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Valuation of American Call Option Considering Uncertain Volatility

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Abstract. The parabolic variational inequality for simulating the valuation of American option is used to analyze a continuous dependence of the solution with respect to the uncertain volatility parameter. Three kinds of the continuity are proved, enabling us to employ the maximum range method for the uncertain parameter, under the condition that the criterion-functional has the corresponding property.

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1 Introduction

The problem of pricing American options is important both in theory and in practice. It has been shown by the Nobel Prize laureates Merton [7] and Black and Scholes [4] that the valuation of American call option can be simulated by a free boundary problem for a degenerate parabolic equation. A weak solution of the problem has been defined by Badea and Wang [2]. They proved the existence and uniqueness of the weak solution and some regularity results by a detailed analysis based on the use of maximum principles.

Efficient numerical methods for the solution of the problem using finite elements in "space" and backward differences in "time" have been proposed by Allegretto et al. [1] and Lin et al. [8]. These authors started from an equivalent variational inequality, which can be derived by a suitable change of all variables and which avoids the degeneracy.

The aim of the present paper is to complete the results of Badea and Wang [2, 3] by an analysis of a continuous dependence of the weak solution with respect to the volatility. The latter parameter appears to be the only parameter, which is not observable directly in the market. On the basis of a variational inequality for the weak

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solution [3], we prove three kinds of the continuous dependence, provided the volatility belongs to a given compact interval of positive numbers. Then one can employ the maximum range method [5,6], if a criterion-functional is prescribed, which has properties, corresponding to the three kinds of the continuity, mentioned above.

2 Formulation by a parabolic variational inequality

The original mathematical model of American call option proposed by Merton [7] is represented by a free boundary problem for a parabolic equation

$$w_t - \frac{1}{2}\sigma^2 x^2 w_{xx} - (r-d)xw_x + rw = 0, \qquad \text{in } D, \tag{2.1}$$

with the initial condition

$$w(x,0) = 0,$$

and the boundary conditions

$$w(0,t) = 0, \quad w\big(s(t),t\big) = (x-Z)^+, \quad w_x\big(s(t),t\big) = 1,$$

for all $t \in (0,T]$. Here $w(x,t)$ denotes the value of the American call option, T is the maturity date (the time at which the American call option expires), r is the interest rate, d the dividend rate, σ the volatility, Z the exercise price, x denotes the stock price, $t \equiv T - t_r$, where t_r is the real time,

$$D = \Big\{(x,t):\ 0 < x < s(t), \quad t \in (0,T]\Big\}.$$

The free boundary $x = s(t)$ denotes the optimal exercise curve,

$$w_x \equiv \frac{\partial w}{\partial x}, \quad w_t \equiv \frac{\partial w}{\partial t}, \quad \text{and} \quad (u)^+ = \max\{u, 0\}.$$

We assume that r, d, Z, σ are positive real constants.

If we define a new function $u(x,t)$ by

$$u = w - (x-Z)^+,$$

we can extend the function u outside the domain D by zero. In this way a weak solution has been defined by Badea and Wang in [2], where an upper bound

$$S_0(\sigma) = \frac{Z\lambda(\sigma)}{\big(\lambda(\sigma)-1\big)},$$

with

$$\lambda(\sigma) = \sigma^{-2}\Big\{\frac{\sigma^2}{2} - r + d + \Big[\Big(\frac{\sigma^2}{2} - r + d\Big)^2 + 2r\sigma^2\Big]^{\frac{1}{2}}\Big\},$$

was found for $s(t)$, so that

$$\sup_{t\in(0,T]} s(t) \le S_0. \tag{2.2}$$

Remark 2.1. Since $d > 0$ and $\sigma^2 > 0$, $\lambda(\sigma) > 1$ follows. One can choose any constant $S \ge S_0(\sigma)$ and define $\Omega = (0, S)$, $\quad I = [0, T]$, the space

$$W(I) = \Big\{ v \in L^2\big(I, H_0^1(\Omega)\big) : \ x^{-1} v_t \in L^2\big(I, L^2(\Omega)\big) \Big\},$$

the bilinear form

$$a(\sigma; u, v) = \frac{\sigma^2}{2(u_x, v_x)} + (d-r)(u_x, x^{-1}v) + r(x^{-1}u, x^{-1}v),$$

and the function

$$q(x) = (x^{-1}d - x^{-2}rZ)H(x - Z),$$

where (f, g) denotes the inner product $\int_\Omega fg\,dx$ and $H(\cdot)$ is the Heaviside function.

Definition 2.1. *We say that u is a weak solution if $u \in W(I)$ is such that $u(x, 0) = 0$ and*

$$(x^{-2}u_t, v) + a(\sigma; u, v) + \big(qH(u), v\big) = \frac{1}{2}\sigma^2 v(Z), \tag{2.3}$$

holds for all $v \in H_0^1(\Omega)$ and almost all $t \in I$.

Badea and Wang proved in [2] that there exists a unique weak solution,

$$\begin{aligned}
&x^{-1}u \in L^\infty\big(I, L^2(\Omega)\big), && u_x \in L^\infty\big(I, L^2(\Omega)\big),\\
&\sup_{t\in(0,T]} \{s(t)\} \le S_0, && \inf_{t\in I} \{s(t)\} \ge Z \max\Big\{\frac{r}{d}, 1\Big\},\\
&u_t \ge 0, \quad \text{in } \ \Omega \times I, && u > 0, \quad \text{in } \ D,\\
&u = 0, \quad \text{in } \ (\Omega \times I)\backslash D.
\end{aligned}$$

Moreover, the values of $u(x, t)$ do not depend on the value of S. It is easy to deduce that the weak solution satisfies the following parabolic variational inequality (see [3]):

$$\begin{aligned}
&(x^{-2}u_t, v - u) + a(\sigma; u, v - u)\\
\ge &\frac{\sigma^2\big(v(Z) - u(Z)\big)}{2} - \int_Z^S q(v - u)\,dx,
\end{aligned} \tag{2.4}$$

for all $v \in K$ and

$$\text{almost all} \quad t \in I, \quad u \in L^2(I, K), \quad x^{-1}u_t \in L^2\big(I, L^2(\Omega)\big),$$

and $u(x, 0) = 0$, where

$$K = \Big\{ v \in H_0^1(\Omega) : \ v \ge 0, \quad \text{in } \ \Omega \Big\}.$$

In what follows we will start the analysis on the basis of problem (2.4).

3 Continuous dependence of solution with respect to volatility

We will consider that the volatility σ is uncertain and belongs to a given interval

$$\mathcal{U}_{ad} = \big[\sigma_{min}, \sigma_{max}\big], \qquad 0 < \sigma_{min} < \sigma_{max} < \infty.$$

A question arises about the dependence of the solution $u \equiv u(\sigma)$ of problem (2.4) on the parameter $\sigma \in \mathcal{U}_{ad}$. Let us choose

$$S \geq \max_{\sigma \in \mathcal{U}_{ad}} S_0(\sigma),$$

Then we will prove the following main result.

Theorem 3.1. *Let $\sigma_n \in \mathcal{U}_{ad}$, $\sigma_n \to \sigma$, as $n \to \infty$. Then*

$$\begin{aligned} &u(\sigma_n) \to u(\sigma), && in\ L^2\big(I; H_0^1(\Omega)\big),\\ &x^{-1}u(\sigma_n)(t) \to x^{-1}u(\sigma)(t), && for\ a.a.\ t \in I,\\ &x^{-1}u(\sigma_n) \rightharpoonup x^{-1}u(\sigma) \quad (weak\ star), && in\ L^\infty\big(I; L^2(\Omega)\big). \end{aligned}$$

For the proof we shall need the following lemmas and a proposition.

Lemma 3.1. *There exist positive constants C_0, C_1 such that*

$$\begin{aligned} &a(\sigma; v, v) = \frac{\sigma^2}{2\,\|v_x\|_0^2} + \frac{r+d}{2\,\|x^{-1}v\|_0^2} \geq C_0\,\|v\|_1^2,\\ &\big|a(\sigma; u, v)\big| \leq C_1\,\|u\|_1\,\|v\|_1, \end{aligned}$$

for any $u, v \in H_0^1(\Omega)$ and any $\sigma \in \mathcal{U}_{ad}$.

Henceforth $\|v\|_0$ denotes the norm in $L^2(\Omega)$ and $\|v\|_1$ the standard norm in the Sobolev space $H^1(\Omega)$.

Proof is a consequence of the inequalities

$$S^{-1}\|v\|_0 \leq \ \|x^{-1}v\|_0 \leq \ 2\|v_x\|_0, \qquad \forall v \in H_0^1(\Omega).$$

Proposition 3.2. *There exists a constant C_3 such that*

$$\big\|x^{-1}u_t(\sigma)\big\|_{L^2\left(I; L^2(\Omega)\right)} \leq C_3, \qquad \forall \sigma \in \mathcal{U}_{ad}.$$

Proof. By an analysis of the proof of Lemmas 3.4 and 3.5 in [2]-II, we infer that

$$\big\|x^{-1}\partial_t u_k(\sigma)\big\|_{L^2\left(I; L^2(\Omega)\right)} \leq C(\sigma_{\max}), \qquad \forall \sigma \in \mathcal{U}_{ad},$$

where $u_k(\sigma)$ is the Rothe piecewise linear approximation of $u(\sigma)$ in the interval I. Then a function ω with

$$x^{-1}\partial_t\omega \in L^2\big(I; L^2(\Omega)\big),$$

can be found as a weak limit of a subsequence of $\{x^{-1}\partial_t u_k(\sigma)\}$ such that the norm of $x^{-1}\partial_t\omega$ in $L^2\big(I; L^2(\Omega)\big)$ is bounded by $C(\sigma_{\max})$. In proving Theorems 4.1 and 2.1 of [2]-II, it was verified that $\omega \equiv u(\sigma)$. $\square$

Lemma 3.2. *The set $L^2(I_t; K)$ is weakly closed in*

$$L^2\big(I_t; H_0^1(\Omega)\big), \qquad \textit{for all } t \in I,$$

where $I_t = (0, T]$.

Proof. It is readily seen that $L^2(I_t; K)$ is convex, since K is convex. Let $u_n \in L^2(I_t; K)$, $u_n \to u$ in $L^2\big(I_t; H_0^1(\Omega)\big)$, as $n \to \infty$. Then from

$$\int_0^t \|u_n(\tau) - u(\tau)\|_1^2 \, d\tau \to 0, \qquad \forall t \in I,$$

we infer that $u_n(\tau) \to u(\tau)$ in $H_0^1(\Omega)$ for a.a. $\tau \in I_t$, so that $u(\tau) \in K$ for a.a. $\tau \in I_t$ and $u \in L^2(I_t; K)$. Since $L^2(I_t; K)$ is convex and closed in $L^2\big(I_t; H_0^1(\Omega)\big)$, it is weakly closed. $\square$

Proof of Theorem 3.1. For brevity, let us denote $u_n = u(\sigma_n)$ and

$$L(\sigma_n; w) = \frac{\sigma^2 w(Z)}{2} - \int_Z^S qw \, dx,$$

From (2.4), we infer that

$$(x^{-2}u_{nt}, v - u_n) + a(\sigma_n; u_n, v - u_n) \geq L(\sigma_n; v - u_n), \tag{3.1}$$

holds for any $v \in K$ and a.a. $t \in I$.

Let us insert $v = 0$ to obtain

$$(x^{-1}\partial_t u_n, x^{-1}u_n) + a(\sigma; u_n, u_n) \leq L(\sigma_n; u_n). \tag{3.2}$$

By using Lemma 3.1, the embedding $H_0^1(\Omega) \hookrightarrow C(\overline{\Omega})$ and the inequality

$$|fg| \leq \frac{\epsilon}{2f^2} + (2\epsilon)^{-1}g^2,$$

with $\epsilon > 0$, we arrive at

$$\frac{d}{dt}\|x^{-1}u_n(t)\|_0^2 + 2C_0\|u_n\|_1^2 \leq C_0\|u_n\|_1^2 + \overline{C},$$

Integrating over the interval I_t, we obtain

$$\|x^{-1}u_n(t)\|_0^2 + C_0 \int_0^t \|u_n(\tau)\|_1^2 \, d\tau \le \overline{C}T, \qquad \forall t \in I, \quad n \ge 1, \tag{3.3}$$

Then there exists a subsequence $\{u_m\} \subset \{u_n\}$ and u^* such that

$$x^{-1}u^*(t) \in L^2(\Omega), \quad u^* \in L^2\big(I; H_0^1(\Omega)\big),$$

and

$$\begin{aligned}
&x^{-1}u_m(t) \rightharpoonup x^{-1}u^*(t) \quad (weakly), && \text{in } L^2(\Omega), \ \forall t \in I, && (3.4)\\
&x^{-1}u_m \rightharpoonup x^{-1}u^* \quad (weak\ star), && \text{in } L^\infty\big(I; L^2(\Omega)\big), && (3.5)\\
&u_m \rightharpoonup u^* \quad (weakly), && \text{in } L^2\big(I; H_0^1(\Omega)\big). && (3.6)
\end{aligned}$$

Let us verify that $u^* = u(\sigma)$. To this end we choose an arbitrary $v \in L^2(I_t; K)$. We have

$$(x^{-2}\partial_t u_m, u_m - v) + a(\sigma_m; u_m, u_m - v) \le L(\sigma_m; u_m - v),$$

for almost all $t \in I$. From this inequality, we obtain

$$\begin{aligned}
&\frac{d}{2dt}\|x^{-1}u_m\|_0^2 + a(\sigma_m; u_m, u_m)\\
\le&(x^{-1}\partial_t u_m, v) + a(\sigma_m; u_m, v) + L(\sigma_m; u_m - v),
\end{aligned} \tag{3.7}$$

so that

$$\begin{aligned}
&\frac{1}{2}\|x^{-1}u_m(t)\|_0^2 + \int_0^t a\big(\sigma_m; u_m(\tau), u_m(\tau)\big)\, d\tau\\
\le &\int_0^t \Big[\big(x^{-1}\partial_t u_m(\tau), v(\tau)\big) + a\big(\sigma_m; u_m(\tau), v(\tau)\big)\\
&\qquad + L\big(\sigma_m; u_m(\tau) - v(\tau)\big)\Big] d\tau,
\end{aligned} \tag{3.8}$$

holds for all $t \in I$.

Let us pass to the lim inf with $m \to \infty$. By virtue of (3.4), we arrive at

$$\liminf \|x^{-1}u_m(t)\|_0^2 \ge \|x^{-1}u^*(t)\|_0^2. \tag{3.9}$$

The functional

$$\psi(w) = \int_0^t a\big(\sigma; w(\tau), w(\tau)\big)\, d\tau,$$

is convex and continuous in $L^2\big(I; H_0^1(\Omega)\big)$ due to Lemma 3.1. Hence, it is weakly lower semicontinuous, so that (3.6) yields

$$\liminf \int_0^t a\big(\sigma; u_m(\tau), u_m(\tau)\big)\, d\tau \ge \int_0^t a\big(\sigma; u^*(\tau), u^*(\tau)\big)\, d\tau. \tag{3.10}$$

Moreover, we observe that

$$
\begin{aligned}
&\Big| \int_0^t a(\sigma_m; u_m, u_m)\, d\tau - \int_0^t a(\sigma; u_m, u_m)\, d\tau \Big| \\
\le & \frac{1}{2} \int_0^t \left|\sigma_m^2 - \sigma^2\right| \|u_m(\tau)\|_1^2 \, d\tau \to 0,
\end{aligned} \tag{3.11}
$$

follows from (3.3).

By using Proposition 3.1, we infer that a subsequence of $\{u_m\}$ exists (and we shall denote it by the same symbol), such that

$$
x^{-1}\partial_t u_m \rightharpoonup x^{-1}\partial_t u^* \ (weakly), \quad \text{in } \ L^2\big(I; L^2(\Omega)\big), \quad \text{as } \ m \to \infty.
$$

Then

$$
\int_0^t (x^{-1}\partial_t u_m, x^{-1}v)\, d\tau \to \int_0^t (x^{-1}\partial_t u^*, x^{-1}v)\, d\tau. \tag{3.12}
$$

Since

$$
x^{-1}v(\tau) \in L^2(\Omega), \qquad \text{if } \ v(\tau) \in H_0^1(\Omega),
$$

so that the integral represents a linear continuous functional in $L^2\big(I, L^2(\Omega)\big)$. By virtue of (3.6)

$$
\int_0^t a(\sigma; u_m, v)\, d\tau \to \int_0^t a(\sigma; u^*, v)\, d\tau,
$$

and using an analogue of (3.11), we arrive at

$$
\int_0^t a(\sigma; u_m, v)\, d\tau \to \int_0^t a(\sigma; u^*, v)\, d\tau. \tag{3.13}
$$

Next, we have

$$
\begin{aligned}
&\Big| \int_0^t L(\sigma_m; u_m - v)\, d\tau - \int_0^t L(\sigma; u_m - v)\, d\tau \Big| \\
\le & C\big|\sigma_m^2 - \sigma^2\big| \int_0^t \|u_m - v\|_1 \, d\tau \to 0,
\end{aligned}
$$

and

$$
\int_0^t L(\sigma; u_m - v)\, d\tau \to \int_0^t L(\sigma; u^* - v)\, d\tau,
$$

by virtue of (3.6). As a result,

$$
\int_0^t L(\sigma_m; u_m - v)\, d\tau \to \int_0^t L(\sigma; u^* - v)\, d\tau. \tag{3.14}
$$

From (3.7)–(3.14), we obtain

$$\begin{aligned}&\frac{1}{2}\|x^{-1}u^*(t)\|_0^2+\int_0^t a(\sigma;u^*,u^*)\,d\tau\\ \le&\int_0^t\Big[(x^{-1}\partial_t u^*,x^{-1}v)+a(\sigma;u^*,v)+L(\sigma;u^*-v)\Big]\,d\tau.\end{aligned}$$

This inequality can be rewritten as

$$\int_0^t\Big[(x^{-1}\partial_t u^*,x^{-1}v-x^{-1}u^*)+a(\sigma;u^*,v-u^*)-L(\sigma;v-u^*)\Big]\,d\tau\ge 0, \tag{3.15}$$

By Lemma 3.2 and (3.6), we infer that $u^*\in L^2(I_t,K)$.

Let us set

$$v(\tau)=u^*(\tau)+w(x)\chi_\epsilon(\tau),$$

where $w\in K$ is arbitrary and χ_ϵ is the characteristic function of the interval

$$\big[t_0-\epsilon,\ t_0+\epsilon\big],\qquad t_0<T.$$

Then $v\in L^2(I_t,K)$. Since the Lebesgue Theorem implies that

$$(2\epsilon)^{-1}\int_{t_0-\epsilon}^{t_0+\epsilon} g(\tau)\,d\tau\to g(t_0),\quad \text{as}\quad \epsilon\to 0,$$

holds for any measurable function g and almost all $t_0\in I$, from (3.15) we infer that

$$\Big[(x^{-1}\partial_t u^*,x^{-1}w)+a(\sigma;u^*,w)-L(\sigma;w)\Big]_{t=t_0}\ge 0. \tag{3.16}$$

Next, let us take $v=0$ and $v=2u^*$ in (3.15). Then

$$\int_0^t\Big[(x^{-1}\partial_t u^*,x^{-1}u^*)+a(\sigma;u^*,u^*)-L(\sigma;u^*)\Big]\,d\tau=0,\qquad \forall t\in I, \tag{3.17}$$

and

$$\Big[(x^{-1}\partial_t u^*,x^{-1}u^*)+a(\sigma;u^*,u^*)-L(\sigma;u^*)\Big]_{t=t_0}, \tag{3.18}$$

follows by differentiation. From (3.16) and (3.18), we obtain

$$(x^{-2}\partial_t u^*,w-u^*)+a(\sigma;u^*,w-u^*)-L(\sigma;w-u^*)\ge 0,$$

for all $w\in K$ and almost all $t\in I$, so that $u^*=u(\sigma)$.

Since the weak solution is unique (see [2]-I), the whole original sequence $\{u_n\}$ tends to $u(\sigma)$ weakly in $L^2\big(I,H_0^1(\Omega)\big)$, $x^{-1}u_n(t)$ tends weakly to $x^{-1}u(\sigma)(t)$ in $L^2(\Omega)$ for all $t\in I$ and $x^{-1}u_n$ weakly star to $x^{-1}u(\sigma)$ in $L^\infty\big(I,L^2(\Omega)\big)$.

To prove the **strong** convergence, we introduce the following bilinear form in $L^2\big(I_t,H_0^1(\Omega)\big)$

$$\langle u,v\rangle=\frac{1}{2}\big(x^{-1}u(t),x^{-1}v(t)\big)+\int_0^t a(\sigma;u,v)\,d\tau,$$

and let $\|u\|_a^2=\langle u,u\rangle$.

Lemma 3.3. *Let us denote $u = u(\sigma)$, $u_n = u(\sigma_n)$. Then*

$$\|u_n\|_a \to \|u\|_a, \quad \text{as } n \to \infty.$$

Proof. Since $\{u_n\}$ is bounded in $L^2(I, H_0^1(\Omega))$ by virtue of (3.3),

$$\int_0^t \big(a(\sigma_n; u_n, u_n) - a(\sigma; u_n, u_n)\big)\, d\tau \le \frac{1}{2}|\sigma_n^2 - \sigma^2| \int_0^t \|u_n\|_1^2 \, d\tau \to 0, \tag{3.19}$$

and

$$\Big| \int_0^t L(\sigma_n; u_n)\, d\tau - \int_0^t L(\sigma; u_n)\, d\tau \Big| \le C|\sigma_n^2 - \sigma^2| \int_0^t \|u_n\|_1 \, d\tau \to 0. \tag{3.20}$$

In the variational inequality for u_n, we insert $v = 0$ and $v = 2u_n$ to obtain

$$\begin{aligned}\mathcal{B}_n \equiv & \frac{\|x^{-1}u_n(t)\|^2}{2} + \int_0^t a(\sigma_n; u_n, u_n)\, d\tau \\ = & \int_0^t L(\sigma_n; u_n)\, d\tau.\end{aligned} \tag{3.21}$$

Then from (3.19)-(3.21) and the weak convergence of $\{u_n\}$ it follows that

$$\begin{aligned}\|u_n\|_a^2 &= \mathcal{B}_n + \int_0^t \big(a(\sigma; u_n, u_n) - a(\sigma_n; u_n, u_n)\big)\, d\tau \\ &= \int_0^t L(\sigma; u_n)\, d\tau + \int_0^t \big(L(\sigma_n; u_n) - L(\sigma; u_n)\big)\, d\tau \\ &\to \int_0^t L(\sigma; u)\, d\tau.\end{aligned}$$

On the other hand, (3.17) yields that

$$\|u\|_a^2 = \int_0^t L(\sigma; u)\, d\tau,$$

so that

$$\|u_n\|_a^2 \to \|u\|_a^2, \quad \text{as} \quad n \to \infty.$$

Then the lemma is proved. □

Let us observe that

$$\langle u_n, v\rangle = \frac{1}{2}\big(x^{-1}u_n(t), x^{-1}v\big) + \int_0^t a(\sigma; u_n, v)\, d\tau \to \langle u, v\rangle,$$

and $\langle v, u_n\rangle \to \langle v, u\rangle$ holds for any $v \in L^2(I_t, H_0^1(\Omega))$ by virtue of the weak convergences proved above. Then

$$\begin{aligned}\|u_n - u\|_a^2 &= \langle u_n - u, u_n - u\rangle \\ &= \|u_n\|_a^2 + \|u\|_a^2 - \langle u, u_n\rangle - \langle u_n, u\rangle \to 0,\end{aligned} \tag{3.22}$$

follows from Lemma (3.3).

It is readily seen that

$$C_0 \int_0^t \|v\|_1^2 \, d\tau \le \|v\|_a^2 \qquad \text{and} \qquad \frac{1}{2}\big\|x^{-1}v(t)\big\|_0^2 \le \|v\|_a^2,$$

hold for any $v \in L^2\big(I; H_0^1(\Omega)\big)$. Then the strong convergence follows from (3.22). $\square$

4 Maximum range problem

Assume that a criterion-functional

$$\Phi(\sigma; v) : \mathcal{U}_{ad} \times \Big(L^2_{x^{-1}}(\Omega) \times L^2\big(I, H_0^1(\Omega)\big)\Big) \mapsto \mathbf{R},$$

is given, such that if

$$\begin{aligned} &\sigma_n \in \mathcal{U}_{ad}, \quad \sigma_n \to \sigma, \quad \text{as} \quad n \to \infty, && \\ &(i)\ v_n \to v, && \text{in } L^2\big(I, H_0^1(\Omega)\big), \\ &(ii)\ x^{-1}v_n(t) \to x^{-1}v(t), && \text{in } L^2(\Omega),\ \forall t \in I, \\ &(iii)\ x^{-1}v_n \rightharpoonup x^{-1}v\ \ (\text{weak star}), && \text{in } L^\infty\big(I, L^2(\Omega)\big), \end{aligned} \tag{4.1}$$

then

$$\Phi(\sigma_n; v_n) \to \Phi(\sigma; v), \quad \text{as } n \to \infty. \tag{4.2}$$

Theorem 4.1. *Let the criterion-functional* Φ *satisfy conditions* (4.1)-(4.2). *Then there exists at least one solution of the maximization problem*

$$\overline{\sigma} = \arg\max_{\sigma \in \mathcal{U}_{ad}} \Phi\big(\sigma; u(\sigma)\big), \tag{4.3}$$

and at least one solution of the minimization problem

$$\underline{\sigma} = \arg\min_{\sigma \in \mathcal{U}_{ad}} \Phi\big(\sigma; u(\sigma)\big). \tag{4.4}$$

Proof. 1^0. Let $\{\sigma_n\}$ be a maximizing sequence of the functional $J(\sigma) \equiv \Phi\big(\sigma; u(\sigma)\big)$, i.e.,

$$J(\sigma_n) \to \sup_{\sigma \in \mathcal{U}_{ad}} J(\sigma), \quad \text{as} \quad n \to \infty. \tag{4.5}$$

Since $\mathcal{U}_{ad}$ is compact, there exists a subsequence $\{\sigma_m\}$ and $\sigma^* \in \mathcal{U}_{ad}$ such that

$$\sigma_m \to \sigma^*, \quad \text{as} \quad m \to \infty.$$

By Theorem 3.1, the sequence $\{u(\sigma_m)\}$ and $u(\sigma^*)$ satisfy conditions (i)-(iii) of (4.1). By using assumption (4.1)-(4.2), we infer that

$$\begin{aligned} J(\sigma_m) &= \Phi\big(\sigma_m; u(\sigma_m)\big) \to \Phi\big(\sigma^*; u(\sigma^*)\big) \\ &= J(\sigma^*), \quad \text{as} \quad m \to \infty. \end{aligned}$$

From (4.5), we obtain

$$J(\sigma^*) = \sup_{\sigma \in \mathcal{U}_{ad}} J(\sigma),$$

so that σ^* solves problem (4.3), i.e., $\sigma^* = \overline{\sigma}$.

2^0. Let $\{\sigma_n\}$ be a minimizing sequence of $J(\sigma)$, i.e.,

$$J(\sigma_n) \to \inf_{\sigma \in \mathcal{U}_{ad}} J(\sigma), \quad \text{as} \quad n \to \infty. \tag{4.6}$$

There exist $\{\sigma_m\} \subset \{\sigma_n\}$ and σ_* such that

$$\sigma_m \to \sigma_*, \quad \text{as} \quad m \to \infty.$$

Using Theorem 3.1, we obtain $J(\sigma_m) \to J(\sigma_*)$ and by comparing the limit with (4.6), we arrive at

$$J(\sigma_*) = \inf_{\sigma \in \mathcal{U}_{ad}} J(\sigma),$$

so that σ_* solves problem (4.4), i.e., $\sigma_* = \underline{\sigma}$. □

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