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RICHARDSON EXTRAPOLATION AND DEFECT CORRECTION OF FINITE ELEMENT METHODS FOR OPTIMAL CONTROL PROBLEMS *

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Abstract

Asymptotic error expansions in H^1-norm for the bilinear finite element approximation to a class of optimal control problems are derived for rectangular meshes. With the rectangular meshes, the Richardson extrapolation of two different schemes and an interpolation defect correction can be applied. The higher order numerical approximations are used to generate a posteriori error estimators for the finite element approximation.

Mathematics subject classification: 65R20, 65M12, 65M60, 65N30, 76S05, 49J20.
Key words: Optimal control problem, Finite element methods, Asymptotic error expansions, Defect correction, A posteriori error estimates.

1. Introduction

The aim of this paper is to discuss the asymptotic behavior of the finite element approximation for a model optimal control problem described as follows:

$$\begin{cases} \min\limits_{u\in K}\left\{\dfrac{1}{2}||y-z_d||_H^2+\dfrac{1}{2}||u||_U^2\right\} \\ \qquad -\mathrm{div}\,(A\nabla y)=f+Bu \quad \text{in } \Omega, \\ \qquad y|_{\partial\Omega}=0, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain in R^n with Lipschitz boundary $\partial\Omega$, $L^2(\Omega)$ stands for the usual L^2-inner product space, K is a nonempty closed convex set in $L^2(\Omega)$, f, $z_d\in L^2(\Omega)$, B is a continuous linear operator from $U=L^2(\Omega)$ to $L^2(\Omega)$, $H=L^2(\Omega)$, and

$$A(\cdot)=\left(a_{i,j}(\cdot)\right)_{n\times n}\in\left(L^\infty(\Omega)\right)^{n\times n},$$

such that there is a constant $\sigma>0$ satisfying that for any vector $X=(x_1,x_2,\dots,x_n)\in R^n$

$$X^TA(x)X\ge\sigma||X||_{R^n}^2 \quad \text{for almost all } x\in\Omega,$$

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where

$$||X||_{R^n} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

In this paper, we only consider the two dimensional problem, i.e., $n = 2$.

Problem (1.1) is crucial in many engineering applications see, e.g., [26, 32]. Finite element method is one of the efficient numerical methods for solving (1.1); the literature in this aspect is huge (see, e.g., [1–3, 13]). Systematic introduction to the finite element method for partial differential equations and optimal control problems are available in, for example, [10, 26, 32]. At present there are extensive theoretical studies of the finite element approximation for various optimal control problems, see, e.g., [1, 8, 35] for a priori error estimates, and [2, 3, 9, 28, 29] for a posteriori error estimates. Very recently, superconvergence has been considered in [8, 11, 27, 31] for Galerkin finite element methods and in [8] for mixed finite element methods.

In the present paper we study two numerical approaches of higher accuracy, namely, [11, 27, 31] the Richardson extrapolation schemes and an interpolation defect correction method in the H^1-norm.

As an efficient numerical method to increase the accuracy of approximations, the Richardson extrapolation has been demonstrated in [30] for the difference method, in [5–7, 12, 14, 15, 17–22, 24, 25, 33, 34, 37–39] for the (Galerkin and Petrov-Galerkin) finite element method and the mixed finite element method, in [16, 36] for the collocation method and the boundary element method, respectively.

The defect correction of (Galerkin and Petrov-Galerkin) finite elements by means of an interpolation postprocessing technique is another numerical method to obtain approximations of higher accuracy, which has been studied for a wide variety of models. See, for example, [4, 6, 17, 18, 21, 23] and the references cited therein.

This paper is organized as follows. In Section 2, the approximation subspace and the variational formula of (1.1) are provided. Also, the asymptotic expansion of the finite element approximation is presented in this section for the future need. To the best of our knowledge, the asymptotic expansions are new in that they are obtained under the condition that the mesh is uniform in x- or y-direction (not both x- and y-direction), which is different from those presented in the previous literatures (see, e.g., [7]). Section 3 is devoted to investigating the asymptotic expansions of the exact solution to the model problem in the H^1-norm. Two numerical approaches of the Richardson extrapolation schemes are presented in Section 3. Section 4 deals with an interpolation defect correction approximation in the H^1-norm based on the results given in Section 3. Furthermore, at the ends of Sections 3 and 4, a posteriori error estimators are furnished as by-products of these numerical solutions with higher convergence rates. Some related problems are addressed in Section 5.

2. The Asymptotic Expansion

In this section we first give the weak variational formula and the finite element method for the convex distributed optimal control problem (1.1). To this end, we denote the standard Sobolev spaces by $W^{m,q}(\Omega)$ on the domain Ω with the norm $||\cdot||_{m,q}$ and the seminorm $|\cdot|_{m,q}$. Also, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ with the norm $||\cdot||_m$ and the seminorm $|\cdot|_m$. We set $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. In addition, throughout the paper, C stands for a generic positive constant, independent of the mesh size h, whose specific value depends on the context in which it appears. Then, the weak form of (1.1) is

$$\begin{cases} \min\limits_{u\in K}\left\{\frac{1}{2}||y-z_d||_H^2+\frac{1}{2}||u||_U^2\right\} \\ \qquad a(y(u),v)=(f+Bu,v), \quad \forall v\in H_0^1(\Omega), \end{cases} \tag{2.1}$$

where

$$\begin{aligned} a(y,v) &= \int_\Omega (A\nabla y,\nabla v)_{R^n}, && \forall y,\, v\in H^1(\Omega) \\ (u,v) &= \int_\Omega uv, && \forall u,\, v\in L^2(\Omega). \end{aligned}$$

It has been proved in [26] that Problem(2.1) is equivalent to the following optimality conditions: Find the triple (y^*,p^*,u^*) such that

$$\begin{cases} a(y^*,v)=(f+Bu^*,v) & \forall v\in Y=H_0^1(\Omega), \\ a(p^*,q)=(y^*-z_d,q) & \forall q\in Y=H_0^1(\Omega), \\ (u^*+B^*p^*,w-u^*)\geq 0 & \forall w\in K\subset U=L^2(\Omega), \end{cases} \tag{2.2}$$

where B^* is the adjoint operator of B.

In this paper, we only consider the unconstrained case, that is, $K=U=L^2(\Omega)$, which is the special simple case, but the ideas used in analysis are quite general [26]. Thus, it is easy to deduce from (2.2) that $u^*=-B^*p^*$. For the sake of simplicity of analysis, we take $B=I$, and Ω a rectangle in R^2. Then (2.2) can be rewritten into (we denote y^* by y, and p^* by p for simplicity)

$$\begin{cases} a(y,v)+(p,v)=(f,v) & \forall v\in H_0^1(\Omega), \\ a(p,q)-(y,q)=(-z_d,q) & \forall q\in H_0^1(\Omega), \end{cases} \tag{2.3}$$

which is the weak form of the following problem:

$$\begin{cases} -\mathrm{div}(A\nabla y)+p=f & \text{in}\,\Omega, \\ -\mathrm{div}(A\nabla p)-y=-z_d & \text{in}\,\Omega, \\ y=0 & \text{on}\,\partial\Omega, \\ p=0 & \text{on}\,\partial\Omega. \end{cases} \tag{2.4}$$

Now let us consider the finite element approximation to (2.3) in two-dimensional case. To this end, let T_{h_1,h_2} be a finite element partition of $\overline{\Omega}$ into regular rectangles, where h_1 and h_2 are the mesh sizes in x_1- and x_2-axis, respectively. Denote the finite element space by

$$V_{h_1,h_2}=\big\{v\in C(\overline{\Omega}):\, v|_e\in Q_{1,1}(e),\, \forall e\in T_{h_1,h_2}\big\},$$

where $Q_{m,n}$ represents the space of polynomials of degree no more than m and n in x_1 and x_2 on element e, respectively. Moreover, we let

$$V^0_{h_1,h_2}=V_{h_1,h_2}\cap H_0^1(\Omega).$$

Then the finite element approximation of (2.3) is: Find y_{h_1,h_2}, $p_{h_1,h_2}\in V^0_{h_1,h_2}$ such that

$$\begin{cases} a(y_{h_1,h_2},v_{h_1,h_2})+(p_{h_1,h_2},v_{h_1,h_2})=(f,v_{h_1,h_2}) & \forall v_{h_1,h_2}\in V^0_{h_1,h_2}, \\ a(p_{h_1,h_2},q_{h_1,h_2})-(y_{h_1,h_2},q_{h_1,h_2})=(-z_d,q_{h_1,h_2}) & \forall q_{h_1,h_2}\in V^0_{h_1,h_2}, \end{cases} \tag{2.5}$$

which, together with (2.3), leads to the following finite element error equation:

$$\begin{cases} a(y - y_{h_1,h_2}, v_{h_1,h_2}) + (p - p_{h_1,h_2}, v_{h_1,h_2}) = 0 & \forall v_{h_1,h_2} \in V^0_{h_1,h_2}, \\ a(p - p_{h_1,h_2}, v_{h_1,h_2}) - (y - y_{h_1,h_2}, q_{h_1,h_2}) = 0 & \forall q_{h_1,h_2} \in V^0_{h_1,h_2}. \end{cases} \tag{2.6}$$

From [25] we recall the following lemmas.

Lemma 2.1. *Assume that $\alpha \in H^4(\Omega)$ and $u \in H^5(\Omega)$. Then, we have*

$$\begin{aligned} &\int_\Omega \alpha(u - i_{h_1,h_2}u)_{x_1} v_{h_1,h_2,x_1} \\ &= \sum_{e \in T_{h_1,h_2}} \frac{h_{2,e}^2}{3} \int_e (\alpha u_{x_1x_2x_2})_{x_1} v_{h_1,h_2} + \sum_{e \in T_{h_1,h_2}} \frac{h_{1,e}^2}{3} \int_e \alpha_{x_1} u_{x_1x_1} (v_{h_1,h_2})_{x_1} \\ &\quad + \mathcal{O}(h^4)||u||_5 ||v_{h_1,h_2}||_1, \quad v_{h_1,h_2} \in V^0_{h_1,h_2}, \end{aligned}$$

where $h = \max\limits_{e \in T_{h_1,h_2}} \{h_{1,e}, h_{2,e}\}$ and i_{h_1,h_2} stands for the bilinear interpolation operator.

Lemma 2.2. *Assume that $\alpha \in H^3(\Omega)$, $u \in H^5(\Omega)$, and the mesh is uniform in x- or y-direction. Then, we have*

$$\begin{aligned} &\int_\Omega \alpha(u - i_{h_1,h_2}u)_{x_2} (v_{h_1,h_2})_{x_1} \\ &= -\sum_{e \in T_{h_1,h_2}} \frac{h_{1,e}^2}{3} \int_e \alpha u_{x_1x_1x_2} (v_{h_1,h_2})_{x_1} - \sum_{e \in T_{h_1,h_2}} \frac{h_{2,e}^2}{3} \int_e (\alpha u_{x_2x_2})_{x_1} (v_{h_1,h_2})_{x_2} \\ &\quad + \mathcal{O}(h^4)||u||_5 ||v_{h_1,h_2}||_1, \quad v_{h_1,h_2} \in V^0_{h_1,h_2}, \end{aligned}$$

where $h = \max\limits_{e \in T_{h_1,h_2}} \{h_{1,e}, h_{2,e}\}$ and i_{h_1,h_2} stands for the bilinear interpolation operator.

Lemma 2.3. *Assume that $\alpha \in H^1(\Omega)$ and $u \in H^3(\Omega)$. Then, we have*

$$\begin{aligned} &\int_\Omega \alpha(u - i_{h_1,h_2}u) v_{h_1,h_2} \\ &= -\sum_{e \in T_{h_1,h_2}} \frac{h_{1,e}^2}{3} \int_e \alpha u_{x_1x_1} v_{h_1,h_2} - \sum_{e \in T_{h_1,h_2}} \frac{h_{2,e}^2}{3} \int_e \alpha u_{x_2x_2} v_{h_1,h_2} \\ &\quad + \mathcal{O}(h^4)||u||_3 ||v_{h_1,h_2}||_1, \quad v_{h_1,h_2} \in V^0_{h_1,h_2}, \end{aligned}$$

where $h = \max\limits_{e \in T_{h_1,h_2}} \{h_{1,e}, h_{2,e}\}$ and i_{h_1,h_2} stands for the bilinear interpolation operator.

Theorem 2.1. *Assume that $a_{ij} \in H^4(\Omega)$ $(1 \le i,\, j \le 2)$, $y \in H^5(\Omega)$, and the mesh is uniform* *in x- or y-direction. Then, we have*

$$\begin{aligned}
&a(y-i_{h_1,h_2}y,v_{h_1,h_2})\\
=&\sum_{e\in T_{h_1,h_2}}\frac{h_{1,e}^2}{3}\int_e(a_{11})_{x_1}y_{x_1x_1}(v_{h_1,h_2})_{x_1}+\sum_{e\in T_{h_1,h_2}}\frac{h_{1,e}^2}{3}\int_e(a_{22}y_{x_2x_1x_1})_{x_2}v_{h_1,h_2}\\
&+\sum_{e\in T_{h_1,h_2}}\frac{h_{2,e}^2}{3}\int_e(a_{22})_{x_2}y_{x_2x_2}(v_{h_1,h_2})_{x_2}+\sum_{e\in T_{h_1,h_2}}\frac{h_{2,e}^2}{3}\int_e(a_{11}y_{x_1x_2x_2})_{x_1}v_{h_1,h_2}\\
&-\sum_{e\in T_{h_1,h_2}}\frac{h_{1,e}^2}{3}\int_e(a_{21}y_{x_1x_1})_{x_2}(v_{h_1,h_2})_{x_1}-\sum_{e\in T_{h_1,h_2}}\frac{h_{1,e}^2}{3}\int_e a_{12}y_{x_1x_2x_2}(v_{h_1,h_2})_{x_1}\\
&-\sum_{e\in T_{h_1,h_2}}\frac{h_{2,e}^2}{3}\int_e(a_{12}y_{x_2x_2})_{x_1}(v_{h_1,h_2})_{x_2}-\sum_{e\in T_{h_1,h_2}}\frac{h_{2,e}^2}{3}\int_e a_{21}y_{x_2x_1x_1}(v_{h_1,h_2})_{x_2}\\
&+\mathcal{O}(h^4)||y||_5||v_{h_1,h_2}||_1,\quad v_{h_1,h_2}\in V^0_{h_1,h_2}.
\end{aligned}\tag{2.7}$$

Proof: The desired result (2.7) *follows directly from Lemmas* 2.1 *and* 2.2. □

Corollary 2.1. *If* $u \in H^3(\Omega)$, $a_{ij} \in H^1(\Omega)$, *and the mesh is uniform in x- or y-direction, then we have*

$$|a(u-i_{h_1,h_2}u,v_{h_1,h_2})|\le Ch^2||u||_3||v_{h_1,h_2}||_1,\quad v_{h_1,h_2}\in V^0_{h_1,h_2}.$$

Corollary 2.2. *If* $u \in H^2(\Omega)$, *then we have*

$$|(u-i_{h_1,h_2}u,v_{h_1,h_2})|\le Ch^2||u||_2||v_{h_1,h_2}||_0,\quad v_{h_1,h_2}\in V^0_{h_1,h_2}.$$

In addition, we also need the following lemma.

Lemma 2.4. *Assume that the matrix A is positive definite. Then the seminorms*

$$|\sigma|_1^2:=\int_\Omega(\nabla\sigma,\nabla\sigma)_{R^2}\ \text{ and }\ |\sigma|_A^2:=\int_\Omega(A\nabla\sigma,\nabla\sigma)_{R^2}$$

are equivalent.

3. The Richardson Extrapolation

On the basis of Theorem 2.1 and Lemma 2.3, we discuss in this section the asymptotic expansion of the error between the finite element solution and the bilinear interpolation of the exact solution of (2.3) in order to establish the asymptotic error expansion of the finite element approximation in H^1-norm. The Richardson extrapolation of two different schemes will be performed to generate high order approximations to the exact solution of (2.3).

3.1. The Richardson extrapolation in two directions

We first discuss the global extrapolation method of finite element approximation for (2.3) in both x_1 and x_2 directions in this subsection. In order to do it, we first consider the asymptotic expansion of finite element approximation for (2.3) in both x_1 and x_2 directions as follows.

Theorem 3.1. *Suppose that (y, p) and $(y_{h_1,h_2}, p_{h_1,h_2})$ are the exact solution of (2.3) and its finite element solution, respectively, and $a_{ij} \in H^4(\Omega)$ $(1 \le i, j \le 2)$, y, $p \in H^5(\Omega)$. Then, in the sense of the H^1-norm we have the following asymptotic expansions:*

$$y_{h_1,h_2} - i_{h_1,h_2}y = h^2\xi_{h_1,h_2} + \mathcal{O}(h^4),$$
$$p_{h_1,h_2} - i_{h_1,h_2}p = h^2\eta_{h_1,h_2} + \mathcal{O}(h^4),$$

where $(\xi_{h_1,h_2}, \eta_{h_1,h_2}) \in V^0_{h_1,h_2} \times V^0_{h_1,h_2}$ and will be specified in the proof, and i_{h_1,h_2} is the bilinear interpolation operator.

Proof. Set

$$\rho_{h_1,h_2} := y_{h_1,h_2} - i_{h_1,h_2}y \quad \text{and} \quad \theta_{h_1,h_2} := p_{h_1,h_2} - i_{h_1,h_2}p.$$

Then, it follows from (2.6) that

$$\begin{aligned}&a(\rho_{h_1,h_2}, v_{h_1,h_2}) + (\theta_{h_1,h_2}, v_{h_1,h_2})\\ =&a(y - i_{h_1,h_2}y, v_{h_1,h_2}) + (p - i_{h_1,h_2}p, v_{h_1,h_2}) \quad \forall v_{h_1,h_2} \in V^0_{h_1,h_2},\end{aligned} \tag{3.1a}$$

and

$$\begin{aligned}&a(\theta_{h_1,h_2}, q_{h_1,h_2}) - (\rho_{h_1,h_2}, q_{h_1,h_2})\\ =&a(p - i_{h_1,h_2}p, q_{h_1,h_2}) - (y - i_{h_1,h_2}y, q_{h_1,h_2}) \quad \forall q_{h_1,h_2} \in V^0_{h_1,h_2}.\end{aligned} \tag{3.1b}$$

Furthermore, from Theorem 2.1 and Lemma 2.3 we know that

$$\begin{aligned}&a(y - i_{h_1,h_2}y, v_{h_1,h_2}) + (p - i_{h_1,h_2}p, v_{h_1,h_2})\\ =&h^2G_{h_1,h_2}(v_{h_1,h_2}) + \mathcal{O}(h^4)||y||_5||v_{h_1,h_2}||_1 \quad \forall v_{h_1,h_2} \in V^0_{h_1,h_2},\end{aligned} \tag{3.2a}$$

and

$$\begin{aligned}&a(p - i_{h_1,h_2}p, q_{h_1,h_2}) - (y - i_{h_1,h_2}y, q_{h_1,h_2})\\ =&h^2L_{h_1,h_2}(q_{h_1,h_2}) + \mathcal{O}(h^4)||p||_5||q_{h_1,h_2}||_1 \quad \forall q_{h_1,h_2} \in V^0_{h_1,h_2},\end{aligned} \tag{3.2b}$$

where

$$\begin{aligned}G_{h_1,h_2}(\phi) =& \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{1,e}}{h}\right)^2 \int_e (a_{11})_{x_1} y_{x_1x_1}\phi_{x_1} + \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{1,e}}{h}\right)^2 \int_e (a_{22}y_{x_2x_1x_1})_{x_2}\phi\\ &+ \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{2,e}}{h}\right)^2 \int_e (a_{22})_{x_2} y_{x_2x_2}\phi_{x_2} + \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{2,e}}{h}\right)^2 \int_e (a_{11}y_{x_1x_2x_2})_{x_1}\phi\\ &- \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{1,e}}{h}\right)^2 \int_e (a_{21}y_{x_1x_1})_{x_2}\phi_{x_1} - \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{1,e}}{h}\right)^2 \int_e a_{12}y_{x_1x_2x_2}\phi_{x_1}\\ &- \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{2,e}}{h}\right)^2 \int_e (a_{12}y_{x_2x_2})_{x_1}\phi_{x_2} - \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{2,e}}{h}\right)^2 \int_e a_{21}y_{x_2x_1x_1}\phi_{x_2}\\ &- \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{1,e}}{h}\right)^2 \int_e y_{x_1x_1}\phi - \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{2,e}}{h}\right)^2 \int_e y_{x_2x_2}\phi,\end{aligned}$$

and

$$
\begin{aligned}
L_{h_1,h_2}(\psi)=&\sum_{e\in T_{h_1,h_2}}\frac{1}{3}\left(\frac{h_{1,e}}{h}\right)^2\int_e(a_{11})_{x_1}p_{x_1x_1}\psi_{x_1}+\sum_{e\in T_{h_1,h_2}}\frac{1}{3}\left(\frac{h_{1,e}}{h}\right)^2\int_e(a_{22}p_{x_2x_1x_1})_{x_2}\psi\\
&+\sum_{e\in T_{h_1,h_2}}\frac{1}{3}\left(\frac{h_{2,e}}{h}\right)^2\int_e(a_{22})_{x_2}p_{x_2x_2}\psi_{x_2}+\sum_{e\in T_{h_1,h_2}}\frac{1}{3}\left(\frac{h_{2,e}}{h}\right)^2\int_e(a_{11}p_{x_1x_2x_2})_{x_1}\psi\\
&-\sum_{e\in T_{h_1,h_2}}\frac{1}{3}\left(\frac{h_{1,e}}{h}\right)^2\int_e(a_{21}p_{x_1x_1})_{x_2}\psi_{x_1}-\sum_{e\in T_{h_1,h_2}}\frac{1}{3}\left(\frac{h_{1,e}}{h}\right)^2\int_e a_{12}p_{x_1x_2x_2}\psi_{x_1}\\
&-\sum_{e\in T_{h_1,h_2}}\frac{1}{3}\left(\frac{h_{2,e}}{h}\right)^2\int_e(a_{12}p_{x_2x_2})_{x_1}\psi_{x_2}-\sum_{e\in T_{h_1,h_2}}\frac{1}{3}\left(\frac{h_{2,e}}{h}\right)^2\int_e a_{21}p_{x_2x_1x_1}\psi_{x_2}\\
&+\sum_{e\in T_{h_1,h_2}}\frac{1}{3}\left(\frac{h_{1,e}}{h}\right)^2\int_e p_{x_1x_1}\psi+\sum_{e\in T_{h_1,h_2}}\frac{1}{3}\left(\frac{h_{2,e}}{h}\right)^2\int_e p_{x_2x_2}\psi.
\end{aligned}
$$

Obviously, we have

$$
G_{h_1/2,h_2/2}(\phi)=G_{h_1,h_2}(\phi)\quad\text{and}\quad L_{h_1/2,h_2/2}(\psi)=L_{h_1,h_2}(\psi). \tag{3.3}
$$

Let $(\xi,\eta)\in H_0^1(\Omega)\times H_0^1(\Omega)$ and $(\xi_{h_1,h_2},\eta_{h_1,h_2})\in V_{h_1,h_2}^0\times V_{h_1,h_2}^0$ be the solution and the finite element solution of the following auxiliary problem, respectively,

$$
\begin{cases}
a(\xi,v)+(\eta,v)=G_{h_1,h_2}(v) & \forall v\in H_0^1(\Omega),\\
a(\eta,q)-(\xi,q)=L_{h_1,h_2}(q) & \forall q\in H_0^1(\Omega).
\end{cases} \tag{3.4}
$$

Then, from (3.1), (3.2), and (3.4) one finds that

$$
\begin{aligned}
&a(\rho_{h_1,h_2}-h^2\xi_{h_1,h_2},v_{h_1,h_2})+(\theta_{h_1,h_2}-h^2\eta_{h_1,h_2},v_{h_1,h_2})\\
=&\mathcal{O}(h^4)||y||_5||v_{h_1,h_2}||_1\quad\forall v_{h_1,h_2}\in V_{h_1,h_2}^0,\\
&a(\theta_{h_1,h_2}-h^2\eta_{h_1,h_2},q_{h_1,h_2})-(\rho_{h_1,h_2}-h^2\xi_{h_1,h_2},q_{h_1,h_2})\\
=&\mathcal{O}(h^4)||p||_5||q_{h_1,h_2}||_1\quad\forall v_{h_1,h_2}\in V_{h_1,h_2}^0.
\end{aligned}
$$

Let

$$
\rho^*_{h_1,h_2}:=\rho_{h_1,h_2}-h^2\xi_{h_1,h_2}\quad\text{and}\quad\theta^*_{h_1,h_2}:=\theta_{h_1,h_2}-h^2\eta_{h_1,h_2}.
$$

Thus, we have

$$
a(\rho^*_{h_1,h_2},v_{h_1,h_2})+(\theta^*_{h_1,h_2},v_{h_1,h_2})=\mathcal{O}(h^4)||y||_5||v_{h_1,h_2}||_1\quad\forall v_{h_1,h_2}\in V_{h_1,h_2}^0, \tag{3.5a}
$$

$$
a(\theta^*_{h_1,h_2},q_{h_1,h_2})-(\rho^*_{h_1,h_2},q_{h_1,h_2})=\mathcal{O}(h^4)||p||_5||q_{h_1,h_2}||_1\quad\forall q_{h_1,h_2}\in V_{h_1,h_2}^0. \tag{3.5b}
$$

Moreover, take $v_{h_1,h_2}=\rho^*_{h_1,h_2}$ and $q_{h_1,h_2}=\theta^*_{h_1,h_2}$ in (3.5) to obtain

$$
a(\rho^*_{h_1,h_2},\rho^*_{h_1,h_2})+a(\theta^*_{h_1,h_2},\theta^*_{h_1,h_2})=\mathcal{O}(h^4)\big(||y||_5||\rho^*_{h_1,h_2}||_1+||p||_5||\theta^*_{h_1,h_2}||_1\big),
$$

which, together with Lemma 2.4, Poincare inequality and Schwartz inequality, yields that there are positive constants α and β such that

$$
\begin{aligned}
&\alpha\left(||\rho^*_{h_1,h_2}||_1^2+||\theta^*_{h_1,h_2}||_1^2\right)\\
\leq&|\rho^*_{h_1,h_2}|_1^2+|\theta^*_{h_1,h_2}|_1^2\\
\leq&\beta\big(a(\rho^*_{h_1,h_2},\rho^*_{h_1,h_2})+a(\theta^*_{h_1,h_2},\theta^*_{h_1,h_2})\big)\\
\leq&Ch^8\big((||y||_5^2+||p||_5^2)+\frac{\alpha}{2}(||\rho^*_{h_1,h_2}||_1^2+||\theta^*_{h_1,h_2}||_1^2)\big).
\end{aligned}
$$

This implies

$$||\rho^*_{h_1,h_2}||_1^2 + ||\theta^*_{h_1,h_2}||_1^2 \leq Ch^8\big(||y||_5^2 + ||p||_5^2\big).$$

Therefore, we have

$$\begin{aligned} ||\rho^*_{h_1,h_2}||_1 &\leq Ch^4\left(||y||_5 + ||p||_5\right), \\ ||\theta^*_{h_1,h_2}||_1 &\leq Ch^4\left(||y||_5 + ||p||_5\right). \end{aligned}$$

□

Following the procedure for Theorem 3.1 and utilizing Corollaries 2.1 and 2.2 we can also prove the following result.

Lemma 3.1. *If* $(\xi, \eta) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *and* $(\xi_{h_1,h_2}, \eta_{h_1,h_2}) \in V^0_{h_1,h_2} \times V^0_{h_1,h_2}$ *are the solution and the finite element solution of* (3.4)*, respectively, and if the mesh is uniform unidirectionally, then we have the superconvergent estimate*

$$||\xi_{h_1,h_2} - i_{h_1,h_2}\xi||_1 + ||\eta_{h_1,h_2} - i_{h_1,h_2}\eta||_1 \leq Ch^2(||\xi||_3 + ||\eta||_3).$$

Now we use the interpolation postprocessing technique to get a global extrapolation approximation of higher accuracy. Let us consider the global extrapolation method of finite element approximation for (2.3) in both x_1 and x_2 directions. Analogous to [12], [15], [17], we need to define a postprocessing interpolation operator $I^4_{4h_1,4h_2}$ to satisfy

$$I^4_{4h_1,4h_2} i_{h_1,h_2} = I^4_{4h_1,4h_2}, \tag{3.6a}$$

$$||I^4_{4h_1,4h_2} v_{h_1,h_2}||_1 \leq C||v_{h_1,h_2}||_1 \quad \forall v_{h_1,h_2} \in V^0_{h_1,h_2}, \tag{3.6b}$$

$$||I^4_{4h_1,4h_2} u - u||_1 \leq Ch^4||u||_5 \qquad \forall u \in H^5(\Omega). \tag{3.6c}$$

To this end, we assume that the rectangular partition T_{h_1,h_2} has been obtained from $T_{4h_1,4h_2}$ with mesh size $4h$ by subdividing each element of $T_{4h_1,4h_2}$ into 16 small congruent rectangles. Let $\tau := \bigcup_{i=1}^{16} e_i$ with $e_i \in T_{h_1,h_2}$. Then, we can define a postprocessing interpolation operator $I^4_{4h_1,4h_2}$ associated with $T_{4h_1,4h_2}$ of degree at most 4 in x_1 and x_2 on τ according to the following conditions:

$$\begin{aligned} &I^4_{4h_1,4h_2} u|_\tau \in Q_{4,4}(\tau), \\ &\left(I^4_{4h_1,4h_2} u\right)(z_i) = u(z_i), \qquad i = 1, 2, \ldots, 25, \end{aligned} \tag{3.7}$$

where z_i $(i = 1, \ldots, 25)$ is one of the 25 vertices of the 16 small elements e_i $(i = 1, \ldots, 16)$. It is easy to check that the operator $I^4_{4h_1,4h_2}$ defined by (3.7) is of the properties described in (3.6).

We are now in a position to assert our main result in this section.

Theorem 3.2. *We have under the conditions of Theorem* 3.1 *that*

$$\begin{aligned} I^4_{4h_1,4h_2} y_{h_1,h_2} - y &= h^2\xi + r_{h_1,h_2}, \quad ||r_{h_1,h_2}||_1 \leq Ch^4, \\ I^4_{4h_1,4h_2} p_{h_1,h_2} - p &= h^2\eta + r^*_{h_1,h_2}, \quad ||r^*_{h_1,h_2}||_1 \leq Ch^4, \end{aligned}$$

where $(\xi, \eta) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *is the solution of* (3.4).

Proof. Let

$$\overline{r}_{h_1,h_2} := y_{h_1,h_2} - i_{h_1,h_2} y - h^2 i_{h_1,h_2}\xi.$$

Then, it follows from Theorem 3.1 and Lemma 3.1 that

$$||\overline{r}_{h_1,h_2}||_1 \leq Ch^4.$$

Thus, we find from (3.6) that

$$
\begin{aligned}
& I^4_{4h_1,4h_2} y_{h_1,h_2} - y \\
=& I^4_{4h_1,4h_2}(y_{h_1,h_2} - i_{h_1,h_2} y) + (I^4_{4h_1,4h_2} y - y) \\
=& I^4_{4h_1,4h_2}(h^2 i_{h_1,h_2}\xi + \overline{r}_{h_1,h_2}) + (I^4_{4h_1,4h_2} y - y) \\
=& h^2 I^4_{4h_1,4h_2}\xi + I^4_{4h_1,4h_2}\overline{r}_{h_1,h_2} + (I^4_{4h_1,4h_2} y - y) \\
=& h^2\xi + h^2(I^4_{4h_1,4h_2}\xi - \xi) + I^4_{4h_1,4h_2}\overline{r}_{h_1,h_2} + (I^4_{4h_1,4h_2} y - y) \\
=& h^2\xi + r_{h_1,h_2},
\end{aligned}
$$

where

$$r_{h_1,h_2} := h^2(I^4_{4h_1,4h_2}\xi - \xi) + I^4_{4h_1,4h_2}\overline{r}_{h_1,h_2} + (I^4_{4h_1,4h_2} y - y)$$

with $||r_{h_1,h_2}||_1 \le Ch^4$. Analogously, we can also get the second equality in the theorem. □

Theorem 3.2 guarantees that we can use low order finite element solutions to generate high order approximations by the Richardson extrapolation. And thus, we employ, in addition to $V^0_{h_1,h_2} \times V^0_{h_1,h_2}$, the finite element space $V^0_{h_1/2,h_2/2} \times V^0_{h_1/2,h_2/2}$ gained by subdividing each element $e_i \in T_{2h_1,2h_2}$ into 16 small congruent element $\hat{e}_j \in T_{h_1/2,h_2/2}$ $(j = 1, 2, \ldots, 16)$. Denote by $(y_{h_1/2,h_2/2}, p_{h_1/2,h_2/2}) \in V^0_{h_1/2,h_2/2} \times V^0_{h_1/2,h_2/2}$ and $I^4_{2h_1,2h_2}$ the finite element approximation and the postprocessing interpolation operator of degree at most 4 in x_1 and x_2 with respect to this new partition. From Theorem 3.2 we know under the H^1-norm that

$$I^4_{2h_1,2h_2} y_{h_1/2,h_2/2} - y = \left(\frac{h}{2}\right)^2 \xi + \mathcal{O}(h^4),$$

which produces by applying the Richardson extrapolation that under the H^1-norm

$$\frac{1}{3}\left(4I^4_{2h_1,2h_2} y_{h_1/2,h_2/2} - I^4_{4h_1,4h_2} y_{h_1,h_2}\right) = y + \mathcal{O}(h^4). \tag{3.8}$$

Similarly, we have under the H^1-norm that

$$\frac{1}{3}\left(4I^4_{2h_1,2h_2} p_{h_1/2,h_2/2} - I^4_{4h_1,4h_2} p_{h_1,h_2}\right) = p + \mathcal{O}(h^4). \tag{3.9}$$

It is very important for a finite element method to have a computable a posteriori error estimator so that we can assess the accuracy of the approximate solutions. The superconvergent approximations generated above in (3.8) and (3.9) can be used naturally to produce efficient a posteriori error estimators. In fact, we can obtain by using the same way as Theorem 5.3 in [12] the following result.

Theorem 3.3. *Under the assumptions of Theorem* 3.2*, we have*

$$
\begin{aligned}
& ||y - I^4_{2h_1,2h_2} y_{h_1/2,h_2/2}||_1 \\
=& \frac{1}{3}||I^4_{2h_1,2h_2} y_{h_1/2,h_2/2} - I^4_{4h_1,4h_2} y_{h_1,h_2}||_1 + \mathcal{O}(h^4),
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
& ||p - I^4_{2h_1,2h_2} p_{h_1/2,h_2/2}||_1 \\
=& \frac{1}{3}||I^4_{2h_1,2h_2} p_{h_1/2,h_2/2} - I^4_{4h_1,4h_2} p_{h_1,h_2}||_1 + \mathcal{O}(h^4).
\end{aligned}
\tag{3.11}
$$

In addition, if there exist positive constants C_1, C_2, ϵ_1, and ϵ_2 such that

$$||y - I^4_{2h_1,2h_2} y_{h_1/2,h_2/2}||_1 \geq C_1 h^{4-\epsilon_1}, \tag{3.12}$$
$$||p - I^4_{2h_1,2h_2} p_{h_1/2,h_2/2}||_1 \geq C_2 h^{4-\epsilon_2}, \tag{3.13}$$

then we have

$$\lim_{h\to 0} \frac{||I^4_{2h_1,2h_2} y_{h_1/2,h_2/2} - I^4_{4h_1,4h_2} y_{h_1,h_2}||_1}{3||y - I^4_{2h_1,2h_2} y_{h_1/2,h_2/2}||_1} = 1, \tag{3.14}$$
$$\lim_{h\to 0} \frac{||I^4_{2h_1,2h_2} p_{h_1/2,h_2/2} - I^4_{4h_1,4h_2} p_{h_1,h_2}||_1}{3||p - I^4_{2h_1,2h_2} p_{h_1/2,h_2/2}||_1} = 1. \tag{3.15}$$

From (3.10) we see that the computable error estimator

$$\frac{1}{3}||I^4_{2h_1,2h_2} y_{h_1/2,h_2/2} - I^4_{4h_1,4h_2} y_{h_1,h_2}||_1$$

is the principal part of the error $||y - I^4_{2h_1,2h_2} y_{h_1/2,h_2/2}||_1$, and can be used as an a posteriori error indicator to assess the accuracy of the finite element error $||y - I^4_{2h_1,2h_2} y_{h_1/2,h_2/2}||_1$. Meanwhile, the condition (3.12) seems to be a reasonable assumption because $\mathcal{O}(h^2)$ is the optimal convergence rate of $||y - I^4_{2h_1,2h_2} y_{h_1/2,h_2/2}||_1$ according to Theorem 3.2. Then, it can be further seen from (3.14) that the a posteriori error estimator

$$\frac{1}{3}||I^4_{2h_1,2h_2} y_{h_1/2,h_2/2} - I^4_{4h_1,4h_2} y_{h_1,h_2}||_1$$

is quite reliable. The same comments are also valid for (3.11), (3.13), and (3.15).

3.2. The Richardson extrapolation in one direction

The approach introduced in the last subsection has a limitation in that it requires a global and uniform refinement in both the x_1- and x_2-directions, and hence, it wastes computing time and memory. To overcome this shortcoming, here we propose an extrapolation method of a partial refinement (see [17] and [38]), in which the meshes are refined just in either the x_1- or x_2-direction. Thus, this method is more efficient and is also more suitable for parallel computations.

Theorem 3.4. *Under the conditions of Theorem* 3.1 *we have in the sense of the H^1-norm that*

$$y_{h_1,h_2} - i_{h_1,h_2} y = h_1^2 \xi^1_{h_1,h_2} + h_2^2 \xi^2_{h_1,h_2} + \mathcal{O}(h^4),$$
$$p_{h_1,h_2} - i_{h_1,h_2} p = h_1^2 \eta^1_{h_1,h_2} + h_2^2 \eta^2_{h_1,h_2} + \mathcal{O}(h^4),$$

where $(\xi^1_{h_1,h_2}, \eta^1_{h_1,h_2})$, $(\xi^2_{h_1,h_2}, \eta^2_{h_1,h_2}) \in V^0_{h_1,h_2} \times V^0_{h_1,h_2}$.

Proof. Let (ξ^1, η^1), $(\xi^2, \eta^2) \in H^1_0(\Omega) \times H^1_0(\Omega)$ and $(\xi^1_{h_1,h_2}, \eta^1_{h_1,h_2})$, $(\xi^2_{h_1,h_2}, \eta^2_{h_1,h_2}) \in V^0_{h_1,h_2} \times V^0_{h_1,h_2}$ be the exact solutions and the finite element solutions of the following two auxiliary variational problems, respectively:

$$\begin{cases} a(\xi^1, v) + (\eta^1, v) = L_{1,h_1}(v), & v \in H^1_0(\Omega), \\ a(\eta^1, q) - (\xi^1, q) = L_{3,h_1}(q), & q \in H^1_0(\Omega), \end{cases} \tag{3.16}$$

and

$$\begin{cases} a(\xi^2, v) + (\eta^2, v) = L_{2,h_2}(v), & v \in H_0^1(\Omega), \\ a(\eta^2, q) - (\xi^2, q) = L_{4,h_2}(q), & q \in H_0^1(\Omega), \end{cases} \tag{3.17}$$

where

$$\begin{aligned}
L_{1,h_1}(\phi) = & \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{1,e}}{h_1}\right)^2 \int_e (a_{11})_{x_1} y_{x_1x_1}\phi_{x_1} + \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{1,e}}{h_1}\right)^2 \int_e (a_{22}y_{x_2x_1x_1})_{x_2}\phi \\
& - \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{1,e}}{h_1}\right)^2 \int_e (a_{21}y_{x_1x_1})_{x_2}\phi_{x_1} - \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{1,e}}{h_1}\right)^2 \int_e a_{12}y_{x_1x_2x_2}\phi_{x_1} \\
& - \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{1,e}}{h_1}\right)^2 \int_e y_{x_1x_1}\phi, \\
L_{2,h_2}(\phi) = & \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{2,e}}{h_2}\right)^2 \int_e (a_{22})_{x_2} y_{x_2x_2}\phi_{x_2} + \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{2,e}}{h_2}\right)^2 \int_e (a_{11}y_{x_1x_2x_2})_{x_1}\phi \\
& - \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{2,e}}{h_2}\right)^2 \int_e (a_{12}y_{x_2x_2})_{x_1}\phi_{x_2} - \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{2,e}}{h_2}\right)^2 \int_e a_{21}y_{x_2x_1x_1}\phi_{x_2} \\
& - \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{2,e}}{h_2}\right)^2 \int_e y_{x_2x_2}\phi, \\
L_{3,h_1}(\psi) = & \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{1,e}}{h_1}\right)^2 \int_e (a_{11})_{x_1} p_{x_1x_1}\psi_{x_1} + \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{1,e}}{h_1}\right)^2 \int_e (a_{22}p_{x_2x_1x_1})_{x_2}\psi \\
& - \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{1,e}}{h_1}\right)^2 \int_e (a_{21}p_{x_1x_1})_{x_2}\psi_{x_1} - \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{1,e}}{h_1}\right)^2 \int_e a_{12}p_{x_1x_2x_2}\psi_{x_1} \\
& + \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{1,e}}{h_1}\right)^2 \int_e p_{x_1x_1}\psi, \\
L_{4,h_2}(\psi) = & \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{2,e}}{h_2}\right)^2 \int_e (a_{22})_{x_2} p_{x_2x_2}\psi_{x_2} + \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{2,e}}{h_2}\right)^2 \int_e (a_{11}p_{x_1x_2x_2})_{x_1}\psi \\
& - \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{2,e}}{h_2}\right)^2 \int_e (a_{12}p_{x_2x_2})_{x_1}\psi_{x_2} - \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{2,e}}{h_2}\right)^2 \int_e a_{21}p_{x_2x_1x_1}\psi_{x_2} \\
& + \sum_{e\in T_{h_1,h_2}} \frac{1}{3}\left(\frac{h_{2,e}}{h_2}\right)^2 \int_e p_{x_2x_2}\psi.
\end{aligned}$$

From (3.1) and (3.2) one finds that

$$\begin{aligned}
& a(\rho_{h_1,h_2}, v_{h_1,h_2}) + (\theta_{h_1,h_2}, v_{h_1,h_2}) \\
=& h_1^2 L_{1,h_1}(v_{h_1,h_2}) + h_2^2 L_{2,h_2}(v_{h_1,h_2}),\ v_{h_1,h_2} \in V^0_{h_1,h_2},
\end{aligned}$$

and

$$\begin{aligned}
& a(\theta_{h_1,h_2}, q_{h_1,h_2}) - (\rho_{h_1,h_2}, q_{h_1,h_2}) \\
=& h_1^2 L_{3,h_1}(v_{h_1,h_2}) + h_2^2 L_{4,h_2}(v_{h_1,h_2}),\ q_{h_1,h_2} \in V^0_{h_1,h_2},
\end{aligned}$$

which, together with (3.16) and (3.17), implies

$$\begin{aligned}&a(\rho_{h_1,h_2}-h_1^2\xi^1_{h_1,h_2}-h_2^2\xi^2_{h_1,h_2},v_{h_1,h_2})+(\theta_{h_1,h_2}-h_1^2\eta^1_{h_1,h_2}-h_2^2\eta^2_{h_1,h_2},v_{h_1,h_2})\\ =&\mathcal{O}(h^4)||y||_5||v_{h_1,h_2}||_1,\ v_{h_1,h_2}\in V^0_{h_1,h_2},\end{aligned} \tag{3.18a}$$

and

$$\begin{aligned}&a(\theta_{h_1,h_2}-h_1^2\eta^1_{h_1,h_2}-h_2^2\eta^2_{h_1,h_2},q_{h_1,h_2})-(\rho_{h_1,h_2}-h_1^2\xi^1_{h_1,h_2}-h_2^2\xi^2_{h_1,h_2},q_{h_1,h_2})\\ =&\mathcal{O}(h^4)||p||_5||q_{h_1,h_2}||_1,\ q_{h_1,h_2}\in V^0_{h_1,h_2},\end{aligned} \tag{3.18b}$$

where $(\xi^1_{h_1,h_2},\eta^1_{h_1,h_2})$ and $(\xi^2_{h_1,h_2},\eta^2_{h_1,h_2})$ are the finite element solutions of (3.16) and (3.17), respectively.

Set

$$\hat{\rho}_{h_1,h_2}:=\rho_{h_1,h_2}-h_1^2\xi^1_{h_1,h_2}-h_2^2\xi^2_{h_1,h_2},\ \hat{\theta}_{h_1,h_2}:=\theta_{h_1,h_2}-h_1^2\eta^1_{h_1,h_2}-h_2^2\eta^2_{h_1,h_2}.$$

Then, it follows from (3.16)–(3.18) that

$$a(\hat{\rho}_{h_1,h_2},v_{h_1,h_2})+(\hat{\theta}_{h_1,h_2},v_{h_1,h_2})=\mathcal{O}(h^4)||y||_5||v_{h_1,h_2}||_1,\qquad v_{h_1,h_2}\in V^0_{h_1,h_2}, \tag{3.19a}$$
$$a(\hat{\theta}_{h_1,h_2},q_{h_1,h_2})-(\hat{\rho}_{h_1,h_2},q_{h_1,h_2})=\mathcal{O}(h^4)||p||_5||q_{h_1,h_2}||_1,\qquad q_{h_1,h_2}\in V^0_{h_1,h_2}. \tag{3.19b}$$

Following the steps for the estimates of $\rho^*_{h_1,h_2}$ and $\theta^*_{h_1,h_2}$ in the proof of Theorem 3.1 yields by means of (3.19) that

$$||\hat{\rho}_{h_1,h_2}||_1\le Ch^4\quad\text{and}\quad||\hat{\theta}_{h_1,h_2}||_1\le Ch^4.$$ □

By the same argument as that for Theorem 3.2, we can establish the following result.

Theorem 3.5. *We have under the conditions of Theorem* 3.4 *that*

$$\begin{aligned}I^4_{4h_1,4h_2}y_{h_1,h_2}-y&=h_1^2\xi_1+h_2^2\xi_2+\tilde{r}_{h_1,h_2},\quad ||\tilde{r}_{h_1,h_2}||_1\le Ch^4,\\ I^4_{4h_1,4h_2}p_{h_1,h_2}-p&=h_1^2\eta_1+h_2^2\eta_2+\hat{r}_{h_1,h_2},\quad ||\hat{r}_{h_1,h_2}||_1\le Ch^4,\end{aligned}$$

where (ξ_1,η_1), $(\xi_2,\eta_2)\in H^1_0(\Omega)\times H^1_0(\Omega)$ *are the variational solutions of* (3.16) *and* (3.17), *respectively.*

From Theorem 3.5 one can obtain the following unidirectional Richardson extrapolation results under the H^1-norm:

$$\frac{4}{3}(I^4_{2h_1,4h_2}y_{h_1/2,h_2}+I^4_{4h_1,2h_2}y_{h_1,h_2/2})-\frac{5}{3}(I^4_{4h_1,4h_2}y_{h_1,h_2})=y+\mathcal{O}(h^4), \tag{3.20a}$$
$$\frac{4}{3}(I^4_{2h_1,4h_2}p_{h_1/2,h_2}+I^4_{4h_1,2h_2}p_{h_1,h_2/2})-\frac{5}{3}(I^4_{4h_1,4h_2}p_{h_1,h_2})=p+\mathcal{O}(h^4), \tag{3.20b}$$

where $(y_{h_1/2,h_2},p_{h_1/2,h_2})$, $(y_{h_1,h_2/2},p_{h_1,h_2/2})$ and $(y_{h_1,h_2},p_{h_1,h_2})$ are the finite element solutions corresponding to the meshes $T_{h_1/2,h_2}$, $T_{h_1,h_2/2}$ and T_{h_1,h_2}, respectively. Here, $T_{h_1/2,h_2}$ and $T_{h_1,h_2/2}$ are the meshes gained by subdividing each element of T_{h_1,h_2} into two small congruent rectangles in x_1- and x_2-direction, respectively.

Similar to (3.8) and (3.9), we can also construct a posteriori error estimators by virtue of (3.20).

Theorem 3.6. *Under the conditions of Theorem* 3.5, *we have*

$$
\begin{aligned}
&||y - I^4_{2h_1,4h_2} y_{h_1/2,h_2}||_1 \\
=&\frac{1}{3}||I^4_{2h_1,4h_2} y_{h_1/2,h_2} + 4I^4_{4h_1,2h_2} y_{h_1,h_2/2} - 5I^4_{4h_1,4h_2} y_{h_1,h_2}||_1 + \mathcal{O}(h^4), \\
&||p - I^4_{2h_1,4h_2} p_{h_1/2,h_2}||_1 \\
=&\frac{1}{3}||I^4_{2h_1,4h_2} p_{h_1/2,h_2} + 4I^4_{4h_1,2h_2} p_{h_1,h_2/2} - 5I^4_{4h_1,4h_2} p_{h_1,h_2}||_1 + \mathcal{O}(h^4), \\
&||y - I^4_{4h_1,2h_2} y_{h_1,h_2/2}||_1 \\
=&\frac{1}{3}||I^4_{4h_1,2h_2} y_{h_1,h_2/2} + 4I^4_{2h_1,4h_2} y_{h_1/2,h_2} - 5I^4_{4h_1,4h_2} y_{h_1,h_2}||_1 + \mathcal{O}(h^4), \\
&||p - I^4_{4h_1,2h_2} p_{h_1,h_2/2}||_1 \\
=&\frac{1}{3}||I^4_{4h_1,2h_2} p_{h_1,h_2/2} + 4I^4_{2h_1,4h_2} p_{h_1/2,h_2} - 5I^4_{4h_1,4h_2} p_{h_1,h_2}||_1 + \mathcal{O}(h^4).
\end{aligned}
$$

In addition, if there exist positive constants C_1, C_2, C_3, C_4 *and* ϵ_1, ϵ_2, ϵ_3, ϵ_4 *such that*

$$
\begin{aligned}
&||y - I^4_{2h_1,4h_2} y_{h_1/2,h_2}||_1 \geq C_1 h^{4-\epsilon_1}, \\
&||p - I^4_{2h_1,4h_2} p_{h_1/2,h_2}||_1 \geq C_2 h^{4-\epsilon_2}, \\
&||y - I^4_{4h_1,2h_2} y_{h_1,h_2/2}||_1 \geq C_3 h^{4-\epsilon_3}, \\
&||p - I^4_{4h_1,2h_2} p_{h_1,h_2/2}||_1 \geq C_4 h^{4-\epsilon_4},
\end{aligned}
$$

then we have

$$
\begin{aligned}
&\lim_{h\to 0} \frac{||I^4_{2h_1,4h_2} y_{h_1/2,h_2} + 4I^4_{4h_1,2h_2} y_{h_1,h_2/2} - 5I^4_{4h_1,4h_2} y_{h_1,h_2}||_1}{3||y - I^4_{2h_1,4h_2} y_{h_1/2,h_2}||_1} = 1, \\
&\lim_{h\to 0} \frac{||I^4_{2h_1,4h_2} p_{h_1/2,h_2} + 4I^4_{4h_1,2h_2} p_{h_1,h_2/2} - 5I^4_{4h_1,4h_2} p_{h_1,h_2}||_1}{3||p - I^4_{2h_1,4h_2} p_{h_1/2,h_2}||_1} = 1, \\
&\lim_{h\to 0} \frac{||4I^4_{2h_1,4h_2} y_{h_1/2,h_2} + I^4_{4h_1,2h_2} y_{h_1,h_2/2} - 5I^4_{4h_1,4h_2} y_{h_1,h_2}||_1}{3||y - I^4_{4h_1,2h_2} y_{h_1,h_2/2}||_1} = 1, \\
&\lim_{h\to 0} \frac{||4I^4_{2h_1,4h_2} p_{h_1/2,h_2} + I^4_{4h_1,2h_2} p_{h_1,h_2/2} - 5I^4_{4h_1,4h_2} p_{h_1,h_2}||_1}{3||p - I^4_{4h_1,2h_2} p_{h_1,h_2/2}||_1} = 1.
\end{aligned}
$$

4. The Interpolation Defect Correction

In this section we propose and investigate an interpolation defect correction scheme (see, e.g., [18, 20, 21, 23]) applied to the finite element solution $(y_{h_1,h_2}, p_{h_1,h_2}) \in V^0_{h_1,h_2} \times V^0_{h_1,h_2}$ to obtain approximations with higher convergence rate. Also, these new approximations are naturally used to form a posteriori error estimators in order to estimate the actual accuracy of the finite element solutions.

First of all, for the future need we construct another interpolation operator $I^2_{2h_1,2h_2}$ associated with the mesh $T_{2h_1,2h_2}$ to satisfy

$$I^2_{2h_1,2h_2} i_{h_1,h_2} = I^2_{2h_1,2h_2}, \tag{4.1a}$$

$$||I^2_{2h_1,2h_2} v_{h_1,h_2}||_1 \leq C||v_{h_1,h_2}||_1 \qquad \forall v_{h_1,h_2} \in V^0_{h_1,h_2}, \tag{4.1b}$$

$$||I^2_{2h_1,2h_2} u - u||_1 \leq Ch^2||u||_3 \qquad u \in H^3(\Omega). \tag{4.1c}$$

Then, similar to the last section, it is again assumed that the rectangular partition T_{h_1,h_2} has been obtained from $T_{2h_1,2h_2}$ with mesh size $2h$ by subdividing each element of $T_{2h_1,2h_2}$ into four small congruent rectangles. Set $\hat{e} := \bigcup_{i=1}^4 e_i$ with $e_i \in T_{h_1,h_2}$. And thus, the interpolation operator $I_{2h_1,2h_2}^2$ of degree at most 2 in x_1 and x_2 on $\hat{e}$ is defined as follows:

$$
\begin{aligned}
&I_{2h_1,2h_2}^2 u|_{\hat{e}} \in Q_{2,2}(\hat{e}),\\
&\left(I_{2h_1,2h_2}^2 u\right)(z_i) = u(z_i), \quad i = 1, \ldots, 9,
\end{aligned}
$$

where z_i $(i = 1, \ldots, 9)$ are the nine vertices of the four small elements e_i $(i = 1, 2, 3, 4)$. We can also check that the interpolation operator $I_{2h_1,2h_2}^2$ defined above is of the properties indicated in (4.1).

In addition, we also need a pair of finite element projection operator $R_{h_1,h_2} \times S_{h_1,h_2}$: $H_0^1(\Omega) \times H_0^1(\Omega) \to V_{h_1,h_2}^0 \times V_{h_1,h_2}^0$ defined by

$$
\begin{aligned}
&a(R_{h_1,h_2} y - y, v_{h_1,h_2}) + (S_{h_1,h_2} p - p, v_{h_1,h_2}) = 0, \quad v_{h_1,h_2} \in V_{h_1,h_2}^0,\\
&a(S_{h_1,h_2} p - p, q_{h_1,h_2}) - (R_{h_1,h_2} y - y, q_{h_1,h_2}) = 0, \quad q_{h_1,h_2} \in V_{h_1,h_2}^0.
\end{aligned}
$$

Then, $(R_{h_1,h_2} y, S_{h_1,h_2} p)$ is the solution of (2.5) if (y, p) is the solution of (2.3).

Theorem 4.1. *Suppose that the conditions of Theorem* 3.2 *are fulfilled. Then, we have*

$$
||y_{h_1,h_2}^* - y||_1 + ||p_{h_1,h_2}^* - p||_1 \le Ch^4,
$$

where

$$
\begin{aligned}
y_{h_1,h_2}^* &:= I_{4h_1,4h_2}^4 y_{h_1,h_2} + I_{2h_1,2h_2}^2 y_{h_1,h_2} - I_{2h_1,2h_2}^2 R_{h_1,h_2} I_{4h_1,4h_2}^4 y_{h_1,h_2},\\
p_{h_1,h_2}^* &:= I_{4h_1,4h_2}^4 p_{h_1,h_2} + I_{2h_1,2h_2}^2 p_{h_1,h_2} - I_{2h_1,2h_2}^2 S_{h_1,h_2} I_{4h_1,4h_2}^4 p_{h_1,h_2}.
\end{aligned}
$$

Proof. It has been proved in Theorem 3.2 that

$$
I_{4h_1,4h_2}^4 y_{h_1,h_2} - y = h^2 \xi + r_{h_1,h_2} \quad \text{with} \quad ||r_{h_1,h_2}||_1 \le Ch^4.
$$

Then, multiplying this equality by the operator $(I - I_{2h_1,2h_2}^2 R_{h_1,h_2})$, where I is the identity operator, yields in H^1-norm that

$$
\begin{aligned}
&(I - I_{2h_1,2h_2}^2 R_{h_1,h_2})(I_{4h_1,4h_2}^4 y_{h_1,h_2} - y)\\
=&h^2 (I - I_{2h_1,2h_2}^2 R_{h_1,h_2})\xi + \mathcal{O}(h^4)\\
=&h^2 (\xi - I_{2h_1,2h_2}^2 \xi) + h^2 (I_{2h_1,2h_2}^2 \xi - I_{2h_1,2h_2}^2 \xi_{h_1,h_2}) + \mathcal{O}(h^4)\\
=&h^2 I_{2h_1,2h_2}^2 (i_{h_1,h_2}\xi - \xi_{h_1,h_2}) + \mathcal{O}(h^4),
\end{aligned}
$$

where we have used

$$
||\xi - I_{2h_1,2h_2}^2 \xi||_1 \le Ch^2 ||\xi||_3 \quad \text{and} \quad I_{2h_1,2h_2}^2 i_{h_1,h_2} = I_{2h_1,2h_2}^2
$$

according to the properties of the operator $I_{2h_1,2h_2}^2$ described in (4.1). Furthermore, it follows from Lemma 3.1 and the inequality

$$
||I_{2h_1,2h_2}^2 (i_{h_1,h_2}\xi - \xi_{h_1,h_2})||_1 \le C ||i_{h_1,h_2}\xi - \xi_{h_1,h_2}||_1
$$

that in H^1-norm

$$
(I - I_{2h_1,2h_2}^2 R_{h_1,h_2})(I_{4h_1,4h_2}^4 y_{h_1,h_2} - y) = \mathcal{O}(h^4),
$$

and the left-hand side is nothing but

$$(I - I^2_{2h_1,2h_2} R_{h_1,h_2})(I^4_{4h_1,4h_2} y_{h_1,h_2} - y) = y^*_{h_1,h_2} - y.$$

Similarly, from the equality

$$I^4_{4h_1,4h_2} p_{h_1,h_2} - p = h^2 \eta + r^*_{h_1,h_2} \quad \text{with} \quad ||r^*_{h_1,h_2}||_1 \le Ch^4$$

we can derive that

$$p^*_{h_1,h_2} - p = \mathcal{O}(h^4) \quad \text{in } H^1\text{-norm.} \qquad \square$$

Analogous to Section 3 we can utilize the superconvergent approximation provided in Theorems 4.1 to establish a posteriori error estimators for the finite element solution of the problem (2.3). In fact, we have

Theorem 4.2. *If the conditions of Theorem* 4.1 *are satisfied, then*

$$||y - y_{h_1,h_2}||_1 = ||y^*_{h_1,h_2} - y_{h_1,h_2}||_1 + \mathcal{O}(h^4),$$
$$||p - p_{h_1,h_2}||_1 = ||p^*_{h_1,h_2} - p_{h_1,h_2}||_1 + \mathcal{O}(h^4).$$

Furthermore, if there exist positive constants C_1, C_2, ϵ_1, *and* ϵ_2 *such that*

$$||y - y_{h_1,h_2}||_1 \ge C_1 h^{4-\epsilon_1}, \ \ ||p - p_{h_1,h_2}||_1 \ge C_2 h^{4-\epsilon_2},$$

then we have

$$\lim_{h\to 0} \frac{||y^*_{h_1,h_2} - y_{h_1,h_2}||_1}{||y - y_{h_1,h_2}||_1} = 1, \ \ \lim_{h\to 0} \frac{||p^*_{h_1,h_2} - p_{h_1,h_2}||_1}{||p - p_{h_1,h_2}||_1} = 1.$$

5. Discussions

In this paper, we derived asymptotic error expansions in the sense of H^1-norm for the bilinear finite element approximation to a class of optimal control problems under rectangular meshes. Based on the asymptotic error expansions, the Richardson extrapolation of two different schemes and an interpolation defect correction are provided. Furthermore, as a result of all these higher order numerical approximations, they can be used to generate a posteriori error estimators for the finite element approximation. It should be pointed out that in order to obtain the asymptotic error expansions, the high regularity of the solution to the state and adjoint equation is assumed. This assumption is too strong for many practical problems. However, it is still significant to provide these numerical schemes with high accuracy for optimal control problems in either theory or practice.

There are many important issues remaining to be addressed in this area, including high accuracy analysis for more complicated control problems. Moreover, many computational issues have to be addressed for designing high accurate numerical methods for the optimal control problems.

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