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Modelling and Analysis of a Class of Metal–Forming Problems

T. A. Angelov*

[1] *Institute of Mechanics, Department of Solid Mechanics, Acad. G. Bonchev Street, Block 4, Sofia 1113, Bulgaria*

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Abstract. A class of steady-state metal-forming problems, with rigid-plastic, incompressible, strain-rate dependent material model and nonlocal Coulomb's friction, is considered. Primal, mixed and penalty variational formulations, containing variational inequalities with nonlinear and nondifferentiable terms, are derived and studied. Existence, uniqueness and convergence results are obtained and shortly presented. A priori finite element error estimates are derived and an algorithm, combining the finite element and secant-modulus methods, is utilized to solve an illustrative extrusion problem.

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Key words: Steady-state metal-forming, rigid-plastic material, nonlocal friction, variational formulations, weak solutions, secant-modulus method, FE analysis.

1 Introduction

The computational and experimental study of metal-forming processes has shown that the flow theory of plasticity [1–3] adequately approximates the material behaviour for most of them [4–7], as the frictional contact conditions also significantly influence the results. Due to similarity with the the contact problems in elasticity [8–15], corresponding metal-forming, or plastic flow contact problems, could be formulated and mathematically analysed. This direction of analysis has been followed for example in [16–20] and references therein, where steady-state wire-drawing, extrusion and rolling problems, with linear rigid-viscoplastic Bingham material model [3, 8–10], or nonlinear rigid-viscoplastic material models [4–7] and normal compliance, or nonlocal contact and Coulomb's friction models [11–15], have been formulated and studied. Variational inequality formulations have been derived and existence and uniqueness

*Corresponding author.
URL: `http://www.imbm.bas.bg/index.php?page=150`
Email: taa@imbm.bas.bg (T. A. Angelov)

results have been obtained. The solution of the resulting nonlinear variational problems, requires appropriate succesive linearization methods [9, 11, 12, 21], finite element methods and computational algorithms [11,21,22], to be applied. In [17–20] for example, the secant-modulus method, proposed by Kachanov [9] for solving nonlinear variational problems in the deformation theory of plasticity, has been extended to nonlinear variational inequalities, as in [20] a finite element–secant-modulus computational algorithm is proposed and used.

In this work, a class of metal-forming problems is considered, describing steady-state drawing and extrusion, with nonlocal Coulomb's friction through a rigid die, of an isotropic, rigid-plastic, strain-rate sensitive incompressible metallic strip (workpiece). Primal, mixed and penalty variational inequality formulations, with strongly nonlinear and nondifferentiable terms, are derived and studied. Under restrictions on the material characteristics, existence, uniqueness and convergence results are obtained and shortly presented. Finite element approximations are performed, a priori error estimates are derived and an algorithm, combining the finite element and the secant-modulus method, is utilized to solve an illustrative extrusion problem.

2 Statement of the problem

We suppose that a metallic workpiece occupies the domain $\Omega\subset \mathbb{R}^k$ (k=2, 3), with sufficiently regular boundary Γ, constituting of six open, disjoint subsets (Fig. 1). By Γ_1 and Γ_5 the vertical rear and front ends of the workpiece are denoted. A constant process velocity is prescribed on Γ_1 at extrusion, as Γ_5 is assumed free of tractions, or on Γ_5 at drawing, as then Γ_1 is assumed tractions free. The boundary $\Gamma_2\cup\Gamma_4$ is also assumed tractions free. The contact boundary is denoted by Γ_3. Due to the symmetry, only one half of the workpiece is considered, as by Γ_6 the boundary of symmetry is denoted. We shall further identify the points of $\bar{\Omega}=\Omega\cup\Gamma$ by their cartesian coordinates $\mathbf{x}=\{x_i\}$ and shall use the standard indicial notation and summation convention. Let us denote by

$$\mathbf{u}(\mathbf{x}) = \{u_i(\mathbf{x})\}, \quad \boldsymbol{\sigma}(\mathbf{x}) = \{\sigma_{ij}(\mathbf{x})\}, \quad \dot{\boldsymbol{\varepsilon}}(\mathbf{x}) = \{\dot{\varepsilon}_{ij}(\mathbf{x})\}, \quad (1 \le i, j \le k),$$

the velocity vector, stress and strain-rate tensors respectively and by

$$\bar{\sigma} = \sqrt{\frac{3}{2}s_{ij}s_{ij}}, \qquad \dot{\bar{\varepsilon}} = \sqrt{\frac{2}{3}\dot{e}_{ij}\dot{e}_{ij}}, \tag{2.1}$$

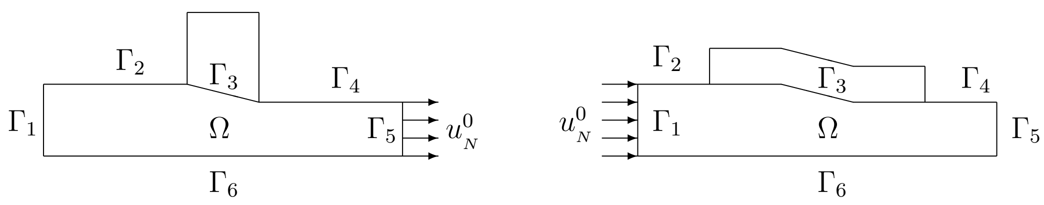


Figure 1: Strip drawing and extrusion problems.

the equivalent stress and strain-rate, where

$$s_{ij} = \sigma_{ij} - \sigma_{H}\delta_{ij}, \qquad \dot{e}_{ij} = \dot{\varepsilon}_{ij} - \frac{1}{3}\dot{\varepsilon}_{V}\delta_{ij},$$

are the components of the deviatoric stress and the strain-rate tensors and

$$\sigma_{H} = \frac{1}{3}\sigma_{ii}, \qquad \dot{\varepsilon}_{V} = \dot{\varepsilon}_{ii},$$

are the hydrostatic pressure and the volume dilatation strain-rate.

Consider the following problem: find the velocity $\mathbf{u}$ and stress $\boldsymbol{\sigma}$ fields, satisfying: equation of equilibrium

$$\sigma_{ij,j} = 0, \quad \text{in } \Omega, \tag{2.2}$$

incompressibility condition

$$\dot{\varepsilon}_{V} = 0, \quad \text{in } \Omega, \tag{2.3}$$

strain-rate-velocity relations

$$\dot{\varepsilon}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.4}$$

yield criterion and flow rule

$$F(\sigma_{ij}, \dot{\bar{\varepsilon}}) \equiv \bar{\sigma}^2 - \sigma_p^2(\dot{\bar{\varepsilon}}) = 0, \qquad \dot{e}_{ij} = \frac{3}{2}\frac{\dot{\bar{\varepsilon}}}{\bar{\sigma}}s_{ij}, \tag{2.5}$$

boundary conditions

$$\begin{aligned}
&\sigma_{N} = 0, \quad \boldsymbol{\sigma}_{T} = \mathbf{0}, \quad \text{on } \Gamma_{1(5)} \cup \Gamma_2 \cup \Gamma_4, && \text{(2.6a)}\\
&\boldsymbol{\sigma}_{T} = \mathbf{0}, \quad u_{N} = u_{N}^0, \quad \text{on } \Gamma_{5(1)}, && \text{(2.6b)}\\
&\boldsymbol{\sigma}_{T} = \mathbf{0}, \quad u_{N} = 0, \quad \text{on } \Gamma_6, && \text{(2.6c)}\\
&u_{N} = 0, && \text{(2.6d)}
\end{aligned}$$

and

$$\begin{cases} \text{if } |\boldsymbol{\sigma}_{T}(\mathbf{u})| < \tau_f(\mathbf{u}), \ \text{then } \mathbf{u}_{T} = \mathbf{0}, \\ \text{if } |\boldsymbol{\sigma}_{T}(\mathbf{u})| = \tau_f(\mathbf{u}), \ \text{then } \exists \text{const } \lambda \geq 0, \ \text{such that } \mathbf{u}_{T} = -\lambda\boldsymbol{\sigma}_{T}(\mathbf{u}), \ \text{on } \Gamma_3. \end{cases}$$

Here δ_{ij} is the Kronecker symbol; $\mathbf{n}=\{n_i\}$ is the unit normal vector outward to Γ;

$$\mathbf{u}_{N} = u_{N}\mathbf{n}, \quad \mathbf{u}_{T} = \{u_{Ti}\}, \quad \text{and} \quad \boldsymbol{\sigma}_{N} = \sigma_{N}\mathbf{n}, \quad \boldsymbol{\sigma}_{T} = \{\sigma_{Ti}\},$$

are the normal and tangential components of the velocity and the stress vector on Γ, where

$$u_{N} = u_i n_i, \quad u_{Ti} = u_i - u_{N} n_i, \tag{2.7a}$$

$$\sigma_{_N}=\sigma_{ij}n_in_j,\quad \sigma_{_{Ti}}=\sigma_{ij}n_j-\sigma_{_N}n_i, \tag{2.7b}$$

$u^0_{_N}$ is the process velocity; $\tau_f(\mathbf{u})$ is the shear strength limit for the interface material on Γ_3, defined by the nonlocal Coulomb friction law

$$\tau_f(\mathbf{u})=\mu_f(\mathbf{x})\bar{\sigma}_{_N}(\mathbf{u}), \tag{2.8}$$

where $\mu_f(\mathbf{x})$ is the coefficient of friction, $\bar{\sigma}_{_N}(\mathbf{u})\geq 0$ is the appropriately mollified normal stress (see [11,19,20]),

$$\bar{\sigma}_{_N}\big(\mathbf{u}(\mathbf{x})\big)=\frac{1}{|\Gamma_h|}\int_{\Gamma_h}w_h(\mathbf{x}-\mathbf{y})\big(-\sigma_{_N}(\mathbf{u}(\mathbf{y}))\big)d\mathbf{y},\quad \mathbf{x}\in\Gamma_3, \tag{2.9}$$

where

$$w_h(\mathbf{x}-\mathbf{y})=\begin{cases}1, & \text{if } |\mathbf{x}-\mathbf{y}|<h,\\ 0, & \text{if } |\mathbf{x}-\mathbf{y}|\geq h,\end{cases} \tag{2.10}$$

$\sigma_p(\dot{\bar{\varepsilon}})$ is the strain-rate dependent, uniaxial yield limit, assumed increasing, almost everywhere differentiable function of $\dot{\bar{\varepsilon}}$, such that

$$\eta_1\leq\sigma_p'(\dot{\bar{\varepsilon}})=\frac{d\sigma_p(\dot{\bar{\varepsilon}})}{d\dot{\bar{\varepsilon}}}\leq\frac{\sigma_p(\dot{\bar{\varepsilon}})}{\dot{\bar{\varepsilon}}}\leq\eta_2,\quad \forall\,\dot{\bar{\varepsilon}}\in[0,\infty), \tag{2.11}$$

where η_1,η_2 are positive constants.

3 Variational formulation and solution method

Let us denote by $\mathbf{V}$ and $\mathbf{H}$ the following Hilbert spaces

$$\begin{aligned}&\mathbf{V}=\Big\{\mathbf{v}:\mathbf{v}\in\big(H^1(\Omega)\big)^k,\ v_{_N}=0,\ \text{on }\ \Gamma_6\Big\},\\ &\mathbf{H}=\big(H^0(\Omega)\big)^k\equiv\big(L_2(\Omega)\big)^k,\\ &\mathbf{V}\subset\mathbf{H}\equiv\mathbf{H}'\subset\mathbf{V}',\end{aligned}$$

where $\mathbf{V}'$ and $\mathbf{H}'$ are their dual spaces. By $(H^m(\Omega))^k$, with m nonnegative integer, we denote the Hilbert space of vector-valued functions defined in Ω (see [8,11,14,22,25])

$$\big(H^m(\Omega)\big)^k=\Big\{\mathbf{v}=\{v_i\}:D^\alpha v_i\in L_2(\Omega),\ 1\leq i\leq k,\ 0\leq|\boldsymbol{\alpha}|\leq m\Big\},$$

endowed with the inner product and norm

$$\begin{aligned}&(\mathbf{u},\mathbf{v})_m=\int_\Omega\sum_{|\alpha|=0}^{m}\Big(\sum_{i=1}^{k}D^\alpha u_iD^\alpha v_i\Big)d\mathbf{x},\\ &\|\mathbf{u}\|_m=(\mathbf{u},\mathbf{u})_m^{\frac12}=\Big(\int_\Omega\sum_{|\alpha|=0}^{m}\Big(\sum_{i=1}^{k}|D^\alpha u_i|^2\Big)d\mathbf{x}\Big)^{\frac12},\end{aligned}$$

where

$$D^{\alpha}v_i = \frac{\partial^{|\alpha|}v_i}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_k^{\alpha_k}},$$
$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_k) \in \mathbb{Z}^k, \quad \alpha_i \geq 0, \quad |\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_k.$$

We equip $\mathbf{V}$ with the following inner product and norm

$$(\mathbf{u}, \mathbf{v})_V = \int_{\Omega} \dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{v})d\mathbf{x} + \int_{\Gamma_3} u_N v_N d\Gamma, \quad \|\mathbf{u}\|_V = (\mathbf{u}, \mathbf{u})_V^{\frac{1}{2}}, \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{V}. \tag{3.1}$$

Remark 3.1. It can be shown, with the help of Korn′s inequality (see [8, 11])

$$\int_{\Omega} \dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{u})d\mathbf{x} + \int_{\Omega} u_i u_i d\mathbf{x} \geq c_K \|\mathbf{u}\|_1^2, \quad \forall\, \mathbf{u} \in \left(H^1(\Omega)\right)^k,$$

where $c_K>0$ is a constant, that for all $\mathbf{v}\in\mathbf{V}$, the norm $\|\mathbf{v}\|_V$ and the usual $(H^1(\Omega))^k$ norm

$$\|\mathbf{v}\|_1 = \left\{ \int_{\Omega} (v_{i,j}v_{i,j} + v_i v_i)d\mathbf{x} \right\}^{\frac{1}{2}},$$

are equivalent.

Let us further denote by $\mathbf{U}$ and $\mathbf{K}$ the following closed, convex subsets of $\mathbf{V}$

$$\mathbf{U} = \left\{ \mathbf{v} : \mathbf{v} \in \mathbf{V}, \;\; v_N = u_N^0, \;\; \text{on} \;\; \Gamma_{5(1)} \right\},$$
$$\mathbf{K} = \left\{ \mathbf{v} : \mathbf{v} \in \mathbf{U}, \;\; v_{i,i} = 0, \;\; \text{in} \;\; \Omega, \quad v_N = 0, \;\; \text{on} \;\; \Gamma_3 \right\}.$$

Then for $\mathbf{u}\in\mathbf{K}$ and all $\mathbf{v}\in\mathbf{K}$, after multiplying (2.2) by $(\mathbf{v} - \mathbf{u})$, in the inner product sense, applying Green′s formula and taking into account the boundary conditions, we obtain

$$\int_{\Omega} \sigma_{ij}(\mathbf{u})(\dot{\varepsilon}_{ij}(\mathbf{v}) - \dot{\varepsilon}_{ij}(\mathbf{u}))d\mathbf{x} + \int_{\Gamma_3} \tau_f(\mathbf{u})|\mathbf{v}_T|d\Gamma - \int_{\Gamma_3} \tau_f(\mathbf{u})|\mathbf{u}_T|d\Gamma \geq 0. \tag{3.2}$$

Let us suppose that $\mu_f(\mathbf{x})\in L_\infty(\Gamma_3)$ and introduce, for all $\mathbf{w}, \mathbf{u}, \mathbf{v} \in \mathbf{V}$, the notations

$$a(\mathbf{w}; \mathbf{u}, \mathbf{v}) = \int_{\Omega} \frac{2}{3}\frac{\sigma_p(\mathbf{w})}{\dot{\bar{\varepsilon}}(\mathbf{w})}\dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{v})d\mathbf{x}, \qquad j(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \tau_f(\mathbf{u})|\mathbf{v}_T|d\Gamma. \tag{3.3}$$

Then the following variational problem is associated with the problem (2.2)-(2.6d): find $\mathbf{u}\in\mathbf{K}$, satisfying

$$a(\mathbf{u}; \mathbf{u}, \mathbf{v} - \mathbf{u}) + j(\mathbf{u}, \mathbf{v}) - j(\mathbf{u}, \mathbf{u}) \geq 0, \quad \forall\, \mathbf{v} \in \mathbf{K}. \tag{3.4}$$

Let us also introduce the space

$$\Lambda = \left\{ q : q \in L_2(\Omega), \;\; (q, 1)_0 = 0 \right\}, \quad \Lambda \subset L_2(\Omega),$$

equipped with $L_2(\Omega)$ inner product and norm, the space $S=H^{1/2}(\Gamma_3)\subset L_2(\Gamma_3)$ of traces $v_N=\gamma_0(\mathbf{v})\cdot\mathbf{n}$, on Γ_3 of all $\mathbf{v}\in\mathbf{V}$ and its dual $S'=H^{-1/2}(\Gamma_3)$, with norms correspondingly

$$\|v_N\|_S = \inf_{\mathbf{v}\in\mathbf{V}} \Big\{ \|\mathbf{v}\|_V : v_N = \gamma_0(\mathbf{v})\cdot\mathbf{n} \Big\},$$
$$\|\tau_N\|_{S'} = \sup_{v_N\in S\setminus\{0\}} \frac{\langle \tau_N, v_N\rangle_\Gamma}{\|v_N\|_S},$$

where $\gamma_0 : (H^1(\Omega))^k \to (H^{1/2}(\Gamma))^k$ is the trace operator and $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality pairing. Then the following mixed variational problem is associated with the problem (2.2)-(2.6d), or problem (3.4) correspondingly: find $\mathbf{u}\in\mathbf{U}$, $\sigma_H\in\Lambda$ and $\sigma_N\in S'$, satisfying

$$\begin{aligned} &a(\mathbf{u};\mathbf{u},\mathbf{v}-\mathbf{u}) + \big(\sigma_H, \dot{\varepsilon}_V(\mathbf{v}-\mathbf{u})\big)_0 + j(\mathbf{u},\mathbf{v}) \\ &\qquad - j(\mathbf{u},\mathbf{u}) \geq \langle \sigma_N, v_N - u_N\rangle_\Gamma, \quad \forall\, \mathbf{v}\in\mathbf{U}, &(3.5a)\\ &\big(q,\dot{\varepsilon}_V(\mathbf{u})\big)_0 = 0, \quad \forall\, q\in\Lambda, &(3.5b)\\ &\langle \tau_N, u_N\rangle_\Gamma = 0, \quad \forall\, \tau_N\in S'. &(3.5c)\end{aligned}$$

Assuming further the following relations between the hydrostatic pressure and the volume dilatation strain-rate in Ω and between the normal stress and velocity on Γ_3

$$\sigma_H(\mathbf{u}) = \frac{\dot{\varepsilon}_V(\mathbf{u})}{d}, \qquad \sigma_N(\mathbf{u}) = -\frac{u_N}{d_N}, \tag{3.6}$$

where d and d_N are small positive (penalty) constants, we obtain the following perturbed formulation of the mixed variational problem (3.5): find $\mathbf{u}\in\mathbf{U}$, $\sigma_H\in\Lambda$ and $\sigma_N\in S'$, satisfying

$$\begin{aligned} &a(\mathbf{u};\mathbf{u},\mathbf{v}-\mathbf{u}) + (\sigma_H, \dot{\varepsilon}_V(\mathbf{v}-\mathbf{u}))_0 + j(\mathbf{u},\mathbf{v}) \\ &\qquad - j(\mathbf{u},\mathbf{u}) \geq \langle \sigma_N, v_N - u_N\rangle_\Gamma, \quad \forall\, \mathbf{v}\in\mathbf{U}, &(3.7a)\\ &\big(q,\dot{\varepsilon}_V(\mathbf{u})\big)_0 - d(q,\sigma_H)_0 = 0, \quad \forall\, q\in\Lambda, &(3.7b)\\ &\langle \tau_N, u_N\rangle_\Gamma + d_N\,\langle \tau_N, \sigma_N\rangle_\Gamma = 0, \quad \forall\, \tau_N\in S'. &(3.7c)\end{aligned}$$

Let us now denote

$$b(\mathbf{u};\mathbf{u},\mathbf{v}) = a(\mathbf{u};\mathbf{u},\mathbf{v}) + \int_\Omega \frac{1}{d}\dot{\varepsilon}_V(\mathbf{u})\dot{\varepsilon}_V(\mathbf{v})d\mathbf{x} + \int_{\Gamma_3} \frac{1}{d_N} u_N v_N d\Gamma, \tag{3.8}$$

then we obtain the following penalty formulation of the variational problem (3.4): find $\mathbf{u}\in\mathbf{U}$, satisfying for all $\mathbf{v}\in\mathbf{U}$ the variational inequality

$$b(\mathbf{u};\mathbf{u},\mathbf{v}-\mathbf{u}) + j(\mathbf{u},\mathbf{v}) - j(\mathbf{u},\mathbf{u}) \geq 0. \tag{3.9}$$

It can be shown, as in [17,19,20], that the following properties, of the introduced above functionals, hold.

Proposition 3.1. *For any fixed* $\mathbf{w}\in\mathbf{U}$, $b(\mathbf{w};\mathbf{u},\mathbf{v}) : \mathbf{V}\times\mathbf{V}\rightarrow\mathbb{R}$ *is a symmetric, bilinear form and there exist positive constants* β_0 *and* β_1 *such that*

$$b(\mathbf{w};\mathbf{u},\mathbf{u}) \geq \beta_0\|\mathbf{u}\|_V^2, \quad |b(\mathbf{w};\mathbf{u},\mathbf{v})| \leq \beta_1\|\mathbf{u}\|_V\|\mathbf{v}\|_V. \tag{3.10}$$

Proposition 3.2. *For all* $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{U}$, *there exist positive constants* m *and* M, *such that*

$$b(\mathbf{v};\mathbf{v},\mathbf{v}-\mathbf{u}) - b(\mathbf{u};\mathbf{u},\mathbf{v}-\mathbf{u}) \geq m\|\mathbf{v}-\mathbf{u}\|_V^2, \tag{3.11a}$$
$$|b(\mathbf{v};\mathbf{v},\mathbf{u}) - b(\mathbf{u};\mathbf{u},\mathbf{u})| \leq M\|\mathbf{v}-\mathbf{u}\|_V\|\mathbf{u}\|_V. \tag{3.11b}$$

Proposition 3.3. *There exist positive constants* c_f *and* c, *depending on the friction coefficient, such that for all* $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{U}$,

$$0 \leq j(\mathbf{u},\mathbf{v}) \leq c_f\|\mathbf{u}\|_V\|\mathbf{v}\|_V, \tag{3.12a}$$
$$|j(\mathbf{u},\mathbf{w}) + j(\mathbf{w},\mathbf{v}) - j(\mathbf{u},\mathbf{v}) - j(\mathbf{w},\mathbf{w})| \leq c\|\mathbf{w}-\mathbf{u}\|_V\|\mathbf{w}-\mathbf{v}\|_V. \tag{3.12b}$$

Proposition 3.4. *If* $\mathbf{u}\in\mathbf{U}$ *is a solution of the above stated problems, there exists a positive constant* c_0, *such that*

$$\|\mathbf{u}\|_V \leq c_0|u_N^0|. \tag{3.13}$$

Remark 3.2. If the nondifferentiable at $\mathbf{v}_T=\mathbf{0}$, functional $j(\mathbf{u},\mathbf{v})$ is replaced by some its convex regularization [11,20], e.g., by

$$j_{d_T}(\mathbf{u},\mathbf{v}) = \int_{\Gamma_3} \tau_f(\mathbf{u})\sqrt{|\mathbf{v}_T|^2 + d_T^2}\,d\Gamma, \tag{3.14}$$

where $d_T>0$ is a constant, then the following result holds.

Proposition 3.5. *The functional* $j_{d_T}(\mathbf{u},\mathbf{v})$ *is Gâteaux differentiable*

$$\langle j'_{d_T}(\mathbf{u},\mathbf{u}),\mathbf{v}\rangle = \int_{\Gamma_3} \tau_f(\mathbf{u})\frac{\mathbf{u}_T\cdot\mathbf{v}_T}{\sqrt{|\mathbf{u}_T|^2 + d_T^2}}d\Gamma, \tag{3.15}$$

and such that

$$\langle j'_{d_T}(\mathbf{u},\mathbf{u}),\mathbf{v}-\mathbf{u}\rangle \leq j_{d_T}(\mathbf{u},\mathbf{v}) - j_{d_T}(\mathbf{u},\mathbf{u}), \tag{3.16}$$
$$\langle j'_{d_T}(\mathbf{u},\mathbf{v}) - j'_{d_T}(\mathbf{u},\mathbf{u}),\mathbf{v}-\mathbf{u}\rangle \geq 0. \tag{3.17}$$

Thus we obtain the following regularized penalty problems: find $\mathbf{u}\in\mathbf{U}$, $\sigma_H\in\Lambda$ and $\sigma_N\in S'$, satisfying

$$a(\mathbf{u};\mathbf{u},\mathbf{v}-\mathbf{u}) + \big(\sigma_H,\dot{\varepsilon}_V(\mathbf{v}-\mathbf{u})\big)_0 + \langle j'_{d_T}(\mathbf{u},\mathbf{u}),\mathbf{v}-\mathbf{u}\rangle$$
$$= \langle\sigma_N, v_N - u_N\rangle_\Gamma, \quad \forall\,\mathbf{v}\in\mathbf{U}, \tag{3.18a}$$
$$\big(q,\dot{\varepsilon}_V(\mathbf{u})\big)_0 - d(q,\sigma_H)_0 = 0, \qquad \forall\, q\in\Lambda, \tag{3.18b}$$
$$\langle\tau_N,u_N\rangle_\Gamma + d_N\,\langle\tau_N,\sigma_N\rangle_\Gamma = 0, \quad \forall\,\tau_N\in S', \tag{3.18c}$$

and find $\mathbf{u}\in\mathbf{U}$, satisfying for all $\mathbf{v}\in\mathbf{U}$,

$$b(\mathbf{u};\mathbf{u},\mathbf{v}-\mathbf{u})+\langle j'_{d_T}(\mathbf{u},\mathbf{u}),\mathbf{v}-\mathbf{u}\rangle=0. \tag{3.19}$$

It should be also noted that the solutions of the problems (3.7), (3.9), (3.18) and (3.19) depend on the introduced penalty and regularization constants.

Let now $\mathbf{u}_0\in\mathbf{U}$ be an arbitrary element and consider the following problem: find $\mathbf{u}_{n+1}\in\mathbf{U}$, $n=0,1,\cdots$, satisfying

$$b(\mathbf{u}_n;\mathbf{u}_{n+1},\mathbf{v}-\mathbf{u}_{n+1})+j(\mathbf{u}_n,\mathbf{v})-j(\mathbf{u}_n,\mathbf{u}_{n+1})\geq 0, \quad \forall\, \mathbf{v}\in\mathbf{U}. \tag{3.20}$$

Then, assuming that the properties established by Proposition 3.1–Proposition 3.4 hold, the following results can be obtained, analogously to [17,19,20].

Proposition 3.6. *The problem* (3.20) *has an unique solution* $\mathbf{u}_{n+1}\in\mathbf{U}$.

Theorem 3.1. *If the the coefficient of friction is sufficiently small, the sequence* $\{\mathbf{u}_n\}$, *defined by problem* (3.20), *converges strongly to the unique solution* $\mathbf{u}$ *of problem* (3.9).

Since the solution of problem (3.9) depends on the penalty constants, for all sufficiently small $d>0$ and $d_N>0$, we obtain a sequence $\{\mathbf{u}^{d,d_N}\}$ of solutions, bounded in $\mathbf{U}\subset\mathbf{V}$. Taking then, without loss of generality, $d_N=c_N d$, where c_N is a positive constant, we can construct by diagonalization a subsequence $\{\mathbf{u}^d\}$, which is weakly convergent in $\mathbf{U}$, such that the following result holds.

Theorem 3.2. *At d tending to zero, the solution* $\mathbf{u}^d\in\mathbf{U}$ *of problem* (3.9) *tends to the unique solution* $\mathbf{u}\in\mathbf{K}$ *of problem* (3.4).

Remark 3.3. Existence and uniqueness results for problem (3.19) can be obtained analogously to problem (3.9). Since the solution depends on the regularization constant, the sequence of solutions $\{\mathbf{u}^{d_T}\}$, obtained for all sufficiently small $d_T>0$, is bounded in $\mathbf{U}\subset\mathbf{V}$ and therefore there exists a subsequence $\{\mathbf{u}^{d_T}\}$, weakly convergent in $\mathbf{V}$, such that the following result holds.

Theorem 3.3. *Let* $\mathbf{u}\in\mathbf{U}$ *and* $\mathbf{u}^{d_T}\in\mathbf{U}$ *be the solutions of the problem* (3.9) *and problem* (3.19) *respectively. Then there exists a positive constant C_0, independent of d_T, such that*

$$\|\mathbf{u}-\mathbf{u}^{d_T}\|_V\leq C_0\sqrt{d_T}. \tag{3.21}$$

Let us now suppose that there exist constants $\alpha_b>0$ and $\beta_b>0$, such that the following stability (Babuška-Brezzi) condition holds [11]

$$\alpha_b\|\tau_N\|_{S'}+\beta_b\|q\|_0\leq \sup_{\mathbf{v}\in\mathbf{U}}\frac{\langle\tau_N,v_N\rangle_\Gamma+(q,\dot{\varepsilon}_V(\mathbf{v}))_0}{\|\mathbf{v}\|_V}, \quad \forall\,\tau_N\in S',\ \forall\, q\in\Lambda. \tag{3.22}$$

Let us also assume $d_N=c_N d$. Then the following result holds:

Theorem 3.4. *Let* $\{\mathbf{u},\sigma_H,\sigma_N\}\in\mathbf{U}\times\Lambda\times S'$ *and* $\{\mathbf{u}^d,\sigma_H^d,\sigma_N^d\}\in\mathbf{U}\times\Lambda\times S'$ *be the unique solutions of the problem* (3.5) *and problem* (3.7) *respectively. Then there exist, independent of d positive constants* $C_\alpha, C_\beta, C_\gamma$, *such that*

$$\|\mathbf{u}-\mathbf{u}^d\|_V\le C_\alpha d,\quad \|\sigma_H-\sigma_H^d\|_0\le C_\beta d,\quad \|\sigma_N-\sigma_N^d\|_{S'}\le C_\gamma d. \tag{3.23}$$

Proof. From problem (3.5) and problem (3.7), for all $\mathbf{v}\in\mathbf{U}$, we obtain correspondingly

$$\begin{aligned}&[a(\mathbf{u};\mathbf{u},\mathbf{u})+j(\mathbf{u},\mathbf{u})+(\sigma_H,\dot{\varepsilon}_V(\mathbf{u}))_0-\langle\sigma_N,u_N\rangle_\Gamma]\\ &\quad+\langle\sigma_N,v_N\rangle_\Gamma-(\sigma_H,\dot{\varepsilon}_V(\mathbf{v}))_0\le a(\mathbf{u};\mathbf{u},\mathbf{v})+j(\mathbf{u},\mathbf{v}), \qquad (3.24)\\ &\big[a(\mathbf{u}^d;\mathbf{u}^d,\mathbf{u}^d)+j(\mathbf{u}^d,\mathbf{u}^d)+(\sigma_H^d,\dot{\varepsilon}_V(\mathbf{u}^d))_0-\langle\sigma_N^d,u_N^d\rangle_\Gamma\big]\\ &\quad+\langle\sigma_N^d,v_N\rangle_\Gamma-(\sigma_H^d,\dot{\varepsilon}_V(\mathbf{v}))_0\le a(\mathbf{u}^d;\mathbf{u}^d,\mathbf{v})+j(\mathbf{u}^d,\mathbf{v}). \qquad (3.25)\end{aligned}$$

Since the quantities in brackets in the left-hand sides of (3.24) and (3.25) are nonnegative, we obtain correspondingly

$$\langle\sigma_N,v_N\rangle_\Gamma+(-\sigma_H,\dot{\varepsilon}_V(\mathbf{v}))_0\le a(\mathbf{u};\mathbf{u},\mathbf{v})+j(\mathbf{u},\mathbf{v})\le C_1\|\mathbf{u}\|_V\|\mathbf{v}\|_V, \tag{3.26}$$

$$\langle\sigma_N^d,v_N\rangle_\Gamma+(-\sigma_H^d,\dot{\varepsilon}_V(\mathbf{v}))_0\le a(\mathbf{u}^d;\mathbf{u}^d,\mathbf{v})+j(\mathbf{u}^d,\mathbf{v})\le C_2\|\mathbf{u}^d\|_V\|\mathbf{v}\|_V, \tag{3.27}$$

where C_1, C_2 are positive constants. We have then

$$\sup_{\mathbf{v}\in\mathbf{U}}\frac{\langle\sigma_N,v_N\rangle_\Gamma+(-\sigma_H,\dot{\varepsilon}_V(\mathbf{v}))_0}{\|\mathbf{v}\|_V}\le C_1\|\mathbf{u}\|_V, \tag{3.28}$$

$$\sup_{\mathbf{v}\in\mathbf{U}}\frac{\langle\sigma_N^d,v_N\rangle_\Gamma+(-\sigma_H^d,\dot{\varepsilon}_V(\mathbf{v}))_0}{\|\mathbf{v}\|_V}\le C_2\|\mathbf{u}^d\|_V, \tag{3.29}$$

and after applying the Babuška-Brezzi condition and taking into account that $\|\mathbf{u}\|_V$ and $\|\mathbf{u}^d\|_V$ are bounded in $\mathbf{U}\subset\mathbf{V}$, we obtain

$$\alpha_b\|\sigma_N\|_{S'}+\beta_b\|\sigma_H\|_0\le C_3,\quad \alpha_b\|\sigma_N^d\|_{S'}+\beta_b\|\sigma_H^d\|_0\le C_4, \tag{3.30}$$

where C_3 and C_4 are positive constants. Therefore from the sequences $\{\sigma_H^d\}$ and $\{\sigma_N^d\}$, obtained for all sufficiently small $d>0$, can be extracted weakly convergent subsequences, such that at $d\to 0$, $\{\sigma_H^d\}\to\sigma_H\in\Lambda$ and $\{\sigma_N^d\}\to\sigma_N\in S'$. Setting further $\mathbf{v}=\mathbf{u}^d$ in problem (3.5) and $\mathbf{v}=\mathbf{u}$ in problem (3.7), adding the inequalities and taking into account (3.6), Propositions 3.1–3.4 and that $\dot{\varepsilon}_V(\mathbf{u})=0$ and $u_N=0$, we obtain

$$\begin{aligned}&m\|\mathbf{u}-\mathbf{u}^d\|_V^2\le a(\mathbf{u};\mathbf{u},\mathbf{u}-\mathbf{u}^d)-a(\mathbf{u}^d;\mathbf{u}^d,\mathbf{u}-\mathbf{u}^d)\le j(\mathbf{u},\mathbf{u}^d)+j(\mathbf{u}^d,\mathbf{u})\\ &-j(\mathbf{u},\mathbf{u})-j(\mathbf{u}^d,\mathbf{u}^d)+\langle\sigma_N-\sigma_N^d,u_N-u_N^d\rangle_\Gamma-(\sigma_H-\sigma_H^d,\dot{\varepsilon}_V(\mathbf{u})-\dot{\varepsilon}_V(\mathbf{u}^d))_0\\ &\le c\|\mathbf{u}-\mathbf{u}^d\|_V^2+c_Nd\|\sigma_N-\sigma_N^d\|_{S'}\|\mathcal{R}\sigma_N^d\|_S+d\|\sigma_H-\sigma_H^d\|_0\|\sigma_H^d\|_0, \qquad (3.31)\end{aligned}$$

where $\mathcal{R}: S' \to S$ is the Riesz map. Since by the Riesz representation theorem we have $\|\sigma_N^d\|_{S'} = \|\mathcal{R}\sigma_N^d\|_S$, using (3.31), at a sufficiently small coefficient of friction, we obtain

$$(m-c)\|\mathbf{u}-\mathbf{u}^d\|_V^2 \le C_5 d\big(\|\sigma_N - \sigma_N^d\|_{S'} + \|\sigma_H - \sigma_H^d\|_0\big), \tag{3.32}$$

where C_5 is a positive constants. From (3.24) and (3.25), for any $\mathbf{v}\in\mathbf{U}$, we have that holds either

$$\begin{aligned} 0 &\le a(\mathbf{u}^d;\mathbf{u}^d,\mathbf{v}) + j(\mathbf{u}^d,\mathbf{v}) - \langle \sigma_N^d, v_N\rangle_\Gamma + \big(\sigma_H^d, \dot{\varepsilon}_V(\mathbf{v})\big)_0 \\ &\le a(\mathbf{u};\mathbf{u},\mathbf{v}) + j(\mathbf{u},\mathbf{v}) - \langle \sigma_N, v_N\rangle_\Gamma + \big(\sigma_H, \dot{\varepsilon}_V(\mathbf{v})\big)_0, \end{aligned} \tag{3.33}$$

or

$$\begin{aligned} 0 &\le a(\mathbf{u};\mathbf{u},\mathbf{v}) + j(\mathbf{u},\mathbf{v}) - \langle \sigma_N, v_N\rangle_\Gamma + \big(\sigma_H, \dot{\varepsilon}_V(\mathbf{v})\big)_0 \\ &< a(\mathbf{u}^d;\mathbf{u}^d,\mathbf{v}) + j(\mathbf{u}^d,\mathbf{v}) - \langle \sigma_N^d, v_N\rangle_\Gamma + \big(\sigma_H^d, \dot{\varepsilon}_V(\mathbf{v})\big)_0. \end{aligned} \tag{3.34}$$

Therefore, for all $\mathbf{v}\in\mathbf{U}$, we have that

$$\begin{aligned} &\langle \pm(\sigma_N - \sigma_N^d), v_N\rangle_\Gamma + \big(\mp(\sigma_H - \sigma_H^d), \dot{\varepsilon}_V(\mathbf{v})\big)_0 \\ \le &\big|a(\mathbf{u};\mathbf{u},\mathbf{v}) - a(\mathbf{u}^d;\mathbf{u}^d,\mathbf{v})\big| + \big|j(\mathbf{u},\mathbf{v}) - j(\mathbf{u}^d,\mathbf{v})\big| \le C_6\|\mathbf{u}-\mathbf{u}^d\|_V\|\mathbf{v}\|_V, \end{aligned} \tag{3.35}$$

where C_6 is a positive constant. Then, after applying the Babuška-Brezzi condition to

$$\sup_{\mathbf{v}\in\mathbf{U}} \frac{\langle \pm(\sigma_N - \sigma_N^d), v_N\rangle_\Gamma + \big(\mp(\sigma_H - \sigma_H^d), \dot{\varepsilon}_V(\mathbf{v})\big)_0}{\|\mathbf{v}\|_V} \le C_6\|\mathbf{u}-\mathbf{u}^d\|_V, \tag{3.36}$$

we obtain

$$\alpha_b\|\sigma_N - \sigma_N^d\|_{S'} + \beta_b\|\sigma_H - \sigma_H^d\|_0 \le C_6\|\mathbf{u}-\mathbf{u}^d\|_V. \tag{3.37}$$

Finally, from (3.32) and (3.37) we obtain

$$\|\mathbf{u}-\mathbf{u}^d\|_V \le C_\alpha d, \tag{3.38}$$

and from (3.37) and (3.38) it follows that

$$\|\sigma_N - \sigma_N^d\|_{S'} \le C_\beta d, \qquad \|\sigma_H - \sigma_H^d\|_0 \le C_\gamma d, \tag{3.39}$$

where C_α, C_β and C_γ are positive constants. □

Remark 3.4. At some regularity conditions, stronger forms of the Babuška-Brezzi stability condition (3.22) might be expected to hold, for example

$$\begin{aligned} &\hat{\alpha}_b\|\tau_N\|_{0,\Gamma} + \beta_b\|q\|_0 \\ \le &\sup_{\mathbf{v}\in\mathbf{U}} \frac{(\tau_N, v_N)_{0,\Gamma} + (q, \dot{\varepsilon}_V(\mathbf{v}))_0}{\|\mathbf{v}\|_V}, \quad \forall\, \tau_N \in S'\cap L_2(\Omega),\ \ \forall\, q\in\Lambda, \end{aligned}$$

or

$$\begin{aligned} &\hat{\alpha}_b\|\tau_N\|_{0,\Gamma} + \hat{\beta}_b\|\nabla q\|_0 \\ \le &\sup_{\mathbf{v}\in\mathbf{U}} \frac{(\tau_N, v_N)_{0,\Gamma} + |(\nabla q, \mathbf{v})_0|}{\|\mathbf{v}\|_V}, \quad \forall \tau_N \in S'\cap L_2(\Omega),\quad \forall\, q\in \Lambda\cap H_0^1(\Omega), \end{aligned}$$

where $\hat{\alpha}_b$, β_b and $\hat{\beta}_b$ are positive constants.

4 Finite element approximation

Let $\mathcal{C}_h$ be a regular partition of $\bar{\Omega}=\cup_{K\in\mathcal{C}_h}K$ into finite elements K and construct the finite element spaces

$$\mathbf{V}_h=\Big\{\mathbf{v}^h:\mathbf{v}^h\in\mathbf{V}\cap\big(C^0(\bar{\Omega})\big)^k,\quad \mathbf{v}^h{}_{|_K}=\hat{\mathbf{v}}^h\circ F_K^{-1},\quad \hat{\mathbf{v}}^h\in\big(Q_l(\hat{K})\big)^k\Big\},$$

where h is the mesh parameter approaching zero, $F_K\colon\hat{K}\to K$, $F_K\in(Q_l(\hat{K}))^k$ is the isoparametric transformation, $\hat{K}$ is the reference element and $(Q_l(\hat{K}))^k$ is the space of polynomials on $\hat{K}$ of order not greater then $l=1,2$ in each variable [16]. Let us also suppose that the following standard approximation properties of $\mathbf{V}_h$ hold (see [11,22,25]):

$$\begin{cases}\forall\,\mathbf{v}\in(H^m(\Omega))^k\cap\mathbf{V},\ \exists\mathbf{v}^h\in\mathbf{V}_h, & \text{such that}\\ \|\mathbf{v}-\mathbf{v}^h\|_s\le c_\Omega h^{r_1}\|\mathbf{v}\|_m,\ r_1=\min\{l+1-s,m-s\},\ m\ge s,\\ \text{if } \gamma_0(\mathbf{v})\in\big(H^{m-\frac{1}{2}}(\Gamma)\big)^k, & \text{then}\\ \|\gamma_0(\mathbf{v})-\gamma_0(\mathbf{v}^h)\|_{s-\frac{1}{2},\Gamma}\le c_\Gamma h^{r_1}\|\gamma_0(\mathbf{v})\|_{m-\frac{1}{2},\Gamma},\end{cases}\tag{4.1}$$

where $c_\Omega>0$ and c_Γ are independent of h and $\mathbf{v}$ positive constants. Then from problem (3.8) we obtain in $\mathbf{U}_h\subset\mathbf{V}_h$, the following finite-dimensional problem: find $\mathbf{u}^h\in\mathbf{U}_h$, satisfying for all $\mathbf{v}^h\in\mathbf{U}_h$ the inequality

$$b\big(\mathbf{u}^h;\mathbf{u}^h,\mathbf{v}^h-\mathbf{u}^h\big)+j(\mathbf{u}^h,\mathbf{v}^h)-j(\mathbf{u}^h,\mathbf{u}^h)\ge 0.\tag{4.2}$$

Remark 4.1. Existence, uniqueness and convergence results for the discrete problem (4.2), at fixed h, can be obtained analogously to the continuous problem (3.9).

Remark 4.2. Essential, for the considered class of problems, is that they are usually defined in domains, ranged from nonconvex L-shaped up to convex conical, almost rectangular ones. Therefore, if all other data of the problems are sufficiently smooth, the following range of regularity of their solutions should be expected

$$\mathbf{u}\in\big(H^\alpha(\Omega)\big)^k\cap\mathbf{U},\quad \frac{5}{3}-\epsilon\le\alpha\le 3-\epsilon,\quad \epsilon>0,$$

small, [23–25].

Theorem 4.1. *Let* $\mathbf{u}\in(H^\alpha(\Omega))^k\cap\mathbf{U}$ *and* $\mathbf{u}^h\in\mathbf{U}_h$ *be the solutions of problem* (3.9) *and problem* (4.2) *respectively and let also*

$$\sigma_N(\mathbf{u})\in H^{\alpha-3/2}(\Gamma_3),\quad \textit{and}\quad \boldsymbol{\sigma}_T(\mathbf{u})\in\big(H^{\alpha-3/2}(\Gamma_3)\big)^k.$$

Then there exists a positive constant C, independent of h, such that

$$\|\mathbf{u}-\mathbf{u}^h\|_V\le Ch^r,\quad r=\min\{l,\alpha-1\}.\tag{4.3}$$

Proof. Adding (4.2) to (3.9) with $\mathbf{v}{=}\mathbf{u}^h$, we obtain after rearranging

$$\begin{aligned}
&b(\mathbf{u};\mathbf{u},\mathbf{u}-\mathbf{u}^h)-b(\mathbf{u}^h;\mathbf{u}^h,\mathbf{u}-\mathbf{u}^h)\\
\leq& b(\mathbf{u};\mathbf{u},\mathbf{u}-\mathbf{v}^h)-b(\mathbf{u}^h;\mathbf{u}^h,\mathbf{u}-\mathbf{v}^h)+j(\mathbf{u}^h,\mathbf{v}^h)+j(\mathbf{u},\mathbf{u})\\
&-j(\mathbf{u},\mathbf{v}^h)-j(\mathbf{u}^h,\mathbf{u})+j(\mathbf{u},\mathbf{u}^h)+j(\mathbf{u}^h,\mathbf{u})-j(\mathbf{u},\mathbf{u})\\
&-j(\mathbf{u}^h,\mathbf{u}^h)+b(\mathbf{u};\mathbf{u},\mathbf{v}^h-\mathbf{u})+j(\mathbf{u},\mathbf{v}^h)-j(\mathbf{u},\mathbf{u}).
\end{aligned} \tag{4.4}$$

Applying then Green's formula to the terms in last line of (4.4) and using relations (3.6) and properties (3.10)–(3.12), for a sufficiently small coefficient of friction such that $m{>}c$, we obtain

$$\begin{aligned}
&(m-c)\|\mathbf{u}-\mathbf{u}^h\|_V^2\\
\leq&(M+c)\|\mathbf{u}-\mathbf{u}^h\|_V\|\mathbf{u}-\mathbf{v}^h\|_V+C_1\|\sigma_T(\mathbf{u})\|_{\alpha-\frac{3}{2},\Gamma}\|\mathbf{u}_T-\mathbf{v}_T^h\|_{-\alpha+\frac{3}{2},\Gamma}\\
&+C_2\|\sigma_N(\mathbf{u})\|_{\alpha-\frac{3}{2},\Gamma}\|\mathbf{u}_T-\mathbf{v}_T^h\|_{-\alpha+\frac{3}{2},\Gamma},
\end{aligned} \tag{4.5}$$

where C_1, C_2 are positive constants. After using the finite elements approximation properties (4.1), we obtain

$$\|\mathbf{u}-\mathbf{u}^h\|_V^2\leq C_3h^r\|\mathbf{u}\|_\alpha\|\mathbf{u}-\mathbf{u}^h\|_V+C_4h^{2r}\|\mathbf{u}\|_\alpha, \tag{4.6}$$

where C_3, C_4 are positive constants. Applying then, to the first term in the right hand side of (4.6), the ε-inequality

$$|ab|\leq\varepsilon a^2+\frac{b^2}{4\varepsilon},\quad\forall\varepsilon>0,\;\; a,b\in\mathbb{R},$$

with appropriately chosen ε, we conclude that there exists an independent of h positive constant C, such that (4.3) holds. □

Let us consider the finite element approximation of problem (3.7). We introduce the finite dimensional spaces

$$\begin{aligned}
\Lambda_h&=\Big\{q^h: q^h\in\Lambda,\; q^h_{|_K}\in R_j(K)\Big\},\\
S'_h&=\Big\{\tau_N^h:\tau_N^h\in S'\cap C^0(\bar\Gamma_3),\;\; \tau^h_{N|_L}=\hat\tau_N^h\circ F_L^{-1},\;\; \hat\tau_N^h\in P_l(\hat L)\Big\},
\end{aligned}$$

where $R_j(K)$, $j{=}0,1$, is the space of constants $P_0(K)$, or the space of bilinear polynomials $Q_1(K)$ defined on K, $P_l(\hat L)$ is the space of polynomials of order not greater than l, $l{=}1,2$, defined on the line element $\hat L$, with the following approximation properties:

$$\begin{aligned}
&\forall q\in H^{m-1}(\Omega),\; m\geq 1,\; \exists q^h\in\Lambda_h, \text{ such that}\\
&\|q-q^h\|_0\leq c'_\Omega h^{r_2}\|q\|_{m-1},\quad r_2=\min\{j+1,m-1\},
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
&\forall\tau_N\in H^{m-\frac{3}{2}}(\Gamma_3),\; \exists\tau_N^h\in S'_h, \text{ such that}\\
&\|\tau_N-\tau_N^h\|_{S'}\leq c'_\Gamma h^{r_3}\|\tau_N\|_{m-\frac{3}{2},\Gamma},\quad r_3=\min\{l+1,m-1\},
\end{aligned} \tag{4.8}$$

where c'_Ω and c'_Γ are positive constants, independent of h and q and h and τ_N correspondingly. We state the following finite dimensional analog of problem (3.7): find $\mathbf{u}^h \in \mathbf{U}_h$, $\sigma_H^h \in \Lambda_h$ and $\sigma_N^h \in S'_h$, satisfying

$$\begin{aligned} &a(\mathbf{u}^h;\mathbf{u}^h,\mathbf{v}^h-\mathbf{u}^h)+\big(\sigma_H^h,\dot{\varepsilon}_V(\mathbf{v}^h-\mathbf{u}^h)\big)_0+j(\mathbf{u}^h,\mathbf{v}^h) \\ &\qquad -j(\mathbf{u}^h,\mathbf{u}^h)\geq\langle\sigma_N^h,v_N^h-u_N^h\rangle_\Gamma,\quad \forall\,\mathbf{v}^h\in\mathbf{U}_h, \end{aligned} \tag{4.9a}$$

$$\big(q^h,\dot{\varepsilon}_V(\mathbf{u}^h)\big)_0-d(q^h,\sigma_H^h)_0=0,\quad \forall\,q^h\in\Lambda_h, \tag{4.9b}$$

$$\langle\tau_N^h,u_N^h\rangle_\Gamma+d_N\langle\tau_N^h,\sigma_N^h\rangle_\Gamma=0,\qquad \forall\,\tau_N^h\in S'_h. \tag{4.9c}$$

Let us further assume that the following discrete analog of the Babuška-Brezzi condition (3.22) holds:

$$\begin{aligned} &\alpha_b^h\|\tau_N^h\|_{S'}+\beta_b^h\|q^h\|_0 \\ \leq&\sup_{\mathbf{v}^h\in\mathbf{U}_h}\frac{\langle\tau_N^h,v_N^h\rangle_\Gamma+\big(q^h,\dot{\varepsilon}_V(\mathbf{v}^h)\big)_0}{\|\mathbf{v}^h\|_V},\quad \forall\,\tau_N^h\in S'_h,\ \forall\,q^h\in\Lambda_h, \end{aligned} \tag{4.10}$$

where α_b^h and β_b^h are positive numbers [11,24]. Then the following result holds.

Theorem 4.2. *Let*

$$\{\mathbf{u},\sigma_H,\sigma_N\}\in\mathbf{U}\cap\big(H^\alpha(\Omega)\big)^k\times\Lambda\cap H^{\alpha-1}(\Omega)\times S'\cap H^{\alpha-\frac{3}{2}}(\Gamma_3),$$

and

$$\{\mathbf{u}^h,\sigma_H^h,\sigma_N^h\}\in\mathbf{U}_h\times\Lambda_h\times S'_h,$$

be the unique solutions of problem (3.7) *and problem* (4.9) *respectively and let also*

$$\boldsymbol{\sigma}_T(\mathbf{u})\in\big(H^{\alpha-\frac{3}{2}}(\Gamma_3)\big)^k.$$

Then there exist independent of h positive constants C_0, C'_0 and C''_0, such that

$$\begin{aligned} &\|\mathbf{u}-\mathbf{u}^h\|_V \\ \leq& C_0h^r\times\Big[\big(1+\alpha_b^{h^{-\frac{1}{2}}}+\beta_b^{h^{-\frac{1}{2}}}\big)\big(1+h^{\frac{s-r}{2}}\big)+\alpha_b^{h^{-1}}+\beta_b^{h^{-1}}\Big], \end{aligned} \tag{4.11a}$$

$$\begin{aligned} &\|\sigma_N-\sigma_N^h\|_{S'} \\ \leq& C'_0h^r\times\Big\{1+\alpha_b^{h^{-1}}\Big[h^{s-r}+\big(1+\alpha_b^{h^{-\frac{1}{2}}}+\beta_b^{h^{-\frac{1}{2}}}\big)\big(1+h^{\frac{s-r}{2}}\big) \\ &+\alpha_b^{h^{-1}}+\beta_b^{h^{-1}}\Big]\Big\}, \end{aligned} \tag{4.11b}$$

$$\begin{aligned} &\|\sigma_H-\sigma_H^h\|_0 \\ \leq& C''_0h^r\times\Big\{h^{s-r}+\beta_b^{h^{-1}}\Big[h^{s-r}+\big(1+\alpha_b^{h^{-\frac{1}{2}}}+\beta_b^{h^{-\frac{1}{2}}}\big)\big(1+h^{\frac{s-r}{2}}\big) \\ &+\alpha_b^{h^{-1}}+\beta_b^{h^{-1}}\Big]\Big\}, \end{aligned} \tag{4.11c}$$

where

$$r=\min\{l,\alpha-1\},\quad s=\min\{j+1,\alpha-1\}.$$

Proof. For all $\mathbf{v}^h \in \mathbf{U}_h$, from problem (4.9) we obtain

$$\begin{aligned}&\big[a(\mathbf{u}^h;\mathbf{u}^h,\mathbf{u}^h)+j(\mathbf{u}^h,\mathbf{u}^h)+\big(\sigma^h_H,\dot{\varepsilon}_V(\mathbf{u}^h)\big)_0-\langle\sigma^h_N,u^h_N\rangle_\Gamma\big]\\&\quad+\langle\sigma^h_N,v^h_N\rangle_\Gamma-\big(\sigma^h_H,\dot{\varepsilon}_V(\mathbf{v}^h)\big)_0\le a(\mathbf{u}^h;\mathbf{u}^h,\mathbf{v}^h)+j(\mathbf{u}^h,\mathbf{v}^h).\end{aligned}\tag{4.12}$$

Since the quantity in brackets in the left-hand side of (4.12) is nonnegative, we obtain

$$\langle\sigma^h_N,v^h_N\rangle_\Gamma+\big(-\sigma^h_H,\dot{\varepsilon}_V(\mathbf{v}^h)\big)_0\le a(\mathbf{u}^h;\mathbf{u}^h,\mathbf{v}^h)+j(\mathbf{u}^h,\mathbf{v}^h)\le C_1\|\mathbf{u}^h\|_V\|\mathbf{v}^h\|_V,\tag{4.13}$$

where C_1 is a positive constant. Applying then the discrete Babuška-Brezzi condition to

$$\sup_{\mathbf{v}^h\in\mathbf{U}_h}\frac{\langle\sigma^h_N,v^h_N\rangle_\Gamma+\big(-\sigma^h_H,\dot{\varepsilon}_V(\mathbf{v}^h)\big)_0}{\|\mathbf{v}^h\|_V}\le C_1\|\mathbf{u}^h\|_V,\tag{4.14}$$

and since $\|\mathbf{u}^h\|_V$ is bounded in $\mathbf{U}_h\subset\mathbf{V}_h$, we obtain

$$\alpha^h_b\|\sigma^h_N\|_{S'}+\beta^h_b\|\sigma^h_H\|_0\le C_2,\tag{4.15}$$

where C_2 is a positive constant. Setting further $\mathbf{v}=\mathbf{u}^h$ in problem (3.7) and adding it to problem (4.9), we obtain

$$\begin{aligned}&a(\mathbf{u};\mathbf{u},\mathbf{u}-\mathbf{u}^h)-a(\mathbf{u}^h;\mathbf{u}^h,\mathbf{u}-\mathbf{u}^h)+\big(\sigma_H-\sigma^h_H,\dot{\varepsilon}_V(\mathbf{u})-\dot{\varepsilon}_V(\mathbf{u}^h)\big)_0\\&-\langle\sigma_N-\sigma^h_N,u_N-u^h_N\rangle_\Gamma\le a(\mathbf{u};\mathbf{u},\mathbf{u}-\mathbf{v}^h)-a(\mathbf{u}^h;\mathbf{u}^h,\mathbf{u}-\mathbf{v}^h)\\&+\big(\sigma_H-\sigma^h_H,\dot{\varepsilon}_V(\mathbf{u})-\dot{\varepsilon}_V(\mathbf{v}^h)\big)_0-\langle\sigma_N-\sigma^h_N,u_N-v^h_N\rangle_\Gamma+j(\mathbf{u}^h,\mathbf{v}^h)\\&+j(\mathbf{u},\mathbf{u})-j(\mathbf{u},\mathbf{v}^h)-j(\mathbf{u}^h,\mathbf{u})+j(\mathbf{u},\mathbf{u}^h)+j(\mathbf{u}^h,\mathbf{u})-j(\mathbf{u},\mathbf{u})\\&-j(\mathbf{u}^h,\mathbf{u}^h)+a(\mathbf{u};\mathbf{u},\mathbf{v}^h-\mathbf{u})+\big(\sigma_H,\dot{\varepsilon}_V(\mathbf{v}^h)-\dot{\varepsilon}_V(\mathbf{u})\big)_0+j(\mathbf{u},\mathbf{v}^h)\\&-j(\mathbf{u},\mathbf{u})-\langle\sigma_N,v^h_N-u_N\rangle_\Gamma.\end{aligned}\tag{4.16}$$

Using then properties (3.10)–(3.12), relations (3.6) and applying Green's formula for the terms in last two lines of (4.16), for a sufficiently small coefficient of friction, we obtain

$$\begin{aligned}&(m-c)\|\mathbf{u}-\mathbf{u}^h\|^2_V\\\le&(M+c)\|\mathbf{u}-\mathbf{u}^h\|_V\|\mathbf{u}-\mathbf{v}^h\|_V+C_3\|\sigma_H-\sigma^h_H\|_0\|\mathbf{u}-\mathbf{v}^h\|_V\\&+C_4\|\sigma_N-\sigma^h_N\|_{S'}\|\mathbf{u}-\mathbf{v}^h\|_V+C_5\|\boldsymbol{\sigma}_T(\mathbf{u})\|_{\alpha-\frac{3}{2},\Gamma}\|\mathbf{u}_T-\mathbf{v}^h_T\|_{-\alpha+\frac{3}{2},\Gamma}\\&+C_6\|\sigma_N(\mathbf{u})\|_{\alpha-\frac{3}{2},\Gamma}\|\mathbf{u}_T-\mathbf{v}^h_T\|_{-\alpha+\frac{3}{2},\Gamma},\end{aligned}\tag{4.17}$$

where C_3, C_4, C_5 and C_6 are positive constants. Applying then the finite elements approximation properties (4.1) we obtain

$$\begin{aligned}&\|\mathbf{u}-\mathbf{u}^h\|^2_V\\\le&C_7h^r\|\mathbf{u}-\mathbf{u}^h\|_V+C_8h^r\|\sigma_H-\sigma^h_H\|_0+C_9h^r\|\sigma_N-\sigma^h_N\|_{S'}+C_{10}h^{2r},\end{aligned}\tag{4.18}$$

where C_7, C_8, C_9 and C_{10} are positive constants. Further, replacing $\mathbf{v}$ with $\mathbf{v}^h$ in (3.7), we obtain

$$\begin{aligned}&\big[a(\mathbf{u};\mathbf{u},\mathbf{u})+j(\mathbf{u},\mathbf{u})+\big(\sigma_H,\dot{\varepsilon}_V(\mathbf{u})\big)_0-\langle\sigma_N,u_N\rangle_\Gamma\big]\\&\quad+\langle\sigma_N,v^h_N\rangle_\Gamma-\big(\sigma_H,\dot{\varepsilon}_V(\mathbf{v}^h)\big)_0\le a(\mathbf{u};\mathbf{u},\mathbf{v}^h)+j(\mathbf{u},\mathbf{v}^h).\end{aligned}\tag{4.19}$$

Then, from (4.12) and (4.19), taking into account that quantities in brackets are non-negative, for any $\mathbf{v}^h\in\mathbf{U}_h$, we have that holds either

$$\begin{aligned}0&\le a(\mathbf{u}^h;\mathbf{u}^h,\mathbf{v}^h)+j(\mathbf{u}^h,\mathbf{v}^h)-\langle\sigma^h_N,v^h_N\rangle_\Gamma+\big(\sigma^h_H,\dot{\varepsilon}_V(\mathbf{v}^h)\big)_0\\&\le a(\mathbf{u};\mathbf{u},\mathbf{v}^h)+j(\mathbf{u},\mathbf{v}^h)-\langle\sigma_N,v^h_N\rangle_\Gamma+\big(\sigma_H,\dot{\varepsilon}_V(\mathbf{v}^h)\big)_0,\end{aligned}\tag{4.20}$$

or

$$\begin{aligned}0&\le a(\mathbf{u};\mathbf{u},\mathbf{v}^h)+j(\mathbf{u},\mathbf{v}^h)-\langle\sigma_N,v^h_N\rangle_\Gamma+\big(\sigma_H,\dot{\varepsilon}_V(\mathbf{v}^h)\big)_0\\&< a(\mathbf{u}^h;\mathbf{u}^h,\mathbf{v}^h)+j(\mathbf{u}^h,\mathbf{v}^h)-\langle\sigma^h_N,v^h_N\rangle_\Gamma+\big(\sigma^h_H,\dot{\varepsilon}_V(\mathbf{v}^h)\big)_0.\end{aligned}\tag{4.21}$$

Therefore, after adding and subtracting additional terms, it follows that

$$\begin{aligned}&\big\langle\pm(\tau^h_N-\sigma^h_N),v^h_N\big\rangle_\Gamma+\big(\mp(q^h-\sigma^h_H),\dot{\varepsilon}_V(\mathbf{v}^h)\big)_0\\\le&\big|a(\mathbf{u};\mathbf{u},\mathbf{v}^h)-a(\mathbf{u}^h;\mathbf{u}^h,\mathbf{v}^h)\big|+\big|j(\mathbf{u},\mathbf{v}^h)-j(\mathbf{u}^h,\mathbf{v}^h)\big|+\big|\big\langle-(\sigma_N-\tau^h_N),v^h_N\big\rangle_\Gamma\big|\\&+\big|\big(\sigma_H-q^h,\dot{\varepsilon}_V(\mathbf{v}^h)\big)_0\big|\le C'_1\|\mathbf{u}-\mathbf{u}^h\|_V\|\mathbf{v}^h\|_V+C'_2\|\sigma_N-\tau^h_N\|_{S'}\|\mathbf{v}^h\|_V\\&+C'_3\|\sigma_H-q^h\|_0\|\mathbf{v}^h\|_V,\end{aligned}\tag{4.22}$$

where C'_1, C'_2, and C'_3 are positive constants. Applying then the discrete Babuška-Brezzi condition to

$$\begin{aligned}&\sup_{\mathbf{v}^h\in\mathbf{U}_h}\frac{\big\langle\pm(\tau^h_N-\sigma^h_N),v^h_N\big\rangle_\Gamma+\big(\mp(q^h-\sigma^h_H),\dot{\varepsilon}_V(\mathbf{v}^h)\big)_0}{\|\mathbf{v}^h\|_V}\\\le&C'_1\|\mathbf{u}-\mathbf{u}^h\|_V+C'_2\|\sigma_N-\tau^h_N\|_{S'}+C'_3\|\sigma_H-q^h\|_0,\end{aligned}\tag{4.23}$$

we obtain

$$\begin{aligned}&\alpha^h_b\|\tau^h_N-\sigma^h_N\|_{S'}+\beta^h_b\|q^h-\sigma^h_H\|_0\\\le&C'_1\|\mathbf{u}-\mathbf{u}^h\|_V+C'_2\|\sigma_N-\tau^h_N\|_{S'}+C'_3\|\sigma_H-q^h\|_0.\end{aligned}\tag{4.24}$$

Using then the triangle inequality, the inequality (4.24) and the finite element approximation properties (4.7), (4.8), we have correspondingly

$$\begin{aligned}\|\sigma_H-\sigma^h_H\|_0&\le\|\sigma_H-q^h\|_0+\|q^h-\sigma^h_H\|_0\\&\le C'_4h^s+\frac{C'_5}{\beta^h_b}(h^s+h^r)+\frac{C'_1}{\beta^h_b}\|\mathbf{u}-\mathbf{u}^h\|_V,\end{aligned}\tag{4.25}$$

$$\begin{aligned}\|\sigma_{N}-\sigma_{N}^{h}\|_{S'} &\leq \|\sigma_{N}-\tau_{N}^{h}\|_{S'}+\|\tau_{N}^{h}-\sigma_{N}^{h}\|_{S'}\\ &\leq C_6' h^r+\frac{C_7'}{\alpha_b^h}(h^s+h^r)+\frac{C_1'}{\alpha_b^h}\|\mathbf{u}-\mathbf{u}^h\|_V,\end{aligned}\tag{4.26}$$

where C_4', C_5', C_6', and C_7' are positive constants. Replacing (4.25) and (4.26) in (4.18), we obtain

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|_V^2 \leq{}& C_7 h^r\|\mathbf{u}-\mathbf{u}^h\|_V+\frac{C_8' h^r}{\alpha_b^h}\|\mathbf{u}-\mathbf{u}^h\|_V+\frac{C_9' h^r}{\beta_b^h}\|\mathbf{u}-\mathbf{u}^h\|_V\\ &+\frac{C_{10}'}{\alpha_b^h}(h^{2r}+h^{s+r})+\frac{C_{11}'}{\beta_b^h}(h^{2r}+h^{s+r})+C_{12}'(h^{2r}+h^{s+r}),\end{aligned}\tag{4.27}$$

where C_8', C_9', C_{10}', C_{11}' and C_{12}' are positive constants. Applying to the first three terms in the right hand side of (4.27), the ε-inequality

$$|ab|\leq \varepsilon a^2+\frac{b^2}{4\varepsilon},\quad \forall\,\varepsilon>0,\quad a,b\in\mathbb{R},$$

with appropriately chosen ε, we obtain (4.11a), where C_0 is a positive constant. From (4.25), (4.26) and (4.11a), we then obtain (4.11b) and (4.11c), where C_0' and C_0'' are positive constants, which completes the proof. □

Remark 4.3. It could be easily checked that, if α_b^h and β_b^h are independent of h, the finite element solutions $\{\mathbf{u}^h,\sigma_H^h,\sigma_N^h\}$ are convergent and the derived error estimates (4.11) are of optimal order. Despite that it could be shown that α_b^h is independent of h for the used here combinations of finite element spaces $\{\mathbf{U}_h,\Lambda_h,S_h'\}$, β_b^h is independent of h however only for the space using $\{Q_2,P_0,P_2\}$ finite elements, providing error estimates of suboptimal order, or for the space using $\{Q_1,P_0,P_1\}$ finite elements, such that on every macroelement, constituting of $(2\times 2)-P_0$ elements, the pressure filter property

$$\sigma_H^{h,1}+\sigma_H^{h,3}=\sigma_H^{h,2}+\sigma_H^{h,4},$$

holds [11,24]. For the finite element combinations $\{Q_2,Q_1,P_2\}$, $\{Q_1,Q_1,P_1\}$ and general $\{Q_1,Q_0,P_1\}$, $\beta_b^h=\mathcal{O}(h)$ as in the first case, despite that $\{\mathbf{u}^h,\sigma_N^h\}$ converges, $\{\sigma_H^h\}$ is numerically unstable, while in the other cases $\{\mathbf{u}^h,\sigma_N^h\}$ is unstable and $\{\sigma_H^h\}$ diverges. Since the rectangular meshes may define a certain kind of pressure filter, satisfactory results can be obtained for some fixed mesh. The instabilities and divergence however, can be easily observed at changing, or refining the mesh.

Remark 4.4. If $\tau_N\in S'\cap L_2(\Omega)$ and also the following discrete Babuška-Brezzi condition is assumed to hold:

$$\begin{aligned}&\hat{\alpha}_b^h\|\tau_N^h\|_{0,\Gamma}+\beta_b^h\|q^h\|_0\\ &\leq \sup_{\mathbf{v}^h\in\mathbf{U}_h}\frac{(\tau_N^h,v_N^h)_{0,\Gamma}+\big(q^h,\dot{\varepsilon}_V(\mathbf{v}^h)\big)_0}{\|\mathbf{v}^h\|_V},\quad \forall\,\tau_N^h\in S_h',\ \forall\,q^h\in\Lambda_h,\end{aligned}\tag{4.28}$$

where $\hat{\alpha}_b^h$ and β_b^h are positive numbers, then it can be proved that $\hat{\alpha}_b^h = \mathcal{O}(h^{1/2})$ and a priori error estimate of suboptimal order for $\|\sigma_{\scriptscriptstyle N} - \sigma_{\scriptscriptstyle N}^h\|_{0,\Gamma}$ also can be derived.

Remark 4.5. If in the finite element approximation of problem (3.7) the following finite dimensional space is used

$$\Lambda_h = \{q^h : q^h \in \Lambda \cap C^0(\bar{\Omega}),\ \ q^h_{|_K} \in Q_1(K)\},$$

where $Q_1(K)$ is the space of bilinear polynomials defined on K, with the approximation properties (4.7), if $\tau_{\scriptscriptstyle N} \in S' \cap L_2(\Omega)$, $q \in \Lambda \cap H_0^1(\Omega)$ and also the following discrete Babuška-Brezzi condition is assumed to hold:

$$\begin{aligned} &\hat{\alpha}_b^h \|\tau_{\scriptscriptstyle N}^h\|_{0,\Gamma} + \hat{\beta}_b^h \|\nabla q^h\|_0 \\ &\le \sup_{\mathbf{v}^h \in \mathbf{U}_h} \frac{(\tau_{\scriptscriptstyle N}^h, v_{\scriptscriptstyle N}^h)_{0,\Gamma} + |(\nabla q^h, \mathbf{v}^h)_0|}{\|\mathbf{v}^h\|_V}, \quad \forall\, \tau_{\scriptscriptstyle N}^h \in S_h',\ \ \forall\, q^h \in \Lambda_h, \end{aligned} \tag{4.29}$$

where $\hat{\alpha}_b^h$ and $\hat{\beta}_b^h$ are positive numbers, then it can be proved that

$$\hat{\alpha}_b^h = \mathcal{O}(h^{\frac{1}{2}}), \quad \hat{\beta}_b^h = \mathcal{O}(h),$$

and following [11] a priori error estimates of optimal order can be derived.

5 Algorithm and results

Applying the secant-modulus method to the regularized problems (3.18), (3.19), after a finite element discretization, we obtain the following, formally equivalent, finite dimensional problems:

1. Find $\{\mathbf{u}_{n+1}^h, \sigma_{\scriptscriptstyle H n+1}^h, \sigma_{\scriptscriptstyle N n+1}^h\} \in \mathbf{U}_h \times \Lambda_h \times S_h'$, $n = 0, 1, 2, \cdots$, satisfying for arbitrary initial $\mathbf{u}_0^h \in \mathbf{U}_h$, the system of equations

$$\begin{aligned} &a(\mathbf{u}_n^h; \mathbf{u}_{n+1}^h, \mathbf{v}^h - \mathbf{u}_{n+1}^h) + \big(\sigma_{\scriptscriptstyle H n+1}^h, \dot{\varepsilon}_V(\mathbf{v}^h - \mathbf{u}_{n+1}^h)\big)_0 \\ &\quad + \big\langle j'_{d_{\scriptscriptstyle T}}(\mathbf{u}_n^h, \mathbf{u}_{n+1}^h), \mathbf{v}^h - \mathbf{u}_{n+1}^h \big\rangle = \big\langle \sigma_{\scriptscriptstyle N n+1}^h, v_{\scriptscriptstyle N}^h - u_{\scriptscriptstyle N n+1}^h \big\rangle_\Gamma, \ \ \forall\, \mathbf{v}^h \in \mathbf{U}_h, \end{aligned} \tag{5.1a}$$

$$\big(q, \dot{\varepsilon}_V(\mathbf{u}_{n+1}^h)\big)_0 - d(q, \sigma_{\scriptscriptstyle H n+1}^h)_0 = 0, \qquad \forall\, q^h \in \Lambda_h, \tag{5.1b}$$

$$\big\langle \tau_{\scriptscriptstyle N}^h, u_{\scriptscriptstyle N n+1}^h \big\rangle_\Gamma + d_{\scriptscriptstyle N} \big\langle \tau_{\scriptscriptstyle N}^h, \sigma_{\scriptscriptstyle N n+1}^h \big\rangle_\Gamma = 0, \quad \forall\, \tau_{\scriptscriptstyle N}^h \in S_h', \tag{5.1c}$$

2. Find $\mathbf{u}_{n+1}^h \in \mathbf{U}_h$, $n = 0, 1, 2, \cdots$, satisfying for arbitrary initial $\mathbf{u}_0^h \in \mathbf{U}_h$, the equation

$$b\big(\mathbf{u}_n^h; \mathbf{u}_{n+1}^h, \mathbf{v}^h - \mathbf{u}_{n+1}^h\big) + \big\langle j'_{d_{\scriptscriptstyle T}}(\mathbf{u}_n^h, \mathbf{u}_{n+1}^h), \mathbf{v}^h - \mathbf{u}_{n+1}^h \big\rangle = 0, \quad \forall\, \mathbf{v}^h \in \mathbf{U}_h, \tag{5.2}$$

until

$$\frac{\|\mathbf{u}_{n+1}^h - \mathbf{u}_n^h\|}{\|\mathbf{u}_{n+1}^h\|} < \delta,$$

where $\|\cdot\|$ is a vector norm and δ is the accuracy tolerance. Problem (5.2) then defines the following algorithm [20].

Algorithm: Find $\{u^h_{n+1}\}$, $n = 0,1,2,\cdots$, satisfying for arbitrary initial $\{u^h_0\}$ the system of equations

$$\mathbf{K}(u^h_n)\{u^h_{n+1}\} = \mathbf{F}(u^h_n), \tag{5.3}$$

until $\|u^h_{n+1} - u^h_n\|/\|u^h_{n+1}\| < \delta$.

Here $\mathbf{K}$ and $\mathbf{F}$ are the velocity dependent stiffness matrix and load vector, $\{u^h_{n+1}\}$ is the vector of nodal velocities.

Remark 5.1. It is important to note that, using some special integration technique for the volumetric strain-rates term in (5.2), we may obtain a complete equivalence between problems (5.1) and (5.2), i.e., the finite element solutions of (5.1) to be also solutions to (5.2) and vice-versa. Thus for example, when (1×1)-Gauss integration of the volumetric strain-rates in (5.2) is used, then Q_1 approximation of (5.2) is equivalent to the unstable $\{Q_1, P_0, P_1\}$ approximation of (5.1). Q_2 approximation of (5.2) with (2×2)-Gauss integration of the volumetric strain-rates is equivalent to $\{Q_2, P_1, P_2\}$ approximation of (5.1) only on rectangular meshes. Even on rectangular meshes however, Q_2 approximation of (5.2) with (1×1)-Gauss integration of the volumetric strain-rates is not equivalent to the stable $\{Q_2, P_0, P_2\}$ approximation of (5.1). In this case, the lower order of integration, leads to a lower order of approximation. Since, if the integration rule, applied to the volumetric strain-rate - hydrostatic pressure terms in both problems, is such that obtained auxiliary problems are equivalent and approximate the continuous problems at $h\to 0$, the order of inexact integration defines the order of approximation, the integration rules may serve to define approximation spaces. A theory for such problems have been developed in [11].

The algorithm (5.3) is applied here, with complete (3×3) and reduced (2×2), or (1×1)-Gauss integration of the volumetric strain-rate term, for solving the following dimensionless extrusion problem [4,5], with known analytical slip-line solution [1,2].

Example: A two-dimensional workpiece with length 20, initial and exit thicknesses 10 and 5 respectively, is extruded through a square die with ram velocity $u^0_N=1$. The following yield limit, satisfying (2.11) for all $\dot{\bar{\varepsilon}}\in[0,\infty)$, is used

$$\sigma_p(\dot{\bar{\varepsilon}}) = \begin{cases} \sigma_p(\dot{\bar{\varepsilon}}_1)\dot{\bar{\varepsilon}}/\dot{\bar{\varepsilon}}_1, & \text{if } \dot{\bar{\varepsilon}} \in [0, \dot{\bar{\varepsilon}}_1], \\ A\dot{\bar{\varepsilon}}^{\alpha}, & \text{if } \dot{\bar{\varepsilon}} \in [\dot{\bar{\varepsilon}}_1, \dot{\bar{\varepsilon}}_2], \\ \sigma_p(\dot{\bar{\varepsilon}}_2)\dot{\bar{\varepsilon}}/\dot{\bar{\varepsilon}}_2, & \text{if } \dot{\bar{\varepsilon}} \in [\dot{\bar{\varepsilon}}_2, \infty), \end{cases} \tag{5.4}$$

where $A>0$, $\alpha\in(0,1]$, $\dot{\bar{\varepsilon}}_1$ and $\dot{\bar{\varepsilon}}_2$ are material constants, depending on the process conditions, with values in our case taken $A=\sqrt{3}$, $\alpha=10^{-3}$; $\dot{\bar{\varepsilon}}_1=10^{-3}$ and $\dot{\bar{\varepsilon}}_2=10^{3}$. Friction coefficient values $\mu=0.0, 0.1, 0.2$ are used.

Two regular finite element meshes have been constructed, *meshI* and *meshII* in Fig. 2, containing quadrilateral finite elements, with sides $h_I\approx 1.667$ and $h_{II}=1$ correspondingly. The values of the regularization and penalty constants are taken, on the

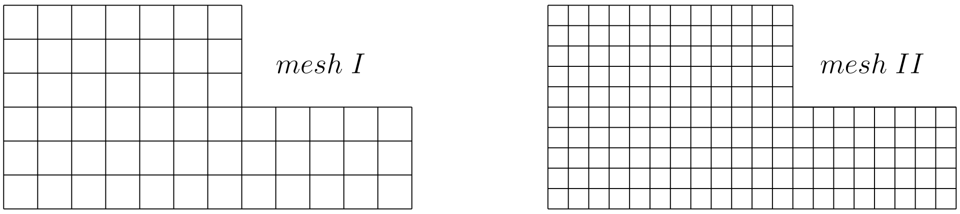


Figure 2: Finite element meshes.

base of computational experiments, respectively $d_T=10^{-6}$, $d=10^{-3}$, $d_N=10^{-6}$. Further decreasing leads to computational instabilities and overconstraining, which is due to the mesh dependence of the penalty parameters in the discrete penalty method.

In the computations, a simple averaging of the effective strain-rates, contact and hydrostatic pressures and friction stresses at finite element nodes (centers) is used. The computed equivalent strain-rates are further multiplied by $\sqrt{3}$, to simplify the comparisons with the results presented in [4,5], where they are defined by the expression

$$\dot{\bar{\varepsilon}} = \sqrt{2\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}},$$

in contrast with the used here expression (2.1). Since also the analytical slip-line solution for the extrusion pressure is $p=2.6\tau_p$, where $p=|\sigma_H|$ and $\tau_p=\sigma_p/\sqrt{3}$ is the shear yield limit [1–4], for the same reason in (5.4) $\sigma_p=\sqrt{3}$ is taken, which gives $p=2.6$. The computational experiments show that the algorithm is fast, results are obtained for about 25-30 iterations, depending on the used friction coefficient, within accuracy $\delta=10^{-4}$. A common practice to simulate singularities, such as the singularity in the strain-rates due to the corner node in our case, is the usage of special finite elements [24], ensuring the corresponding behaviour, or local mesh refinements. The computational experiments performed on *meshI*, on a modified, refined around the corner node, *meshI* and on *meshII* show, as mentioned in [4], that singularity only locally affects the results. The distribution of the effective strain-rates in Ω, obtained for these three meshes for the frictionless case, using bilinear finite elements with complete integration, are illustrated on Fig. 3 and Fig. 4(a). These results also support

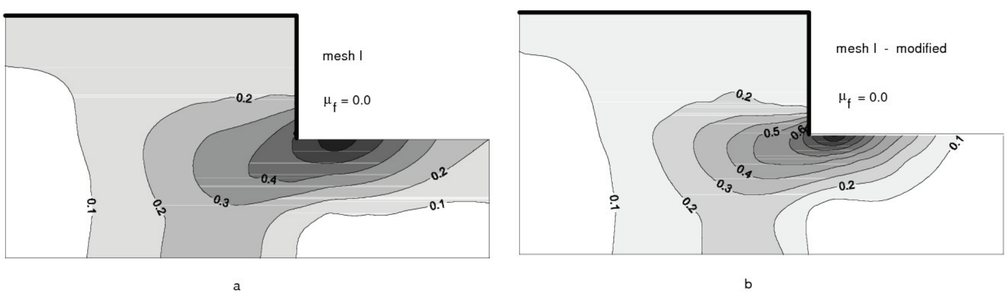
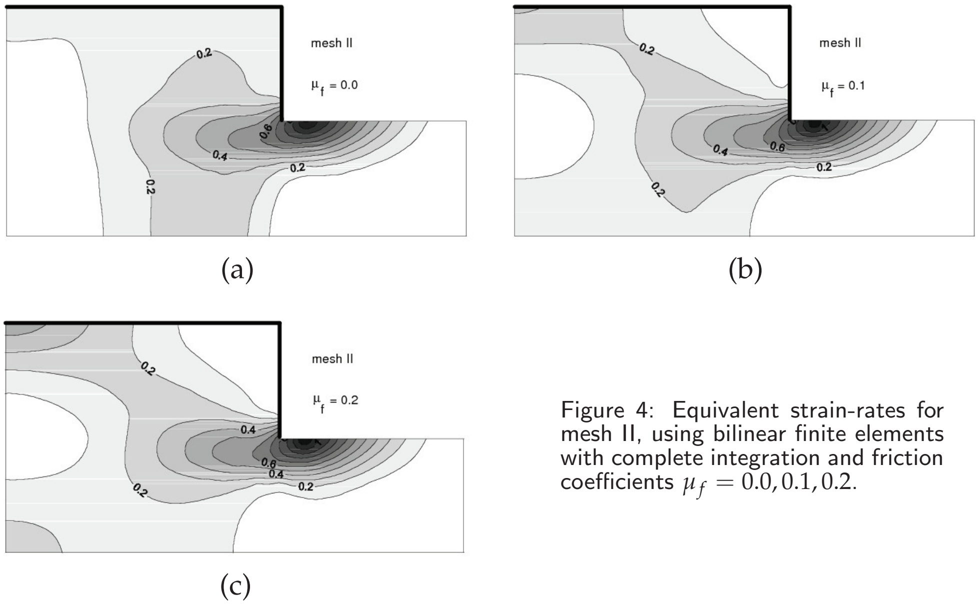


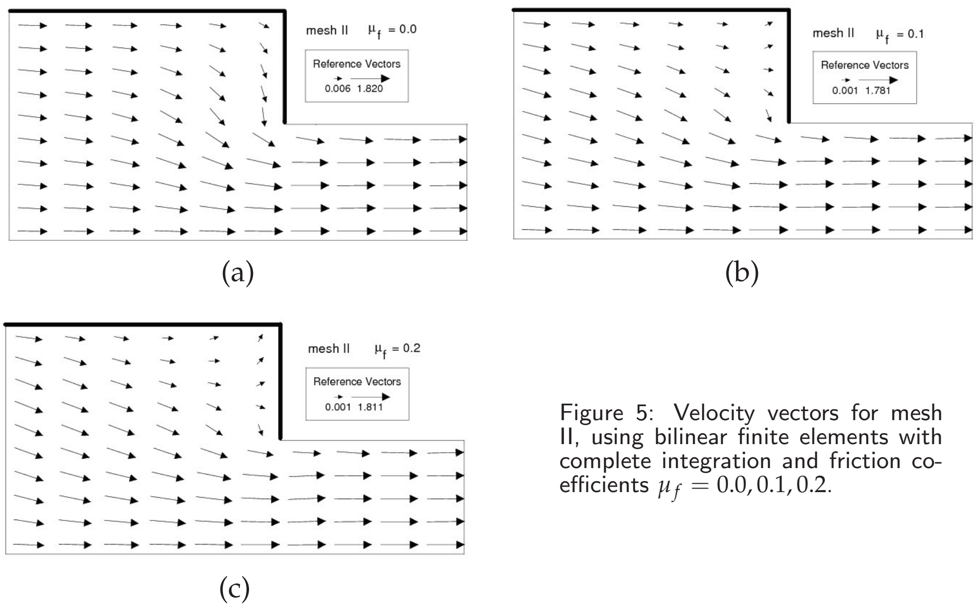
Figure 3: Equivalent strain-rates for mesh I and modified mesh I, using bilinear finite elements with complete integration and friction coefficient $\mu_f = 0.0$.



(a) (b) (c)

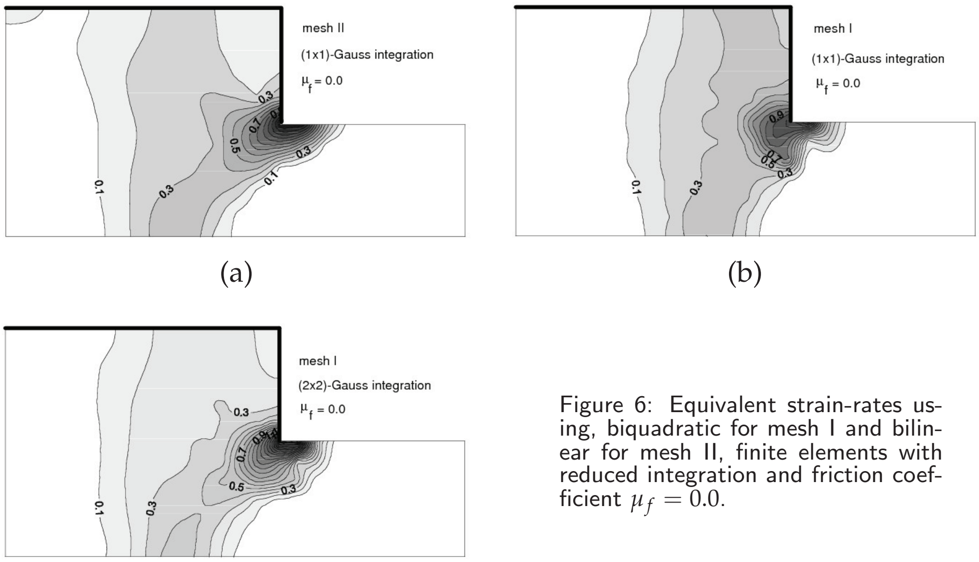
Figure 4: Equivalent strain-rates for mesh II, using bilinear finite elements with complete integration and friction coefficients $\mu_f = 0.0, 0.1, 0.2$.

the obtained theoretical convergence result for the discrete pure penalty problem. Obtained results for the hydrostatic pressure, which is an external variable for the pure penalty problem, show however an overestimation of the analytical result for the extrussion pressure. For the frictionless case on *meshII*, an average hydrostatic pressure value p_{av}=3.7 is obtained. The influence of friction is illustrated on Fig. 4 and Fig. 5, where the obtained distributions of the effective strain-rates and velocity vectors in Ω, using bilinear finite elements with complete integration for *meshII* and friction coeffi-



(a) (b) (c)

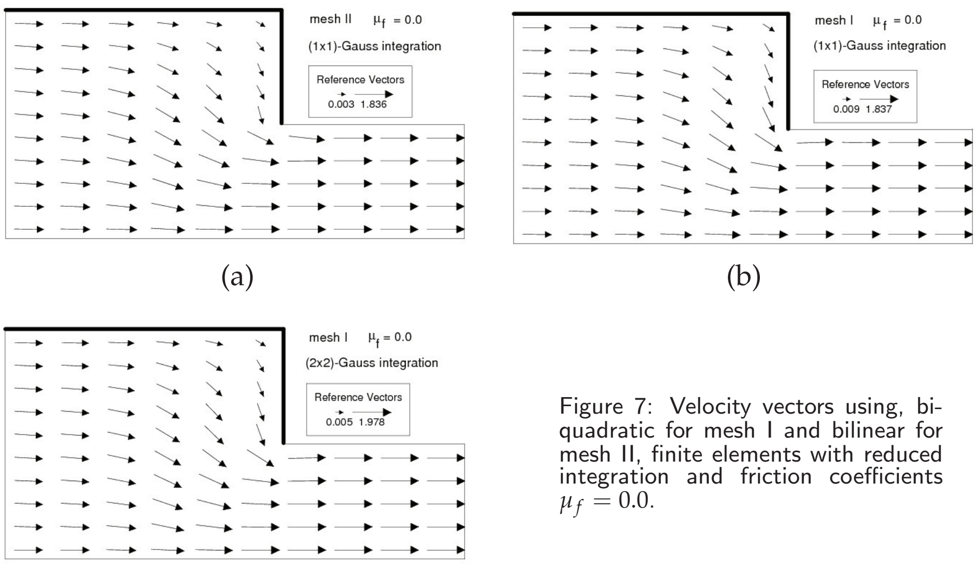
Figure 5: Velocity vectors for mesh II, using bilinear finite elements with complete integration and friction coefficients $\mu_f = 0.0, 0.1, 0.2$.



(a) (b) (c)

Figure 6: Equivalent strain-rates using, biquadratic for mesh I and bilinear for mesh II, finite elements with reduced integration and friction coefficient $\mu_f = 0.0$.

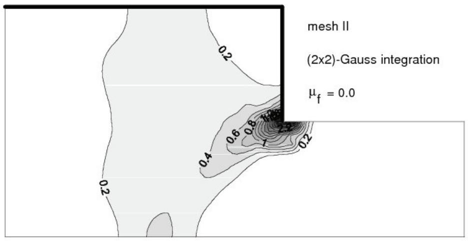
cients μ_f=0.0, μ_f=0.1 and μ_f=0.2, are presented. Fig. 6 and Fig. 7 contain the obtained distributions of the effective strain-rates and velocity vectors in Ω for the frictionless case, using bilinear finite elements with reduced (1×1)-Gauss integration for *meshII* and biquadratic finite elements with reduced (1×1) and (2×2)-Gauss integration for *meshI*. For the average hydrostatic pressure the following values are correspondingly obtained: p_{av}=2.573, p_{av}=2.795 and p_{av}=2.65. It should be mentioned that the results presented on Fig. 6(a), Fig. 7(a) and on Fig. 6(b), Fig. 7(b) are very good for



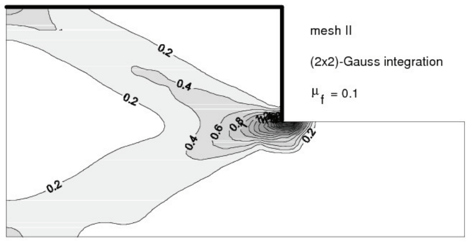
(a) (b) (c)

Figure 7: Velocity vectors using, biquadratic for mesh I and bilinear for mesh II, finite elements with reduced integration and friction coefficients $\mu_f = 0.0$.

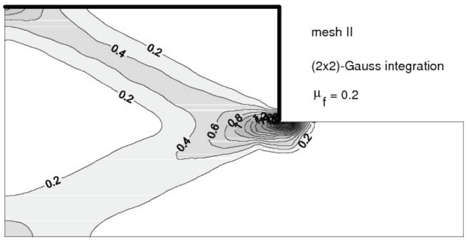
these particular rectangular meshes, despite obtained by using discrete penalty methods that correspond to unstable and stable, but nonequivalent discrete mixed methods, Remark 4.3, Remark 5.1. Fig. 8 and Fig. 9 contain the obtained distributions of the effective strain-rates and velocity vectors in Ω, using biquadratic finite elements with reduced (2×2)-Gauss integration for *meshII* and friction coefficients μ_f=0.0, μ_f=0.1 and μ_f=0.2. These results also support the obtained theoretical convergence resuts and show the influence of friction. For the frictionless case, the obtained aver-



(a)

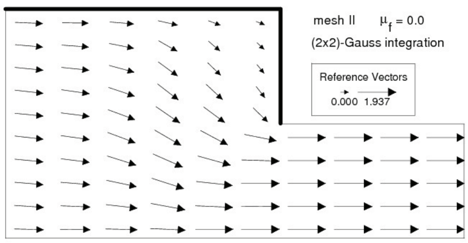


(b)

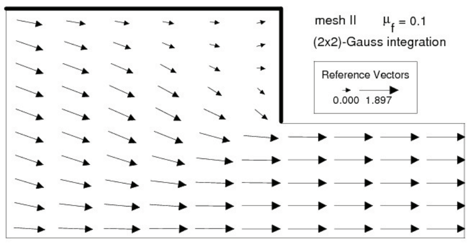


(c)

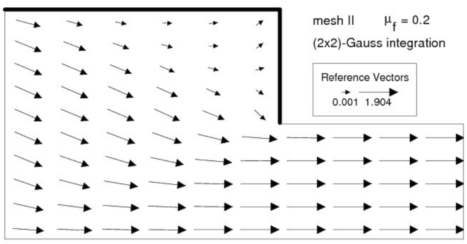
Figure 8: Equivalent strain-rates for mesh II, using biquadratic finite elements with (2×2)-Gauss integration and friction coefficients $\mu_f = 0.0,\ 0.1, 0.2$.



(a)



(b)



(c)

Figure 9: Velocity vectors for mesh II, using biquadratic finite elements with (2×2)-Gauss integration and friction coefficients $\mu_f = 0.0, 0.1, 0.2$.

age hydrostatic pressure p_{av}=2.675 is very close to the analytical extrusion pressure value. It should be mentioned here that, for the frictionless case, the obtained results for the velocities and strain-rates are also very close to these presented in [4,5]. Finally, the computational experiments and comparisons performed, clearly support the obtained theoretical results and demonstrate the applicability and the effectiveness of the proposed method of approach for solving the considered class of metal-forming problems.

6 Conclusions

In this work a variational inequality approach is proposed for analysis of a class of contact problems with friction in the flow theory of plasticity, describing continuous, steady-state metal-forming processes. Existence, uniqueness, approximation and convergence results are obtained and an algorithm, based on the finite element and secant modulus methods, is proposed. The theoretical results are supported by numerical results, which shows the applicability and the effectiveness of the proposed method of approach.

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