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NODAL $\mathcal{O}(h^4)$-SUPERCONVERGENCE IN 3D BY AVERAGING PIECEWISE LINEAR, BILINEAR, AND TRILINEAR FE APPROXIMATIONS*

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Abstract

We construct and analyse a nodal $\mathcal{O}(h^4)$-superconvergent FE scheme for approximating the Poisson equation with homogeneous boundary conditions in three-dimensional domains by means of piecewise trilinear functions. The scheme is based on averaging the equations that arise from FE approximations on uniform cubic, tetrahedral, and prismatic partitions. This approach presents a three-dimensional generalization of a two-dimensional averaging of linear and bilinear elements which also exhibits nodal $\mathcal{O}(h^4)$-superconvergence (ultraconvergence). The obtained superconvergence result is illustrated by two numerical examples.

Mathematics subject classification: 65N30.
Key words: Higher order error estimates, Tetrahedral and prismatic elements, Superconvergence, Averaging operators.

1. Introduction

We consider the Poisson equation with homogeneous Dirichlet boundary condition

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Assume that $\Omega \subset \mathbb{R}^3$ is a bounded rectangular domain and that the right-hand side function $f \in C^4(\overline{\Omega})$.

The weak form of problem (1.1) reads: Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega), \tag{1.2}$$

where $(\cdot,\cdot)$ denotes the scalar products in both $L_2(\Omega)$ and $(L_2(\Omega))^3$.

In [15], Schatz referred about the nodal $\mathcal{O}(h^4)$-superconvergence of quadratic elements on uniform tetrahedral partitions (i.e., for each internal edge e the patch of tetrahedra sharing e

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is a point symmetric set with respect to the midpoint of e). This result is extended by Schatz, Sloan, and Wahlbin [16] to locally symmetric meshes. Since each uniform tetrahedralization is locally point-symmetric with respect to the midpoints of edges, the $\mathcal{O}(h^4)$-superconvergence of quadratic tetrahedral elements holds at these midpoints as well.

Linear triangular elements also exhibit nodal $\mathcal{O}(h^4)$-superconvergence (ultraconvergence) on uniform triangulations consisting solely of equilateral triangles. This result was obtained by Lin and Wang in [14] (see also [2]). It is based on the fact that the corresponding stiffness FE matrix is the same as the matrix associated to the standard 7-point finite difference scheme, which is $\mathcal{O}(h^4)$-accurate. However, this result cannot be extended to three-dimensional space, since the regular tetrahedron is not a space-filler (see, e.g., [4, 11]).

The study of superconvergence by a computer-based approach developed by Babuška et al. [1] requires to examine harmonic polynomials in the plane. Note that the dimension of the space of harmonic polynomials of degree $k \in \{1, 2, \dots\}$ in two variables is only 2, whereas the dimension of such a space in three variables is $2k+1$. This makes superconvergence analysis for $d = 3$ much more difficult (see, e.g., [17]) than for $d = 2$. The likelihood of $2k + 1$ polynomial graphs passing through a common point is much smaller than the probability of two intersecting polynomial graphs.

A suitable averaging of gradients of FE solutions leads to superconvergence, see, e.g., [5,6]. In this paper we show that an averaging of stiffness matrices of several kind of elements exhibits also a superconvergence. In particular, here we will present an averaging of linear algebraic equations arising from FE approximations of problem (1.2) on uniform partitions of $\overline{\Omega}$ into cubes, tetrahedra, and triangular prisms, respectively. The method is an extension of the nodal $\mathcal{O}(h^4)$-superconvergence result for the Poisson equation in two-dimensional domains, where the stiffness matrices corresponding to linear and bilinear elements are appropriately averaged [12,13] to obtain the matrix associated to the standard 9-point finite difference scheme. To the authors' knowledge, extension of this result to the three-dimensional case has not yet been studied. Note that the size (and also the band-width) of the resulting matrix will be the same as for the stiffness matrix corresponding to trilinear finite elements, which produces only $\mathcal{O}(h^2)$-accuracy in the maximum norm at nodes.

2. Construction of the Averaged FE Scheme

2.1. Preliminaries

Assume that $\mathcal{T}_h$ is a uniform face-to-face partition of the domain $\overline{\Omega}$ into cubes. We denote the set of interior nodes of $\mathcal{T}_h$ by $\mathcal{N}_h = \{z_i\}_{i=1}^N$, where $N = N(h)$ and h is the length of any edge.

In order to introduce the relevant FD and FE schemes, we shall use the compact notation from [9]. To this end, the nodes in the FD stencil (see Figure 2.1) are divided into three separate groups (midpoints of faces, vertices, and midpoints of edges) and the following conventional summations

$$\begin{aligned}
\Diamond U_0 &= U_1 + U_2 + U_3 + U_4 + U_{13} + U_{14},\\
\bigcirc U_0 &= U_{19} + U_{20} + U_{21} + U_{22} + U_{23} + U_{24} + U_{25} + U_{26},\\
\Box U_0 &= U_5 + U_6 + U_7 + U_8 + U_9 + U_{10} + U_{11} + U_{12} + U_{15} + U_{16} + U_{17} + U_{18},
\end{aligned}$$

are used, where the value U_0 corresponds to the (central) vertex z_i and $U_1, \dots, U_{26}$ stand for the neighbouring vertices as sketched in Figure 2.1.

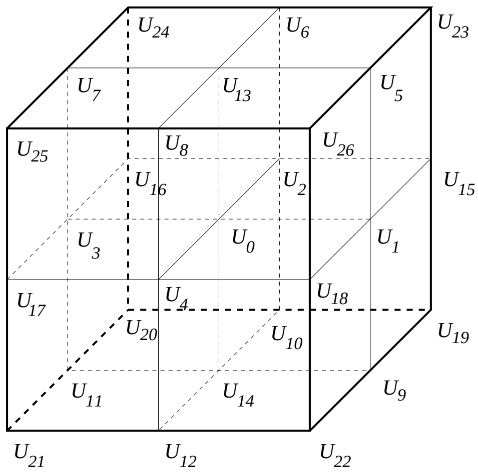

Fig. 2.1. Numbering of nodes with respect to the central node U_0.

Using this notation, the fourth-order accurate 19-point FD scheme can be written as follows (see [9, p. 600])

$$24U_0 - 2\Diamond U_0 - \Box U_0 = 6h^2 f(z_i) + h^4 \Delta f(z_i). \tag{2.1}$$

From now on, we will use the notation u_h for the finite element solution and $\vec{u}_h$ for the vector of its nodal values at z_i, i.e.,

$$(\vec{u}_h)_i = u_h\,(z_i)\,. \tag{2.2}$$

The nodal superconvergence will be measured in the discrete ℓ^2-norm

$$\|u - u_h\|_h = \left(h^3 \sum_{i=1}^{N} (u(z_i) - u_h(z_i))^2 \right)^{1/2}.$$

We will use the same notation also for vectors $\vec{x} = (x_1, \ldots, x_N)^\top$:

$$\|\vec{x}\|_h = \left(h^3 \sum_{i=1}^{N} x_i^2 \right)^{1/2}, \tag{2.3}$$

and for the induced matrix norm. Notice that $\|\vec{x}\|_h = h^{3/2}\|\vec{x}\|_2$, where $\|\cdot\|_2$ is the standard Euclidean norm.

The principal idea in the derivation of the superconvergent FE scheme in our work is to ensure a certain "closeness" between two systems of linear algebraic equations

$$\Delta_h \vec{U}_h = \vec{f}_h \quad \text{and} \quad A_h \vec{u}_h = \vec{F}_h, \tag{2.4}$$

arising from FD scheme (2.1) and from the averaged FE scheme corresponding to (1.2), respectively. In more detail, we will construct the matrix A_h so that

$$A_h = h\Delta_h, \tag{2.5}$$

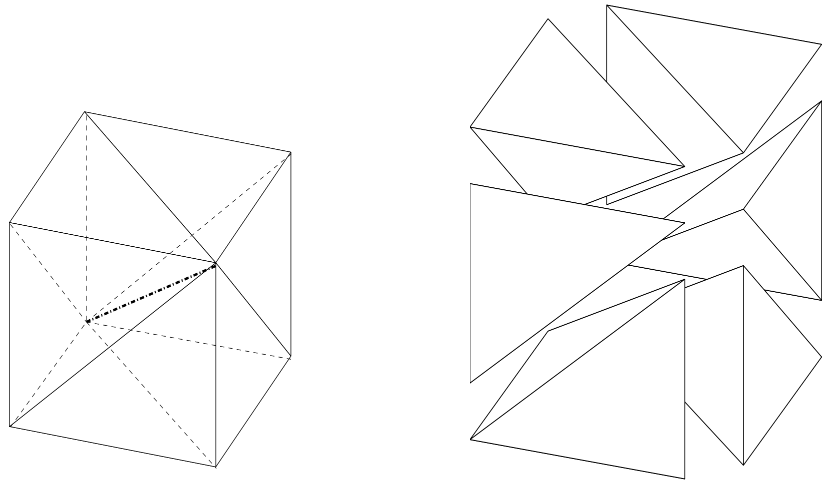

Fig. 2.2. The Kuhn partition of a cube into six non-obtuse tetrahedra.

and we prove that the right-hand side vectors in (2.4) satisfy the estimate

$$\|h\vec{f}_h - \vec{F}_h\|_\infty \le Ch^7, \tag{2.6}$$

where $\|\cdot\|_\infty$ stands for the maximum norm and $(\vec{f}_h)_i = 6h^2 f(z_i) + h^4 \Delta f(z_i)$.

2.2. Averaging of FE approximations

In this subsection, we construct the FE scheme for which requirements (2.5) and (2.6) are satisfied. In order to meet requirement (2.5), we shall use cubic, tetrahedral, and prismatic FE partitions.

Applying the FE discretization on a uniform cubic partition gives the following equation for entries that appear in one row of the stiffness matrix

$$A^c = \frac{8h}{3}\,u_0 - \frac{h}{12} \bigcirc u_0 - \frac{h}{6}\,\Box u_0 \tag{2.7}$$

at each node located at the interior of the domain, where $u_0 = u(z_i)$ for simplicity. The coefficient at the diamond term $\Diamond u_0$ is zero, since the scalar product of the gradients of two basis functions related to adjacent nodes is zero.

A cube can be decomposed into six tetrahedra that share a spatial diagonal (see Figure 2.2). Since each cube has four spatial diagonals, there exists four different such uniform tetrahedral partitions (see Figure 2.3). Although all these partitions yield the same local equation, the averaging requires usage of all associated partitions. This is due to the symmetry requirement, which will be stated later in the proof of Lemma 3.1. Applying the FE method on any of the four tetrahedral partitions gives the following contributions to the local stiffness matrix

$$A^t = 6h\,u_0 - h\Diamond u_0. \tag{2.8}$$

An easy calculation shows that the coefficients standing at the terms $\bigcirc u_0$ and $\Box u_0$ are zero.

We will employ prismatic elements in a similar manner as tetrahedral elements. Again, due to the symmetry requirement we need to use all six possible uniform prismatic partitions (see

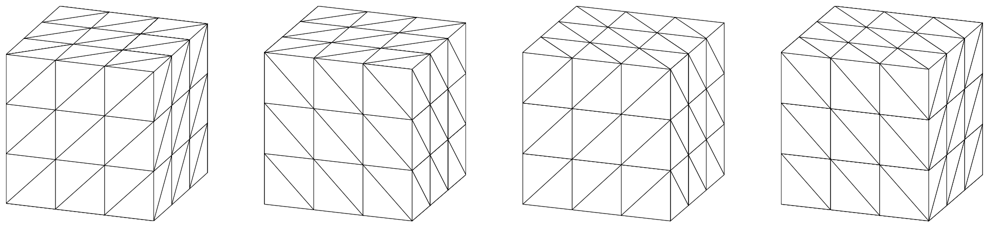

Fig. 2.3. Four different types of tetrahedral partitions.

Figure 2.4). Individual local equations do not fit into our notational framework, but summing all local equations gives

$$A^p = 22hu_0 - \frac{5h}{3}\Diamond u_0 - \frac{h}{4}\bigcirc u_0 - \frac{5h}{6}\Box u_0. \tag{2.9}$$

This local equation will be applied in the averaging.

The global stiffness matrix arising from cubic elements is denoted by A_c, from tetrahedral elements by A_t^j ($j = 1, \ldots, 4$), and from prismatic elements by A_p^k ($k = 1, \ldots, 6$). Notice that

$$A_t^1 = A_t^2 = A_t^3 = A_t^4. \tag{2.10}$$

By summing all these matrices with appropriate weights as follows

$$A_h = -9A_c - \frac{3}{4}\sum_{j=1}^{4} A_t^j + 3\sum_{k=1}^{6} A_p^k, \tag{2.11}$$

we obtain the matrix A_h for the averaged FE scheme. The corresponding combination of local equations (2.7), (2.8), and (2.9) gives the equation

$$A = 24hu_0 - 2h\Diamond u_0 - h\Box u_0 \tag{2.12}$$

for the averaged FE scheme. The above matrix is the same (up to the factor h) as the matrix of the 19-point FD scheme (2.1), i.e., we have $A_h = h\Delta_h$. The matrices A_c, A_t^j, A_p^k are symmetric and positive definite. In Lemma 3.2, we prove that the averaged matrix A_h is also symmetric and positive definite, even though some weights in (2.11) are negative.

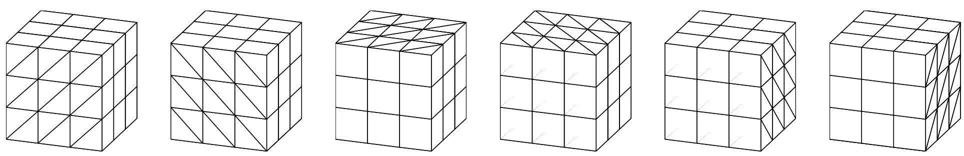

Fig. 2.4. Six different types of prismatic partitions.

3. Superconvergence Properties

In the previous section, we presented an averaged FE scheme with the same system matrix (up to the factor h) as the 19-point finite difference formula. In order to prove the superconvergence property for approximation obtained by the proposed method, we show that condition (2.6) is satisfied.

In what follows, we set

$$v_i = -9c_i - \frac{3}{4}\sum_{j=1}^{4} t_i^j + 3\sum_{k=1}^{6} p_i^k, \tag{3.1}$$

where c_i, t_i^j, and p_i^k are cubic, tetrahedral, and prismatic basis functions related to the node z_i ($i = 1, \dots, N$), respectively. Notice that v_i is piecewise trilinear on each cube that is partitioned into 24 subtetrahedra (see Figure 3.1).

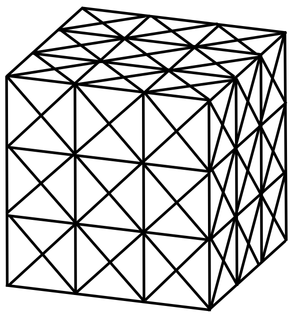

Fig. 3.1. Support of the averaged basis function v_i consists of 27 cubes each of which is partitioned into $24 = 6 \times 4$ tetrahedra.

Now, we prove estimate (2.6).

Lemma 3.1. *Let $f \in C^4(\overline{\Omega})$. Then*

$$\begin{aligned} \|h\vec{f}_h - \vec{F}_h\|_\infty &= \max_{i=1,\dots,N} |\,(f, v_i) - 6h^3 f(z_i) - h^5 \Delta f(z_i)| \\ &\le Ch^7 \|f\|_{C^4(\overline{\Omega})} \ \textit{as } h \to 0, \end{aligned} \tag{3.2}$$

where v_i is the basis function defined in (3.1) *over uniform partitions.*

Proof. Without loss of generality, we can assume $z_i = 0$. Let $S = [-h, h]^3$ be the support of v_i. Due to the symmetry in the partitions, the averaged basis function v_i is even with respect to all coordinate axes, i.e.,

$$v_i(x_1, x_2, x_3) = v_i(-x_1, x_2, x_3) = v_i(x_1, -x_2, x_3) = v_i(x_1, x_2, -x_3). \tag{3.3}$$

Consequently, the integral over the support of the averaged basis function v_i multiplied with any odd function vanishes.

We can expand f as follows

$$\begin{aligned} f(x_1, x_2, x_3) = f(0) &+ \sum_p f_{,p}(0)x_p + \sum_{p,q} \frac{1}{2!} f_{,pq}(0)x_p x_q \\ &+ \sum_{p,q,r} \frac{1}{3!} f_{,pqr}(0)x_p x_q x_r + \sum_{p,q,r,s} R_{pqrs}(x)x_p x_q x_r x_s, \end{aligned} \tag{3.4}$$

where the remainder $R_{pqrs}(x)$ satisfies

$$|R_{pqrs}(x)| \leq \|f\|_{C^4(\overline{\Omega})}, \quad p,q,r,s \in \{1,2,3\}. \tag{3.5}$$

The higher order terms can be bounded with the triangle inequality and (3.5) as

$$\begin{aligned}\left|\int_S \sum_{p,q,r,s} R_{pqrs}(x) x_p x_q x_r x_s \, dx\right| &\leq \int_S \sum_{p,q,r,s} |R_{pqrs}(x)| \, |x_p x_q x_r x_s| \, dx \\ \leq & C\|f\|_{C^4(\overline{\Omega})} \, h^4 \sum_{p,q,r,s} \int_S dx \leq C' h^7 \|f\|_{C^4(\overline{\Omega})}.\end{aligned} \tag{3.6}$$

Using the above expansion (3.4) to compute (f, v_i), all integrals over odd terms $x_p, x_p x_q (p \neq q), x_p x_q x_r, \ldots$ in equation (3.4) vanish. We are left only with the following even terms

$$(f(0), v_i), \ \left(f_{11}(0) x_1^2, v_i\right), \ \left(f_{22}(0) x_2^2, \ v_i\right), \left(f_{33}(0) x_3^2, v_i\right). \tag{3.7}$$

The values of these four terms above can be explicitly computed using

$$(1, c_i) = h^3, \qquad (1, t_i) = 4h^3, \qquad (1, p_i) = 6h^3, \tag{3.8}$$

$$\left(x_i^2, c_i\right) = \frac{1}{6} h^5, \quad \left(x_i^2, t_i\right) = \frac{2}{3} h^5, \quad \left(x_i^2, p_i\right) = h^5, \tag{3.9}$$

where t_i is the sum of four tetrahedral basis functions and p_i is the sum of six prismatic basis functions. By combining the above results with weights from (2.11), we immediately obtain

$$(f, v_i) = 6h^3 f + h^5 \Delta f + H.O.T., \tag{3.10}$$

where higher order terms were estimated in (3.6). This gives

$$|(f, v_i) - 6h^3 f - h^5 \Delta f| \leq C h^7 \|f\|_{C^4(\overline{\Omega})}, \tag{3.11}$$

which complete the proof of the lemma. □

Lemma 3.2. *For the averaged matrix A_h we have*

$$\eta^T A_h \eta \geq 2 \eta^T A_t \eta, \tag{3.12}$$

where the matrix A_t is the global FE matrix arising from the tetrahedral partition.

Proof. First of all notice that A_t is independent of tetrahedral partitions sketched in Figure 2.3 due to (2.10). Using the local equation (2.12), we can compute

$$\eta^T A_h \eta = 24h \sum_{i=1}^N \eta_i^2 - 2h \sum_{i=1}^N \sum_{z_j \in \Diamond z_i} \eta_i \eta_j - h \sum_{i=1}^N \sum_{z_j \in \Box z_i} \eta_i \eta_j, \tag{3.13}$$

where notation $z_j \in \Diamond z_i$ denotes that we perform summation over those j for which the node z_j is a midpoint of some face of the cube centered at z_i. The symbol $z_j \in \Box z_i$ has a similar meaning. Using the estimate $2\eta_i \eta_j \leq \eta_i^2 + \eta_j^2$, we have

$$h \sum_{i=1}^N \sum_{z_j \in \Box z_i} \eta_i \eta_j \leq \frac{h}{2} \sum_{i=1}^N \sum_{z_j \in \Box z_i} \left(\eta_i^2 + \eta_j^2\right), \tag{3.14}$$

As η_i^2 is present in the sum at most 24-times, we have

$$h\sum_{i=1}^{N}\sum_{z_j\in\Box z_i}\eta_i\eta_j \leq 12h\sum_{i=1}^{N}\eta_i^2. \tag{3.15}$$

Combining equations (3.13), (3.15), and (2.8) completes the proof. $\Box$

Theorem 3.1. *If $f \in C^4(\overline{\Omega})$, then for uniform partitions we have*

$$\|u(z_i) - u_h(z_i)\|_h = \mathcal{O}(h^4) \quad as \quad h \to 0. \tag{3.16}$$

Proof. Based on Lemma 3.2, the matrix A_h is symmetric and positive definite. Thus

$$\|A_h^{-1}\|_2 = \lambda_{min}^{-1},$$

where λ_{min} is the smallest eigenvalue of A_h and $\|\cdot\|_2$ is the standard spectral matrix norm. A lower bound for the smallest eigenvalue follows from Lemma 3.2 and the standard estimate of the smallest eigenvalue of A_t,

$$\lambda_{min} = \min_{0\neq x\in\mathbb{R}^N}\frac{x^T A_h x}{x^T x} \geq \min_{0\neq x\in\mathbb{R}^N}\frac{2x^T A_t x}{x^T x}. \tag{3.17}$$

We have $\lambda_{min} \geq Ch^3$, and thus,

$$\|A_h^{-1}\|_2 \leq Ch^{-3}. \tag{3.18}$$

Now, it is easy to see that (2.5) and (2.6) imply the nodal $\mathcal{O}(h^4)$-superconvergence. Indeed, we have by the triangle inequality, formulae (2.2), (2.4), and the embedding $\ell_\infty \hookrightarrow \ell_2$ that

$$\begin{aligned}\|u-u_h\|_h &= \|\vec{u}-\vec{u}_h\|_h \leq \|\vec{u}-\vec{U}_h\|_h + \|\vec{U}_h-\vec{u}_h\|_h\\ &\leq \|\vec{u}-\vec{U}_h\|_h + \|A_h^{-1}\|_2\|h\vec{f}_h-\vec{F}_h\|_h\\ &\leq \|\vec{u}-\vec{U}_h\|_h + C\|A_h^{-1}\|_2\|h\vec{f}_h-\vec{F}_h\|_\infty \leq C^{'}h^4,\end{aligned}$$

where $C = \sqrt{\operatorname{meas}\Omega}$ is independent of h. The last inequality is based on Lemma 3.1, (3.18), and the result by Bramble [3, p. 219-220]. $\Box$

4. Numerical Experiments

In this section, we present two numerical tests which are performed on the cubic domain $\Omega = (0,1)^3$.

Test 1: The load function f is chosen so that the exact solution is

$$u(x,y,z) = x\,(1-x)\,y\,(1-y)\,z\,(1-z)\,. \tag{4.1}$$

Clearly, the resulting load function f has the regularity required in Theorem 3.1, and the nodal superconvergence property should be present. The convergence of the discretization is measured in the stronger norm

$$\|u-u_h\|_\infty = \max_{i=1,\ldots,N}|u(z_i)-u_h(z_i)|$$

Table 4.1: Nodal convergence of different approximations (Test 1).

h	Cubic	Tetrahedral	Prismatic	Averaging
0.250000	0.00162990	0.00140550	0.00050729	4.0509e-05
0.111110	0.00029282	0.00028510	9.70820e-05	1.5284e-06
0.071429	0.00012392	0.00012245	4.11000e-05	2.6828e-07
0.052632	6.66230e-05	6.62270e-05	2.21820e-05	7.8594e-08

Table 4.2: Nodal convergence of different approximations (Test 2).

h	Cubic	Tetrahedral	Prismatic	Averaging
0.250000	0.1075200	0.0967160	0.0315570	0.00020997
0.111110	0.0195730	0.0193270	0.0064288	9.47060e-06
0.071429	0.0084242	0.0083504	0.0027805	1.73330e-06
0.052632	0.0045193	0.0045066	0.0015014	5.09400e-07

and visualized in Figure 4.1. In the end, the $\mathcal{O}(h^4)$-superconvergence in the $\|\cdot\|_\infty$-norm is observed in Table 4.1. This phenomenon is probably due to the existence of an improved bound for the matrix norm of the inverse of the averaged scheme.

Test 2: In the second test, we set the problem whose exact solution is

$$u(x,y,z) = \sin \pi x \, \sin \pi y \, \sin \pi z. \tag{4.2}$$

The error is measured in the same norm as in Test 1. In order to calculate the entries of the stiffness matrix and the load vector, we employed higher order numerical quadrature formulae on tetrahedra from references [7,8,10]. Numerical results are presented in Table 4.2 and Figure 4.2.

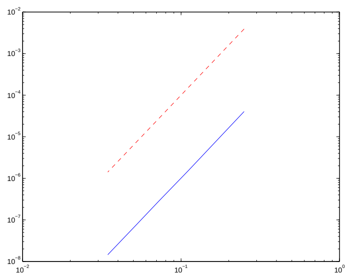

Fig. 4.1. Error measured in the maximum norm for Test 1. The dashed line demonstrates $\mathcal{O}(h^4)$ convergence rate.

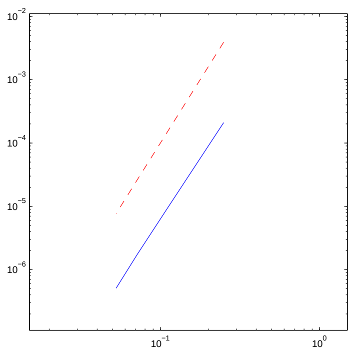

Fig. 4.2. Error measured in the maximum norm for Test 2. The dashed line demonstrates $\mathcal{O}(h^4)$ convergence rate.

Tables 4.1 and 4.2 illustrate also results of Theorem 3.1. Due to limitations of computer memory, we have chosen only $h = 1/4, 1/9, 1/14$, and $1/19$.

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References

[1] I. Babuška, T. Strouboulis, C.S. Upadhyay, S.K. Gangaraj, Computer-based proof of the existence of superconvergence points in the finite element method; superconvergence of the derivatives in finite element solutions of Laplace's, Poisson's, and the elasticity equations, *Numer. Meth. Part. Diff. Eqn.*, **12** (1996), 347–392.

[2] H. Blum, Q. Lin, R. Rannacher, Asymptotic error expansion and Richardson extrapolation for linear finite elements, *Numer. Math.*, **49** (1986), 11–37.

[3] J.H. Bramble, Fourth-order finite difference analogues of the Dirichlet problem for Poisson's equation in three and four dimensions, *Math. Comput.*, **17** (1963), 217–222.

[4] J. Brandts, S. Korotov, M. Křížek, J. Šolc, On nonobtuse simplicial partitions, *SIAM Rev.*, **51** (2009), 317–335.

[5] J. Brandts, M. Křžek, Gradient superconvergence on uniform simplicial partitions of polytopes, *IMA J. Numer. Anal.*, **23** (2003), 489–505.

[6] J. Brandts, M. Křžek, Superconvergence of tetrahedral quadratic finite elements, *J. Comput. Math.*, **23** (2005), 27–36.

[7] R. Cools, Monomial cubature rules since "Stroud": A compliation. II, *J. Comput. Appl. Math.*, **112** (1999), 21–27.

[8] R. Cools, P. Rabinowitz, Monomial cubature rules since "Stroud": A compliation, *J. Comput. Appl. Math.*, **48** (1993), 309–326.

[9] M. M. Gupta, J. Kouatchou, Symbolic derivation of finite difference approximations for the three-dimensional Poisson equation, *Numer. Meth. Part. Diff. Eqn.*, **14** (1998), 593–606.

[10] P. Keast, Moderate-degree tetrahedral quadrature formulas, *Comput. Method. Appl. M.*, **55** (1986), 339–348.

[11] M. Křížek, Superconvergence phenomena on three-dimensional meshes, *Internat. J. Numer. Anal. Model.*, **2** (2005), 43–56.

[12] M. Křížek, Q. Lin, Y. Huang, A nodal superconvergence arising from combination of linear and bilinear elements, *J. Systems Sci. Math. Sci.*, **1** (1988), 191–197.

[13] M. Křížek, P. Neittaanmäki, On $\mathcal{O}(h^4)$-superconvergence of piecewise bilinear FE-approximations, *Mat. Apl. Comput.*, **8** (1989), 49–61.

[14] Q. Lin, J. Wang, Some expansions of the finite element approximations, Research Report IMS-15, Chendu Branch of Acad, Sinica, 1984, 1–11.

[15] A.H. Schatz, Pointwise error estimates, superconvergence and extrapolation, Proc. Conf. Finite Element Methods: Superconvergence, Post-processing and A Posteriori Error Estimates, Univ. of Jyväskylä, July 1–4, 1996, Marcel Dekker, New York, 1998, 237-247.

[16] A.H. Schatz, I.H. Sloan, L.B. Wahlbin, Superconvergence in finite element methods and meshes that are locally symmetric with respect to a point, *SIAM J. Numer. Anal.*, **33** (1996), 505–521.

[17] Z. Zhang, R. Lin, Locating natural superconvergent points of finite element methods in 3D, *Internat. J. Numer. Anal. Model.*, **2** (2005), 19–30.