Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. 2, No. 3, pp. 389-398

DOI: 10.4208/aamm.09-m0982
June 2010

On Lateral-Torsional Buckling of Non-Local Beams

N. Challamel[1,*] and C. M. Wang[2]

[1] *Université Européenne de Bretagne, INSA de Rennes-LGCGM, 20, avenue des Buttes de Coësmes, 35043 Rennes cedex, France*

[2] *Department of Civil Engineering, National University of Singapore, Kent Ridge Crescent, Singapore 119260*

Received 25 August 2009; Accepted (in revised version) 18 March 2010
Available online 25 March 2010

Abstract. Nonlocal continuum mechanics allows one to account for the small length scale effect that becomes significant when dealing with micro- or nano-structures. This paper deals with the lateral-torsional buckling of elastic nonlocal small-scale beams. Eringen's model is chosen for the nonlocal constitutive bending-curvature relationship. The effect of prebuckling deformation is taken into consideration on the basis of the Kirchhoff-Clebsch theory. It is shown that the application of Eringen's model produces small-length scale terms in the nonlocal elastic lateral-torsional buckling moment of a hinged-hinged strip beam. Clearly, the non-local parameter has the effect of reducing the critical lateral-torsional buckling moment. This tendency is consistent with the one observed for the in-plane stability analysis, for the lateral buckling of a hinged-hinged axially loaded column. The lateral buckling solution can be derived from a physically motivated variational principle.

AMS subject classifications: 34D05, 34D20, 74B05, 74B20, 74K15, 74M25.

Key words: Lateral-torsional buckling, Kirchhoff-Clebsch theory, Eringen's model, nonlocal theory, nanostructures.

1 Introduction

Nonlocal elasticity models abandon the classical assumption of locality, and admit that stress depends not only on the strain at that point but on the strains of every point on the body. Local elasticity is inherently size-independent. In contrast, nonlocal continuum mechanics allows one to account for the small length scale effect that can become significant when dealing with small scale structures (typically nanostructures). In fact, small length scale phenomenon is linked to the atomistic structure of the lattice material. Therefore, in recent years, the nonlocal continuum mechanics has attracted the attention of many researchers who worked on the analysis of micro/nano structures.

*Corresponding author.
Email: noel.challamel@insa-rennes.fr (N. Challamel), cvewcm@nus.edu.sg (C. M. Wang)

More specifically, there has been a considerable interest in the extension of local beam theory to nonlocal beams. These articles presented simplified non-local elastic models for the bending, buckling or vibration analyses of small-scale beams/rods, e.g., [1–11]. A systematic rational procedure of non-local elasticity framework is established in Eringen's papers [12, 13]. Non-local field theory of mechanics has been applied to some various engineering problems, such as dispersion of phonon, Rayleigh wave, and stress concentration at the crack tip. The nonlocal model of Eringen [14] is chosen in this paper. This model can be presented in the one-dimensional version as

$$\sigma - l^2\sigma'' = E\varepsilon, \tag{1.1}$$

where σ is the uniaxial stress, ε is the uniaxial strain, E is the Young modulus, l is an additional length scale, specific of the nonlocal constitutive law and the double prime denotes the second order derivative of the quantity with respect to the axial coordinate of the beam. The value of l can be identified from atomistic simulations, or using the dispersive curve of the Born-Kármán model of lattice dynamics (see for instance [15, 16])

$$l \cong 0.386a, \tag{1.2}$$

where a is the distance between atoms. Zhang et al. [17] compared the critical buckling load obtained from of a nonlocal continuum mechanics approach (based on Eringen's nonlocal theory) with some Molecular Dynamics simulation. They discussed the best identification of the length scale of Eringen's nonlocal model.

The differential equation (1.1) clearly shows that the stress can be expressed as an integral of the strain variable where the weighting function is the Green's function of the differential system associated to relevant boundary conditions. The in-plane stability of nonlocal beams was first studied by Sudak [2] who used Eringen's constitutive law [14] for Euler-Bernoulli beam models. The buckling solutions have since been extended to nonlocal Timoshenko beam models (see for instance [6] or [7]). Wang et al. [18] recently investigated the postbuckling problem of cantilevered nano rods/tubes under an end concentrated load. Eringen's nonlocal beam theory is used to account for the small length scale effect.

These studies were mainly focused on straight beams, but the dynamics behaviour of nonlocal arches has been also theoretically investigated [19]. The results obtained for one-dimensional nonlocal media (nonlocal beam mechanics) have been recently extended to two-dimensional media such as nanoplates (see for instance [20] or [21]). However, the out-of-plane stability behaviour of nonlocal beams has never been investigated up to now, to the authors' knowledge. This paper aims to contribute to the understanding of lateral-torsional buckling of nonlocal beams. The behaviour of micro- or nano-cantilevers is potentially concerned by this theoretical study, since they are used as sensors in atomic force microscopy [22]. In Atomic Force Microscopy, a tip located close to the free end of the cantilever beam is utilized as a probe and can interact with a surface. Even if the specific cantilever case will not be treated in this study, the derivation of some theoretical solutions in the simple uniform bending case will be of engineering interest to quantify the small-length effect on the lateral-torsional buckling load. The problem is treated on the basis of Kirchhoff-Clebsch equations of equilibrium for finitely deforming beams. The boundary conditions of the problem allow a complete solution of an intrinsic form of the theory, i.e., a formulation of the theory without any regard to the form of strain displacement relations.

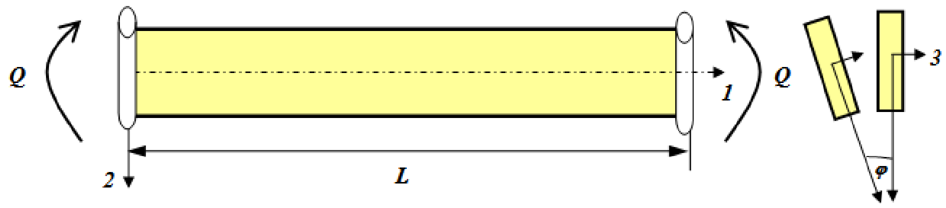

Figure 1: Geometry of the hinged-hinged beam strip.

2 Lateral-torsional buckling problem

We shall derive the linearized lateral-torsional buckling equations of a small scale beam with a rectangular cross-section. The hinged-hinged beam strip of length L is subjected to end moments, Q as shown in Fig. 1. Fig. 1 also shows the cross-section before and after the lateral-torsional deformation. The lateral-torsional buckling behaviour of prismatic homogeneous beams has been extensively investigated since the pioneering studies of Michell (1899) and Prandtl (1899) [23, 24], more than a century ago. Grober (1914) rigorously introduced the in-plane prebuckling deformation effect and obtained the lateral-torsional buckling moment for the hinged-hinged beam strip [25]. In the present paper, the prebuckling effect will be also introduced, extending the rigorous lateral-torsional buckling moment of Grober [25] initially obtained for "local" beams. The Kirchhoff-Clebsch equations are written as [26]

$$\begin{cases} M_1' - \hat{\kappa}_3 M_2 + \kappa_2 \hat{M}_3 = 0, \\ M_2' + \hat{\kappa}_3 M_1 - \kappa_1 \hat{M}_3 = 0, \end{cases} \tag{2.1}$$

where M_i are the bending moments and κ_i the associated curvatures. According to Fig. 1, κ_1 denotes the torsional curvature, κ_2 the out-of-plane curvature and $\hat{\kappa}_3$ the in-plane curvature. The reader is referred to the works of Hodges and Peters [27], Reissner [28], Atanackovic [29] or more recently Simitses and Hodges [30] for the rigorous application of the Kirchhoff-Clebsch theory to lateral-torsional buckling (with a "local" constitutive law). The main advantage of intrinsic equations is that the resulting equations are independent of the way the beam displacement and rotation measures are chosen. The stability analysis is investigated with respect to the in-plane solution

$$\hat{M}_3 - l_3^2 \hat{M}_3'' = EI_3 \hat{\kappa}_3, \tag{2.2}$$

where EI_3 is the in-plane stiffness. l_3 is the additional length scale parameter specific of the nonlocal constitutive law. This length term may be calibrated as in Eq. (1.2) for instance. One recognizes in Eq. (2.2) the Eringen's type differential equation formulated in terms of bending-curvature law. The generalization of Eringen's nonlocal constitutive law to the out-of-plane stability problem of this nonlocal beam may be written as

$$M_1 - l_1^2 M_1'' = GJ\kappa_1, \quad M_2 - l_2^2 M_2'' = EI_2\kappa_2, \quad \text{and} \quad \hat{M}_3 - l_3^2 \hat{M}_3'' = EI_3\hat{\kappa}_3, \tag{2.3}$$

where three characteristic length l_1, l_2 and l_3 have been introduced. For anisotropic beams, the length terms do not necessarily coincide. For isotropic beams, it is reasonable to consider only one single length scale. Independently of the anisotropic nature of the nonlocal elastic beam, the small length term l_3 does not influence the in-plane bending moment, as shown by Peddieson et al. [1] (see more recently [31] for the cantilever case).

$$\hat{M}_3 = Q \Rightarrow \hat{\kappa}_3 = \frac{Q}{EI_3}. \tag{2.4}$$

Therefore, in the general case, only two length scales (l_1, l_2) appear for this lateral-torsional buckling problem. Eq. (2.1) is then written as a system of two linear differential equations

$$\begin{cases} M_1' - \dfrac{Q}{EI_3} M_2 + Q\Big(\dfrac{M_2 - l_2^2 M_2''}{EI_2}\Big) = 0, \\ M_2' + \dfrac{Q}{EI_3} M_1 - Q\Big(\dfrac{M_1 - l_1^2 M_1''}{GJ}\Big) = 0. \end{cases} \tag{2.5}$$

This system of differential equations can also be expressed as

$$\begin{cases} M_1' = \Big(\dfrac{Q}{EI_3} - \dfrac{Q}{EI_2}\Big) M_2 + \dfrac{Ql_2^2}{EI_2} M_2'', \\ M_2'' + \Big(\dfrac{Q}{EI_3} - \dfrac{Q}{GJ}\Big) M_1' + \dfrac{Ql_1^2}{GJ} M_1''' = 0. \end{cases} \tag{2.6}$$

The introduction of the first equation of Eq. (2.6) into the second one leads to the fourth-order linear differential equation of the out-of-plane bending moment M_2

$$\begin{aligned} &\frac{Q^2 l_1^2 l_2^2}{GJEI_2} M_2^{(4)} + \Big[1 + \frac{Ql_2^2}{EI_2}\Big(\frac{Q}{EI_3} - \frac{Q}{GJ}\Big) + \frac{Ql_1^2}{GJ}\Big(\frac{Q}{EI_3} - \frac{Q}{EI_2}\Big)\Big] M_2'' \\ &\qquad + \Big(\frac{Q}{EI_3} - \frac{Q}{GJ}\Big)\Big(\frac{Q}{EI_3} - \frac{Q}{EI_2}\Big) M_2 = 0. \end{aligned} \tag{2.7}$$

The four boundary conditions associated with the hinged-hinged beam are given by

$$M_2(0) = 0, \qquad M_2(L) = 0, \tag{2.8a}$$

$$\theta_1(0) = 0, \qquad \theta_1(L) = 0, \tag{2.8b}$$

where θ_1 is the torsional angle, related to the torsional curvature by $\kappa_1 = \theta_1'$.

3 Solution for critical moment

The solution is sought in the form of

$$M_2(x) = M_2^0 \sin\big(n\pi x/L\big), \tag{3.1a}$$

$$\theta_1(x) = \theta_1^0 \sin\big(n\pi x/L\big). \tag{3.1b}$$

where n is an integer. The boundary conditions are implicitly verified with this solution. Note that the boundary conditions can be also expressed in this case as

$$M_2(0) = 0, \quad M_2(L) = 0, \quad M_2''(0) = 0, \quad M_2''(L) = 0. \tag{3.2}$$

By inserting the out-of-plane bending moment expression of Eq. (3.1) into the fourth-order differential equation (2.7), one obtains the lateral-torsional buckling moment equation

$$\begin{aligned} &\frac{Q^2 l_1^2 l_2^2}{EI_2 GJ}\Big(\frac{n\pi}{L}\Big)^4 - \Big(\frac{n\pi}{L}\Big)^2\Big[1 + \frac{Q l_2^2}{EI_2}\Big(\frac{Q}{EI_3} - \frac{Q}{GJ}\Big) + \frac{Q l_1^2}{GJ}\Big(\frac{Q}{EI_3} - \frac{Q}{EI_2}\Big)\Big] \\ &\qquad + \Big(\frac{Q}{EI_3} - \frac{Q}{GJ}\Big)\Big(\frac{Q}{EI_3} - \frac{Q}{EI_2}\Big) = 0. \end{aligned} \tag{3.3}$$

As a particular case, the local problem is first treated $(l_1 = l_2 = 0)$. In this local case, the differential equation (2.7) reduces to

$$M_2'' + \Big(\frac{Q}{EI_3} - \frac{Q}{GJ}\Big)\Big(\frac{Q}{EI_3} - \frac{Q}{EI_2}\Big)M_2 = 0, \tag{3.4}$$

leading to the lateral-torsional buckling moment of the local case, denoted by $\tilde{Q}$

$$\tilde{Q} = \frac{n\pi}{L}\left(\Big(\frac{1}{EI_3} - \frac{1}{GJ}\Big)\Big(\frac{1}{EI_3} - \frac{1}{EI_2}\Big)\right)^{-\frac{1}{2}}. \tag{3.5}$$

The fundamental buckling moment is obtained with n equal to unity. One recognizes the value given by Grober (1914) [25] (see more recently [29] or [30]). If prebuckling deformation effects are neglected, the lateral-torsional buckling moment is simplified to Prandtl's value [24]

$$\frac{GJ}{EI_3} \ll 1, \ \text{and} \ \frac{EI_2}{EI_3} \ll 1 \Rightarrow \tilde{Q} = \frac{n\pi}{L}\sqrt{EI_2 GJ}. \tag{3.6}$$

Going back to the nonlocal case covered by Eq. (3.3), the nonlocal lateral-torsional buckling moment is calculated as

$$\begin{aligned} Q =& \frac{n\pi}{L}\Big\{\frac{l_1^2 l_2^2}{EI_2 GJ}\Big(\frac{n\pi}{L}\Big)^4 - \Big(\frac{n\pi}{L}\Big)^2\Big[\frac{l_2^2}{EI_2}\Big(\frac{1}{EI_3} - \frac{1}{GJ}\Big) + \frac{l_1^2}{GJ}\Big(\frac{1}{EI_3} - \frac{1}{EI_2}\Big)\Big] \\ &+ \Big(\frac{1}{EI_3} - \frac{1}{GJ}\Big)\Big(\frac{1}{EI_3} - \frac{1}{EI_2}\Big)\Big\}^{-\frac{1}{2}}. \end{aligned} \tag{3.7}$$

If prebuckling deformation effects are neglected, the nonlocal lateral-torsional buckling moment is simplified to

$$\frac{GJ}{EI_3} \ll 1, \ \frac{EI_2}{EI_3} \ll 1 \Rightarrow Q = \frac{n\pi}{L}(EI_2GJ)^{\frac{1}{2}}\Big\{\Big[1+\Big(\frac{l_1}{L}\Big)^2(n\pi)^2\Big]\Big[1+\Big(\frac{l_2}{L}\Big)^2(n\pi)^2\Big]\Big\}^{-\frac{1}{2}}. \tag{3.8}$$

Therefore, when prebuckling deformation effects are neglected, the ratio between the non-local buckling moment and the local one is calculated as

$$\frac{Q}{\tilde{Q}} = \Big\{\Big[1+\Big(\frac{l_1}{L}\Big)^2(n\pi)^2\Big]\Big[1+\Big(\frac{l_2}{L}\Big)^2(n\pi)^2\Big]\Big\}^{-\frac{1}{2}}. \tag{3.9}$$

An asymptotic power can give a good approximate of the lateral-torsional buckling moment

$$\frac{l_1}{L} \ll 1, \ \text{and} \ \frac{l_2}{L} \ll 1 \Rightarrow \frac{Q}{\tilde{Q}} \approx 1 - \frac{1}{2}\pi^2\Big[\Big(\frac{l_1}{L}\Big)^2 + \Big(\frac{l_2}{L}\Big)^2\Big], \tag{3.10}$$

with $n = 1$. Eqs. (3.9) and (3.10) clearly show that the non-local parameters (l_1, l_2) have the effect of reducing the critical lateral-torsional buckling moment

$$\frac{Q}{\tilde{Q}} \leqslant 1. \tag{3.11}$$

Note that the particular case of two identical characteristic lengths gives the simplified formulae (isotropic case)

$$l_1 = l_2 = l \Rightarrow \frac{Q}{\tilde{Q}} = \Big[1+\Big(\frac{l}{L}\Big)^2(n\pi)^2\Big]^{-1}, \tag{3.12}$$

which can be reasonably approximated by

$$\frac{l}{L} \ll 1 \Rightarrow \frac{Q}{\tilde{Q}} \approx 1 - \pi^2\Big(\frac{l}{L}\Big)^2, \tag{3.13}$$

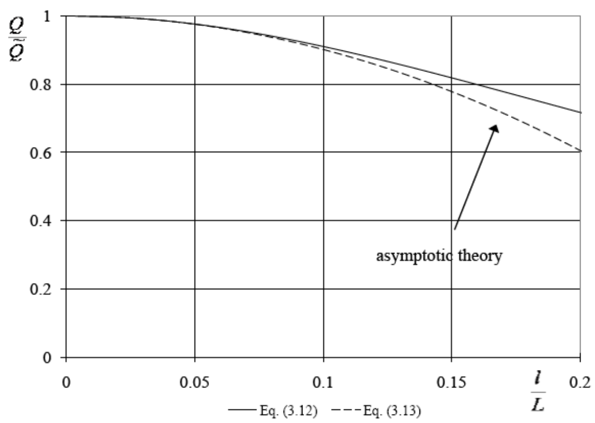

Figure 2: Dimensionless buckling moment $Q/\tilde{Q}$ versus the scale parameter l/L.

with $n = 1$.

Fig. 2 shows the influence of the small scale terms on the normalized lateral-torsional buckling moment. The small scale terms tend to decrease the lateral-torsional buckling moment of the present nonlocal model. For example, when the ratio l/L is equal to 0.1, the difference caused by the nonlocal effect is about 10%. This corresponds to L approximately of $4a$. When the ratio l/L is larger, the length L becomes smaller (approximately $2a$ when l/L is equal to 0.2). For this range of parameters, the limit of the continuum model is certainly reached, and the generic beam modelling can be discussed for the arrangements.

Finally, as we will show, Eq. (3.12) is very similar to the buckling equation obtained for the in-plane analysis.

4 Analogy with in-plane stability problem

Let us consider for instance the buckling of the Euler-Bernoulli hinged-hinged column (Fig. 3), whose constitutive law is given by

$$M_3 - l^2 M_3'' = EI_3 \kappa_3, \quad \text{with } \kappa_3 = w'', \tag{4.1}$$

where w is the deflection. From equilibrium consideration, we have

$$M_3'' = -Pw'', \tag{4.2}$$

and thus the governing equation (4.1) can be written as

$$(EI_3 - Pl^2) w^{(4)} + Pw'' = 0. \tag{4.3}$$

The buckling deflection can be written as

$$w(x) = w^0 \sin\big(n\pi x/L\big). \tag{4.4}$$

After introducing the buckling mode Eq. (4.4) in the differential equation (4.3), the buckling load is finally obtained for the nonlocal Euler-Bernoulli column

$$\frac{P}{\tilde{P}} = \Big[1 + \Big(\frac{l}{L}\Big)^2 (n\pi)^2\Big]^{-1}, \quad \text{with } \tilde{P} = EI_3 \Big(\frac{n\pi}{L}\Big)^2. \tag{4.5}$$

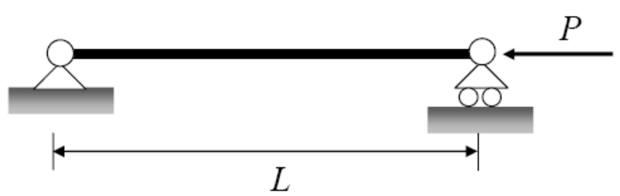

Figure 3: Buckling of a nonlocal Euler-Bernoulli column.

Eq. (4.5) is very similar to Eq. (3.12). Note that the differential equation of Eq. (4.3) can be obtained from a variational principle, using the energy functional given in [31] for the internal elastic energy

$$\delta(U+V)=0, \tag{4.6}$$

with

$$U=\frac{1}{2}\int_0^L EI_3\kappa_3\overline{\kappa_3}dx, \quad V=-\frac{1}{2}P\int_0^L (w')^2dx, \quad \overline{\kappa_3}-l^2\overline{\kappa_3}=\kappa_3,$$

where $\overline{\kappa_3}$ is the nonlocal curvature (see [31] or [32]). In this case, the principle of virtual work leads to

$$\delta U=\frac{1}{2}\int_0^L EI_3\overline{\kappa_3}\delta\kappa_3dx+\frac{1}{2}\int_0^L EI_3\kappa_3\delta\overline{\kappa_3}dx=\int_0^L EI_3\overline{\kappa_3}\delta\kappa_3dx, \tag{4.7}$$

following the Green-type identity associated to the self-adjoint property of the regularized operator for relevant boundary conditions. Application of the stationarity condition of the energy functional Eq. (4.6) leads to the differential equation (4.2)

$$M_3''+Pw''=0, \quad \text{with} \quad M_3=EI_3\overline{\kappa_3}, \quad M_3-l^2M_3''=EI_3w''. \tag{4.8}$$

Note that the elastic potential can also be written in this case as

$$U=\frac{1}{2}\int_0^L EI_3\kappa_3\overline{\kappa_3}dx=\frac{1}{2}\int_0^L EI_3\kappa_3^2dx-\frac{Pl^2}{2}\int_0^L\kappa_3^2dx, \tag{4.9}$$

as suggested for instance in [33]. The proof is detailed below

$$\begin{aligned} U&=\frac{1}{2}\int_0^L EI_3\overline{\kappa_3}\kappa_3dx=\frac{1}{2}\int_0^L M\kappa_3dx=\frac{1}{2}\int_0^L (EI\kappa_3+l^2M_3'')\kappa_3dx \\ &=\frac{1}{2}\int_0^L EI_3\kappa_3^2dx-\frac{P}{2}l^2\int_0^L\kappa_3^2dx. \end{aligned} \tag{4.10}$$

Eq. (4.9) shows that the nonlocal elastic potential U can be expressed from the local elastic potential $\tilde{U}$ corrected by the external loading term

$$U=\tilde{U}\Big(1-\frac{Pl^2}{EI_3}\Big), \tag{4.11}$$

where

$$\tilde{U}=\frac{1}{2}\int_0^L EI_3\kappa_3^2dx.$$

Therefore, the lateral buckling problem can be rigorously treated from a physically motivated variational principle. The normalized lateral buckling load formulae in the case of a hinged-hinged column is very similar to the normalized lateral-torsional buckling moment formulae of a hinged-hinged strip beam.

5 Conclusions

This paper presents the governing equation for the lateral-torsional buckling of elastic nonlocal small-scale beams. In the formulation, Eringen's constitutive model has been adopted for the nonlocal constitutive bending-curvature relationship. Prebuckling deformation effects are taken into consideration via the Kirchhoff-Clebsch equations. It is shown that the application of Eringen's model produces small-length scale terms in the nonlocal elastic lateral-torsional buckling moment of a hinged-hinged beam strip. Clearly, the non-local parameter has the effect of reducing the critical lateral-torsional buckling moment. This tendency is consistent with the one observed for the in-plane stability analysis, for instance in case of lateral buckling of a hinged-hinged axially loaded column. The lateral buckling solution can be derived from a physically motivated variational principle.

References

[1] J. PEDDIESON, G. G. BUCHANAN AND R. P. MCNITT, *Application of nonlocal continuum models to nanotechnology*, Int. J. Eng. Sci., 41 (2003), pp. 305–312.

[2] L. J. SUDAK, *Column buckling of multiwalled carbon nanotubes using nonlocal continuum mechanics*, J. Appl. Phys., 94 (2003), pp. 7281–7287.

[3] Q. WANG, *Wave propagation in carbon nanotubes via nonlocal continuum mechanics*, J. Appl. Phys., 98 (2005), 124301.(doi:10.1063/1.2141648)

[4] P. LU, H. P. LEE, C. LU AND P. Q. ZHANG, *Dynamic properties of flexural beams using a nonlocal elasticity model*, J. Appl. Phys., 99 (2006), 073510.

[5] C. M. WANG, Y. Y. ZHANG AND X. Q. HE, *Vibration of nonlocal Timoshenko beams*, Nanotechnology., 17 (2006), pp. 1–9.

[6] C. M. WANG, Y. Y. ZHANG, S. S. RAMESH AND S. KITIPORNCHAI, *Buckling analysis of micro- and nano-rods/tubes based on nonlocal Timoshenko beam theory*, J. Phys. D: Appl. Phys., 39 (2006), pp. 3904–3909.

[7] J. N. REDDY, *Nonlocal theories for bending, buckling and vibration of beams*, Int. J. Eng. Sci., 45 (2007), pp. 288–307.

[8] C. W. LIM AND C. M. WANG, *Exact variational nonlocal stress modelling with asymptotic higher-order strain gradients for nanobeams*, J. Appl. Phys., 101 (2007), 054312.

[9] C. M. WANG, Y. Y. ZHANG AND S. KITIPORNCHAI, *Vibration of initially stressed micro- and nano-beams*, Int. J. Struct. Stab. Dyn., 7(4) (2007), pp. 555–570.

[10] Q. WANG AND C. M. WANG, *The constitutive relation and small scale parameter of nonlocal continuum mechanics for modelling carbon nanotubes*, Nanotechnology., 18 (2007), 075702.

[11] Q. WANG AND K. M. LIEW, *Application of nonlocal continuum mechanics to static analysis of micro- and nano-structures*, Phys. Lett. A., 363 (2007), pp. 236–242.

[12] A. C. ERINGEN, *Nonlocal polar elastic continua*, Int. J. Eng. Sc., 10 (1972), pp. 1–16.

[13] A. C. ERINGEN AND D. G. B. EDELEN, *On nonlocal elasticity*, Int. J. Eng. Sc., 10 (1972), pp. 233–248.

[14] A. C. ERINGEN, *On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves*, J. Appl. Phys., 54 (1983), pp. 4703–4710.

[15] A. C. ERINGEN, *Theory of nonlocal elasticity and some applications*, Res. Mech., 21 (1987), pp. 313–342.

[16] N. CHALLAMEL, L. RAKOTOMANANA AND L. LE MARREC, *A dispersive wave equation using non-local elasticity*, C. R. Mécanique., 337 (2009), pp. 591–595.

[17] Y. Y. ZHANG, C. M. WANG, W. H. DUAN, Y. XIANG AND Z. ZONG, *Assessment of continuum mechanics models in predicting buckling strains of single-walled carbon nanotubes*, Nanotechnology., 20 (2009), 395707.

[18] C. M. WANG, Y. XIANG AND S. KITIPORNCHAI, *Postbuckling of nano rods/tubes based on nonlocal beam theory*, Int. J. Appl. Mech., 1(2) (2009), pp. 259–266.

[19] C. M. WANG AND W. H. DUAN, *Free vibration of nanorings/arches based on nonlocal elasticity*, J. Appl. Phys., 104 (2008), 014303.

[20] P. LU, P. Q. ZHANG, P. H. LEE, C. M. WANG AND J. N. REDDY, *Non-local elastic plate theories*, P. Roy. Soc. A-Math. Phy., 463 (2007), pp. 3225–3240.

[21] S. C. PRADHAN AND J. K. PHADIKAR, *Nonlocal elasticity theory for vibration of nanoplates*, J. Sound. Vib., 325 (2009), pp. 206–223.

[22] G. BINNIG, C. F. QUATE AND C. GERBER, *Atomic force microscopic*, Phys. Rev. Lett., 56 (1986), pp. 930–933.

[23] A. G. M. MICHELL, *Elastic stability of long beams under transverse forces*, Philos. Mag., 48 (1899), pp. 198–309.

[24] L. PRANDTL, Kipperscheinungen. Dissertation der Universität München, 1899. In: Ludwig Prandtl Gesammelte Abhandlungen, Erster Teil, 10-74, Berlin, Springer-Verlag. (German), 1961.

[25] M. K. GROBER, *Ein beispiel der anwendung der kirchhoffschen stabgleichungen*, Physik. Zeitschr., 15 (1914), pp. 460–462 (German).

[26] A. E. H. LOVE, A Treatise on the Mathematical Theory of Elasticity, 4^{th} Ed. Dover Publications, New-York, 1944.

[27] D. H. HODGES AND D. A. PETERS, *On the lateral buckling of uniform slender cantilever beams*, Int. J. Solids. Struct., 11(12) (1975), pp. 1269–1280.

[28] E. REISSNER, *On lateral buckling of end loaded cantilever beams*, J. Appl. Math. Physics., 30 (1979), pp. 31–40.

[29] T. M. ATANACKOVIC, *Stability theory of elastic rods*, Ser. Sta. Vib. Co. Sy: A., 1997.

[30] G. J. SIMITSES AND D. H. HODGES, Fundamentals of Structural Stability, Elsevier, Burlington, 2006.

[31] N. CHALLAMEL AND C. M. WANG, *The small length scale effect for a non-local cantilever beam: a paradox solved*, Nanotechnology., 19 (2008), 345703.

[32] Y. Y. ZHANG, C. M. WANG AND N. CHALLAMEL, *Bending, buckling and vibration of hybrid nonlocal beams*, J. Eng. Mech., in Press, 2010.

[33] S. ADALI, *Variational principles for multi-walled carbon nanotubes undergoing buckling based on nonlocal elasticity theory*, Phys. Lett. A., 372 (2008), pp. 5701–5705.