}_t(t)\|_0$ are bounded by those left hand side terms of (2.32) and functions (2.23), concludes the proof. $\square$

3 Design of some fully-discrete finite element schemes

To design a finite element method to solve (2.3a)-(2.6b), we partition Ω by a family of regular cubic or tetrahedral meshes T^h with maximum mesh size h. Depending upon the regularity of the solution, we can use a proper order Raviart-Thomas-Nédélec (RTN) mixed finite element space (cf. [17, 18]): For any $l\ge 1$, on a tetrahedral element, we can choose

$$\boldsymbol{U}_h=\big\{\boldsymbol{u}_h\in H(div;\Omega):\ \boldsymbol{u}_h|_K\in(p_{l-1})^3\oplus\tilde{p}_{l-1}\boldsymbol{x},\ \forall K\in T^h\big\}, \tag{3.1a}$$
$$\boldsymbol{V}_h=\big\{\boldsymbol{v}_h\in H(\text{curl};\Omega):\ \boldsymbol{v}_h|_K\in(p_{l-1})^3\oplus S_l,\ \forall K\in T^h\big\}, \tag{3.1b}$$

where the subspace $S_l=\{\vec{p}\in(\tilde{p}_l)^3:\ \boldsymbol{x}\cdot\vec{p}=0\}$; while on a cubic element we choose

$$\boldsymbol{U}_h=\big\{\boldsymbol{u}_h\in H(div;\Omega):\ \boldsymbol{u}_h|_K\in Q_{l,l-1,l-1}\times Q_{l-1,l,l-1}\times Q_{l-1,l-1,l},\ \forall K\in T^h\big\}, \tag{3.2a}$$
$$\boldsymbol{V}_h=\big\{\boldsymbol{v}_h\in H(\text{curl};\Omega):\ \boldsymbol{v}_h|_K\in Q_{l-1,l,l}\times Q_{l,l-1,l}\times Q_{l,l,l-1},\ \forall K\in T^h\big\}. \tag{3.2b}$$

Here $\tilde{p}_k$ denotes the space of homogeneous polynomials of degree k, and $Q_{i,j,k}$ denotes the space of polynomials whose degrees are less than or equal to i,j,k in variables x,y,z, respectively. To accommodate the boundary condition (2.5), we define a subspace of $\boldsymbol{V}_h$:

$$\boldsymbol{V}_h^0 = \big\{ \boldsymbol{v}_h \in \boldsymbol{V}_h :\ \boldsymbol{n} \times \boldsymbol{v}_h = \boldsymbol{0} \big\}. \tag{3.3}$$

Finally, we divide the time interval $I=[0,T]$ by $N+1$ uniform points $t_i = i\Delta t$, where $\Delta t = T/N$, and $i=0,\cdots,N$. Furthermore, we denote $u^k = u(\cdot,t_k)$ and introduce the difference operators:

$$\delta_\tau^2 u^k = (u^{k+1} - 2u^k + u^{k-1})/(\Delta t)^2, \quad \bar{u}^k = \frac{1}{2}(u^{k+1} + u^{k-1}), \quad \delta_{2\tau} u^k = (u^{k+1} - u^{k-1})/(2\Delta t).$$

With the above preparations, we can now develop a fully discrete finite element scheme for solving (2.3a)-(2.6b): Given initial approximations

$$\boldsymbol{E}_h^0,\ \boldsymbol{H}_h^0,\ \boldsymbol{B}_h^0,\ \boldsymbol{D}_h^0,\ \boldsymbol{E}_h^1,\ \boldsymbol{H}_h^1,\ \boldsymbol{B}_h^1,\ \boldsymbol{D}_h^1, \tag{3.4}$$

at time levels t_0 and t_1, for any $n \geq 1$ find $\boldsymbol{E}_h^{n+1} \in V_h^0$, $\boldsymbol{D}_h^{n+1} \in V_h$, $\boldsymbol{H}_h^{n+1}, \boldsymbol{B}_h^{n+1} \in U_h$ such that

$$(i)\ (\boldsymbol{D}_h^{n+1} - \boldsymbol{D}_h^{n-1}, \boldsymbol{\phi}_h) - 2\Delta t(\boldsymbol{H}_h^n, \nabla\times\boldsymbol{\phi}_h) = 0, \quad \forall \boldsymbol{\phi}_h \in V_h^0, \tag{3.5a}$$

$$(ii)\ (\boldsymbol{B}_h^{n+1} - \boldsymbol{B}_h^{n-1}, \boldsymbol{\psi}_h) + 2\Delta t(\nabla\times\boldsymbol{E}_h^n, \boldsymbol{\psi}_h) = 0, \quad \forall \boldsymbol{\psi}_h \in U_h, \tag{3.5b}$$

$$\begin{aligned}(iii)\ &\delta_\tau^2 \boldsymbol{D}_h^n + 2\omega_0\xi\delta_{2\tau}\boldsymbol{D}_h^n + \omega_0^2 \bar{\boldsymbol{D}}^n = \epsilon_0\epsilon_\infty \delta_\tau^2 \boldsymbol{E}_h^n + 2\epsilon_0\epsilon_\infty\omega_0\xi\delta_{2\tau}\boldsymbol{E}_h^n + \epsilon_0\epsilon_s\omega_0^2\bar{\boldsymbol{E}}^n \\ &\qquad + \frac{\chi}{c_v}(\delta_\tau^2 \boldsymbol{H}_h^n + 2\omega_0\xi\delta_{2\tau}\boldsymbol{H}_h^n + \omega_0^2\bar{\boldsymbol{H}}^n) - \frac{\tau\omega_0^2}{c_v}\delta_{2\tau}\boldsymbol{H}_h^n, \end{aligned} \tag{3.5c}$$

$$\begin{aligned}(iv)\ &\delta_\tau^2 \boldsymbol{B}_h^n + 2\omega_0\xi\delta_{2\tau}\boldsymbol{B}_h^n + \omega_0^2 \bar{\boldsymbol{B}}^n = \mu_0\mu_\infty \delta_\tau^2 \boldsymbol{H}_h^n + 2\mu_0\mu_\infty\omega_0\xi\delta_{2\tau}\boldsymbol{H}_h^n + \mu_0\mu_s\omega_0^2\bar{\boldsymbol{H}}^n \\ &\qquad + \frac{\chi}{c_v}(\delta_\tau^2 \boldsymbol{E}_h^n + 2\omega_0\xi\delta_{2\tau}\boldsymbol{E}_h^n + \omega_0^2\bar{\boldsymbol{E}}^n) + \frac{\tau\omega_0^2}{c_v}\delta_{2\tau}\boldsymbol{E}_h^n. \end{aligned} \tag{3.5d}$$

Note that the scheme (3.5a)-(3.5d) is explicit in that at each time step we can first solve (3.5a)-(3.5b) for $\boldsymbol{D}_h^{n+1}$ and $\boldsymbol{B}_h^{n+1}$, independently; then we solve (3.5c)-(3.5d) as a system for $\boldsymbol{E}_h^{n+1}$ and $\boldsymbol{H}_h^{n+1}$.

Theorem 3.1. *Under the assumption* (2.13)*, the system* (3.5c)-(3.5d) *is solvable for* $\boldsymbol{E}_h^{n+1}$ *and* $\boldsymbol{H}_h^{n+1}$.

Proof. Note that the coefficient matrix for $(\boldsymbol{E}_h^{n+1}, \boldsymbol{H}_h^{n+1})'$ can be written as:

$$A = \begin{pmatrix} \dfrac{\epsilon_0\epsilon_\infty}{(\Delta t)^2} + \dfrac{\epsilon_0\epsilon_\infty\omega_0\xi}{\Delta t} + \dfrac{\epsilon_0\epsilon_s\omega_0^2}{2} & \dfrac{\chi}{c_v}\Big(\dfrac{1}{(\Delta t)^2} + \dfrac{\omega_0\xi}{\Delta t} + \dfrac{\omega_0^2}{2}\Big) - \dfrac{\tau\omega_0^2}{2c_v\Delta t} \\ \dfrac{\chi}{c_v}\Big(\dfrac{1}{(\Delta t)^2} + \dfrac{\omega_0\xi}{\Delta t} + \dfrac{\omega_0^2}{2}\Big) + \dfrac{\tau\omega_0^2}{2c_v\Delta t} & \dfrac{\mu_0\mu_\infty}{(\Delta t)^2} + \dfrac{\mu_0\mu_\infty\omega_0\xi}{\Delta t} + \dfrac{\mu_0\mu_s\omega_0^2}{2} \end{pmatrix},$$

whose determinant equals

$$
\begin{aligned}
|A| =& \epsilon_0\mu_0(\epsilon_\infty\mu_\infty-\chi^2)f^2(\Delta t)+\frac{\epsilon_0\mu_0\omega_0^2}{2}f(\Delta t)\big[\epsilon_\infty(\mu_s-\mu_\infty)+\mu_\infty(\epsilon_s-\epsilon_\infty)\big]\\
&+\frac{\epsilon_0\mu_0\omega_0^4(\epsilon_s-\epsilon_\infty)(\mu_s-\mu_\infty)}{4}+\Big(\frac{\tau\omega_0^2}{2c_v\Delta t}\Big)^2,
\end{aligned} \tag{3.6}
$$

where we denote $f(\Delta t)=1/(\Delta t)^2+\omega_0\xi/\Delta t+\omega_0^2/2$. It is easy to see that $|A|>0$ under the assumption (2.13). Hence the matrix A is invertible, which concludes the proof. $\square$

Similarly, we can develop another fully discrete finite element scheme for solving (2.3a)-(2.4b): Given initial approximations (3.4), for any $n\geq 1$ find $\boldsymbol{E}_h^{n+1}\in V_h^0$, $\boldsymbol{D}_h^{n+1}\in V_h$, $\boldsymbol{H}_h^{n+1}$, $\boldsymbol{B}_h^{n+1}\in U_h$ such that

$$
(i)\ (\boldsymbol{D}_h^{n+1}-\boldsymbol{D}_h^{n-1},\boldsymbol{\phi}_h)-2\Delta t(\bar{\boldsymbol{H}}_h^n,\nabla\times\boldsymbol{\phi}_h)=0,\quad \forall\boldsymbol{\phi}_h\in V_h^0, \tag{3.7a}
$$

$$
(ii)\ (\boldsymbol{B}_h^{n+1}-\boldsymbol{B}_h^{n-1},\boldsymbol{\psi}_h)+2\Delta t(\nabla\times\bar{\boldsymbol{E}}_h^n,\boldsymbol{\psi}_h)=0,\quad \forall\boldsymbol{\psi}_h\in U_h, \tag{3.7b}
$$

$$
\begin{aligned}
(iii)\ &\delta_\tau^2\boldsymbol{D}_h^n+2\omega_0\xi\delta_{2\tau}\boldsymbol{D}_h^n+\omega_0^2\boldsymbol{D}^n=\epsilon_0\epsilon_\infty\delta_\tau^2\boldsymbol{E}_h^n+2\epsilon_0\epsilon_\infty\omega_0\xi\delta_{2\tau}\boldsymbol{E}_h^n+\epsilon_0\epsilon_s\omega_0^2\boldsymbol{E}^n\\
&\qquad+\frac{\chi}{c_v}(\delta_\tau^2\boldsymbol{H}_h^n+2\omega_0\xi\delta_{2\tau}\boldsymbol{H}_h^n+\omega_0^2\boldsymbol{H}^n)-\frac{\tau\omega_0^2}{c_v}\delta_{2\tau}\boldsymbol{H}_h^n,
\end{aligned} \tag{3.7c}
$$

$$
\begin{aligned}
(iv)\ &\delta_\tau^2\boldsymbol{B}_h^n+2\omega_0\xi\delta_{2\tau}\boldsymbol{B}_h^n+\omega_0^2\boldsymbol{B}^n=\mu_0\mu_\infty\delta_\tau^2\boldsymbol{H}_h^n+2\mu_0\mu_\infty\omega_0\xi\delta_{2\tau}\boldsymbol{H}_h^n+\mu_0\mu_s\omega_0^2\boldsymbol{H}^n\\
&\qquad+\frac{\chi}{c_v}(\delta_\tau^2\boldsymbol{E}_h^n+2\omega_0\xi\delta_{2\tau}\boldsymbol{E}_h^n+\omega_0^2\boldsymbol{E}^n)+\frac{\tau\omega_0^2}{c_v}\delta_{2\tau}\boldsymbol{E}_h^n.
\end{aligned} \tag{3.7d}
$$

First, we like to remark that this scheme is different from the scheme (3.5a)-(3.5d) in that this scheme is fully coupled for all unknowns $\boldsymbol{E}_h^{n+1}$, $\boldsymbol{H}_h^{n+1}$, $\boldsymbol{B}_h^{n+1}$ and $\boldsymbol{D}_h^{n+1}$. This fact makes implementing this scheme quite challenging, but we shall show below that the unknowns can be separated after some calculations.

Multiplying (3.7a) by $1/(\Delta t)^2+\omega_0\xi/\Delta t$ and using (3.7c) to eliminate $\boldsymbol{D}_h^{n+1}$, we obtain

$$
\begin{aligned}
&\Big(\frac{\epsilon_0\epsilon_\infty}{(\Delta t)^2}+\frac{\epsilon_0\epsilon_\infty\omega_0\xi}{\Delta t}\Big)(\boldsymbol{E}_h^{n+1},\boldsymbol{\phi}_h)-\Big(\frac{1}{\Delta t}+\omega_0\xi\Big)(\boldsymbol{H}_h^{n+1},\nabla\times\boldsymbol{\phi}_h)\\
&\quad+\Big[\frac{\chi}{c_v}\Big(\frac{1}{(\Delta t)^2}+\frac{\omega_0\xi}{\Delta t}\Big)-\frac{\tau\omega_0^2}{2c_v\Delta t}\Big](\boldsymbol{H}_h^{n+1},\boldsymbol{\phi}_h)\\
=&(R_1,\boldsymbol{\phi}_h)+\Big(\frac{1}{\Delta t}+\omega_0\xi\Big)(\boldsymbol{H}_h^{n-1},\nabla\times\boldsymbol{\phi}_h),
\end{aligned} \tag{3.8}
$$

where R_1 is given as

$$
\begin{aligned}
R_1=&-\Big(\epsilon_0\epsilon_s\omega_0^2-\frac{2\epsilon_0\epsilon_\infty}{(\Delta t)^2}\Big)\boldsymbol{E}_h^n-\Big(\frac{\epsilon_0\epsilon_\infty}{(\Delta t)^2}-\frac{\epsilon_0\epsilon_\infty\omega_0\xi}{\Delta t}\Big)\boldsymbol{E}_h^{n-1}-\frac{\chi}{c_v}\Big(\omega_0^2-\frac{2}{(\Delta t)^2}\Big)\boldsymbol{H}_h^n\\
&-\Big[\frac{\chi}{c_v}\Big(\frac{1}{(\Delta t)^2}-\frac{\omega_0\xi}{\Delta t}\Big)+\frac{\tau\omega_0^2}{2c_v\Delta t}\Big]\boldsymbol{H}_h^{n-1}+\frac{2}{(\Delta t)^2}\boldsymbol{D}_h^{n-1}-\Big(\frac{2}{(\Delta t)^2}-\omega_0^2\Big)\boldsymbol{D}_h^n.
\end{aligned} \tag{3.9}
$$

Similarly, using (3.7b) and (3.7d), we can obtain

$$\begin{aligned}&\Big(\frac{\mu_0\mu_\infty}{(\Delta t)^2}+\frac{\mu_0\mu_\infty\omega_0\xi}{\Delta t}\Big)(\boldsymbol{H}_h^{n+1},\boldsymbol{\psi}_h)+\Big(\frac{1}{\Delta t}+\omega_0\xi\Big)(\nabla\times\boldsymbol{E}_h^{n+1},\boldsymbol{\psi}_h)\\&\quad+\Big[\frac{\chi}{c_v}\Big(\frac{1}{(\Delta t)^2}+\frac{\omega_0\xi}{\Delta t}\Big)+\frac{\tau\omega_0^2}{2c_v\Delta t}\Big](\boldsymbol{E}_h^{n+1},\boldsymbol{\psi}_h)\\=&(R_2,\boldsymbol{\psi}_h)-\Big(\frac{1}{\Delta t}+\omega_0\xi\Big)(\nabla\times\boldsymbol{E}_h^{n-1},\boldsymbol{\psi}_h),\end{aligned}\tag{3.10}$$

where R_2 is given as

$$\begin{aligned}R_2=&-\Big(\mu_0\mu_s\omega_0^2-\frac{2\mu_0\mu_\infty}{(\Delta t)^2}\Big)\boldsymbol{H}_h^n-\Big(\frac{\mu_0\mu_\infty}{(\Delta t)^2}-\frac{\mu_0\mu_\infty\omega_0\xi}{\Delta t}\Big)\boldsymbol{H}_h^{n-1}-\frac{\chi}{c_v}\Big(\omega_0^2-\frac{2}{(\Delta t)^2}\Big)\boldsymbol{E}_h^n\\&-\Big[\frac{\chi}{c_v}\Big(\frac{1}{(\Delta t)^2}-\frac{\omega_0\xi}{\Delta t}\Big)-\frac{\tau\omega_0^2}{2c_v\Delta t}\Big]\boldsymbol{E}_h^{n-1}+\frac{2}{(\Delta t)^2}\boldsymbol{B}_h^{n-1}-\Big(\frac{2}{(\Delta t)^2}-\omega_0^2\Big)\boldsymbol{B}_h^n.\end{aligned}\tag{3.11}$$

Hence, at each time step, the scheme (3.7a)-(3.7d) can be implemented as follows: we first solve (3.8) and (3.10) as a system for $\boldsymbol{E}_h^{n+1}$ and $\boldsymbol{H}_h^{n+1}$; then we solve (3.7a) and (3.7b) independently for $\boldsymbol{D}_h^{n+1}$ and $\boldsymbol{B}_h^{n+1}$. Below we assure that the system formed by (3.8) and (3.10) is indeed solvable.

Theorem 3.2. *Under the assumption $\chi<\sqrt{\epsilon_\infty\mu_\infty}$, the system* (3.8) *and* (3.10) *is solvable for $\boldsymbol{E}_h^{n+1}$ and $\boldsymbol{H}_h^{n+1}$.*

Proof. To prove the solvability of the system (3.8) and (3.10), we assume that their right hand sides are zero. Choosing $\boldsymbol{\phi}=\boldsymbol{E}_h^{n+1}$ and $\boldsymbol{\psi}=\boldsymbol{H}_h^{n+1}$ in (3.8) and (3.10) and summing up the resultants, we have

$$\Big(\frac{1}{(\Delta t)^2}+\frac{\omega_0\xi}{\Delta t}\Big)\Big[\epsilon_0\epsilon_\infty\|\boldsymbol{E}_h^{n+1}\|_0^2+\mu_0\mu_\infty\|\boldsymbol{H}_h^{n+1}\|_0^2+2\frac{\chi}{c_v}(\boldsymbol{E}_h^{n+1},\boldsymbol{H}_h^{n+1})\Big]=0,$$

which is equivalent to

$$\epsilon_0\epsilon_\infty\|\boldsymbol{E}_h^{n+1}\|_0^2+\mu_0\mu_\infty\|\boldsymbol{H}_h^{n+1}\|_0^2+2\frac{\chi}{c_v}(\boldsymbol{E}_h^{n+1},\boldsymbol{H}_h^{n+1})=0.\tag{3.12}$$

From (3.12), we easily see that

$$(\boldsymbol{E}_h^{n+1},\boldsymbol{H}_h^{n+1})\le 0.\tag{3.13}$$

On the other hand, we can rewrite (3.12) as

$$\|\sqrt{\epsilon_0\epsilon_\infty}\boldsymbol{E}_h^{n+1}+\sqrt{\mu_0\mu_\infty}\boldsymbol{H}_h^{n+1}\|^2=2\sqrt{\epsilon_0\mu_0}(\sqrt{\epsilon_\infty\mu_\infty}-\chi)(\boldsymbol{E}_h^{n+1},\boldsymbol{H}_h^{n+1})\ge 0,$$

which, combining with (3.13), shows that $(\boldsymbol{E}_h^{n+1},\boldsymbol{H}_h^{n+1})=0$. Hence by (3.12), we have $\|\boldsymbol{E}_h^{n+1}\|_0=\|\boldsymbol{H}_h^{n+1}\|_0=0$, which concludes the solvability of the system (3.8) and (3.10). $\square$

Finally, we want to show that the scheme (3.7a)-(3.7d) is stable.

Theorem 3.3. *Let $c_{inv}>0$ be the constant in the standard inverse estimate* [17]

$$\|\nabla\times\boldsymbol{u}_h\|_0\le c_{inv}h^{-1}\|\boldsymbol{u}_h\|_0,\quad \forall\boldsymbol{u}_h\in\boldsymbol{V}_h. \tag{3.14}$$

Under the time step constraint

$$\frac{c_{inv}\Delta t}{h}=\mathcal{O}(1), \tag{3.15}$$

and the condition $\chi<\sqrt{\epsilon_\infty\mu_\infty}$, the solution $(\boldsymbol{E}_h^{n+1},\boldsymbol{H}_h^{n+1},\boldsymbol{D}_h^{n+1},\boldsymbol{B}_h^{n+1})$ of (3.7a)-(3.7d) *satisfies the following stability: for any $n\ge 1$,*

$$\begin{aligned}&\|\boldsymbol{E}_h^{n+1}\|_0^2+\|\boldsymbol{H}_h^{n+1}\|_0^2+\|\boldsymbol{D}_h^{n+1}\|_0^2+\|\boldsymbol{B}_h^{n+1}\|_0^2\\ \le& C\big[\|\boldsymbol{E}_h^{n}\|_0^2+\|\boldsymbol{E}_h^{n-1}\|_0^2+\|\boldsymbol{H}_h^{n}\|_0^2+\|\boldsymbol{H}_h^{n-1}\|_0^2\\ &+\|\boldsymbol{D}_h^{n}\|_0^2+\|\boldsymbol{D}_h^{n-1}\|_0^2+\|\boldsymbol{B}_h^{n}\|_0^2+\|\boldsymbol{B}_h^{n-1}\|_0^2\big],\end{aligned} \tag{3.16}$$

where the constant $C>0$ is independent of h and Δt.

Proof. Choosing $\boldsymbol{\phi}_h=(\Delta t)^2\boldsymbol{E}_h^{n+1}$ in (3.8), $\boldsymbol{\psi}_h=(\Delta t)^2\boldsymbol{H}_h^{n+1}$ in (3.10), then adding the results together, we obtain

$$\begin{aligned}&(1+\omega_0\xi\Delta t)\Big[\epsilon_0\epsilon_\infty\|\boldsymbol{E}_h^{n+1}\|_0^2+\mu_0\mu_\infty\|\boldsymbol{H}_h^{n+1}\|_0^2+2\frac{\chi}{c_v}(\boldsymbol{E}_h^{n+1},\boldsymbol{H}_h^{n+1})\Big]\\ =&(\Delta t)^2(R_1,\boldsymbol{E}_h^{n+1})+\Delta t(1+\omega_0\xi\Delta t)(\boldsymbol{H}_h^{n-1},\nabla\times\boldsymbol{E}_h^{n+1})+(\Delta t)^2(R_2,\boldsymbol{H}_h^{n+1})\\ &-\Delta t(1+\omega_0\xi\Delta t)(\nabla\times\boldsymbol{E}_h^{n-1},\boldsymbol{H}_h^{n+1}).\end{aligned} \tag{3.17}$$

First, using the assumption $\chi<\sqrt{\epsilon_\infty\mu_\infty}$, we have

$$\begin{aligned}2\frac{\chi}{c_v}(\boldsymbol{E}_h^{n+1},\boldsymbol{H}_h^{n+1})&>-2\sqrt{\epsilon_0\epsilon_\infty\mu_0\mu_\infty}|(\boldsymbol{E}_h^{n+1},\boldsymbol{H}_h^{n+1})|\\ &\ge-\epsilon_0\epsilon_\infty\|\boldsymbol{E}_h^{n+1}\|_0^2-\mu_0\mu_\infty\|\boldsymbol{H}_h^{n+1}\|_0^2,\end{aligned}$$

which makes the left hand side terms of (3.17) bounded below by

$$C(1+\omega_0\xi\Delta t)(\epsilon_0\epsilon_\infty\|\boldsymbol{E}_h^{n+1}\|_0^2+\mu_0\mu_\infty\|\boldsymbol{H}_h^{n+1}\|_0^2).$$

In the following, we just need to estimate those right hand terms of (3.17). Using the Cauchy-Schwarz inequality, the inverse estimate (3.14), and the arithmetic-geometric mean inequality, we have

$$\begin{aligned}\Delta t(1+\omega_0\xi\Delta t)(\boldsymbol{H}_h^{n-1},\nabla\times\boldsymbol{E}_h^{n+1})&\le(1+\omega_0\xi\Delta t)\Delta t\cdot c_{inv}h^{-1}\|\boldsymbol{H}_h^{n-1}\|_0\|\boldsymbol{E}_h^{n+1}\|_0\\ &\le(1+\omega_0\xi\Delta t)\Big[\delta_1\|\boldsymbol{E}_h^{n+1}\|_0^2+\frac{1}{4\delta_1}\cdot(c_{inv}\Delta t/h)^2\|\boldsymbol{H}_h^{n-1}\|_0^2\Big].\end{aligned}$$

Similarly, we have

$$\Delta t(1+\omega_0\xi\Delta t)(\nabla\times \boldsymbol{E}_h^{n-1},\boldsymbol{H}_h^{n+1})\le(1+\omega_0\xi\Delta t)\Big[\delta_2\|\boldsymbol{H}_h^{n+1}\|_0^2+\frac{1}{4\delta_2}\cdot(c_{inv}\Delta t/h)^2\|\boldsymbol{E}_h^{n-1}\|_0^2\Big].$$

By the definition of R_1, we obtain

$$\begin{aligned}
&(\Delta t)^2(R_1,\boldsymbol{E}_h^{n+1})\le\delta_3\|\boldsymbol{E}_h^{n+1}\|_0^2+\frac{1}{4\delta_3}\|(\Delta t)^2R_1\|_0^2\\
\le&\delta_3\|\boldsymbol{E}_h^{n+1}\|_0^2+\frac{1}{4\delta_3}\Big\|(2\epsilon_0\epsilon_\infty-\epsilon_0\epsilon_s\omega_0^2(\Delta t)^2)\boldsymbol{E}_h^n+\epsilon_0\epsilon_\infty(\omega_0\xi\Delta t-1)\boldsymbol{E}_h^{n-1}\\
&+\frac{\chi}{c_v}(2-\omega_0^2(\Delta t)^2)\boldsymbol{H}_h^n-\Big[\frac{\chi}{c_v}(1-\omega_0\xi\Delta t)+\frac{\tau\omega_0^2}{2c_v}\Delta t\Big]\boldsymbol{H}_h^{n-1}\\
&+2\boldsymbol{D}_h^{n-1}-(2-\omega_0^2(\Delta t)^2)\boldsymbol{D}_h^n\Big\|_0^2\\
\le&\delta_3\|\boldsymbol{E}_h^{n+1}\|_0^2+\frac{C}{4\delta_3}\big(\|\boldsymbol{E}_h^n\|_0^2+\|\boldsymbol{E}_h^{n-1}\|_0^2+\|\boldsymbol{H}_h^n\|_0^2+\|\boldsymbol{H}_h^{n-1}\|_0^2+\|\boldsymbol{D}_h^{n-1}\|_0^2+\|\boldsymbol{D}_h^n\|_0^2\big).
\end{aligned}$$

Similarly, by the definition of R_2, we can obtain

$$\begin{aligned}
(\Delta t)^2(R_2,\boldsymbol{H}_h^{n+1})\le\delta_4\|\boldsymbol{H}_h^{n+1}\|_0^2+\frac{C}{4\delta_4}\Big(&\|\boldsymbol{H}_h^n\|_0^2+\|\boldsymbol{H}_h^{n-1}\|_0^2+\|\boldsymbol{E}_h^n\|_0^2\\
&+\|\boldsymbol{E}_h^{n-1}\|_0^2+\|\boldsymbol{B}_h^{n-1}\|_0^2+\|\boldsymbol{B}_h^n\|_0^2\Big).
\end{aligned}$$

Combining the above estimates and choosing δ_i small enough, we have

$$\begin{aligned}
\|\boldsymbol{E}_h^{n+1}\|_0^2+\|\boldsymbol{H}_h^{n+1}\|_0^2\le C\Big(&\|\boldsymbol{H}_h^n\|_0^2+\|\boldsymbol{H}_h^{n-1}\|_0^2+\|\boldsymbol{E}_h^n\|_0^2+\|\boldsymbol{E}_h^{n-1}\|_0^2\\
&+\|\boldsymbol{B}_h^{n-1}\|_0^2+\|\boldsymbol{B}_h^n\|_0^2+\|\boldsymbol{D}_h^{n-1}\|_0^2+\|\boldsymbol{D}_h^n\|_0^2\Big),
\end{aligned}$$

which, along with (3.7c) and (3.7d), completes the proof. □

Optimal error estimates can be proved using ideas similar to our previous work (cf. [10, 12]). Due to its lengthy and technicality, we skip the proof. Similar stability and error estimates can be proved for the scheme (3.5a)-(3.5d).

4 Conclusions

In this paper, we carry out the existence and uniqueness study of time-dependent Maxwell's equations in dispersive lossy bi-isotropic media. Two fully discrete finite element schemes are developed and analyzed. More advanced numerical methods such as hp finite element methods [2,3], discontinuous Galerkin methods [8,9,11], and multiscale finite element methods [4,22] will be considered in the future.

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