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WEIGHTED L^2-NORM A POSTERIORI ERROR ESTIMATION OF FEM IN POLYGONS

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Abstract. In this paper, we generalize well-known results for the L^2-norm a posteriori error estimation of finite element methods applied to linear elliptic problems in convex polygonal domains to the case where the polygons are non-convex. An important factor in our analysis is the investigation of a suitable dual problem whose solution, due to the non-convexity of the domain, may exhibit corner singularities. In order to describe this singular behavior of the dual solution certain weighted Sobolev spaces are employed. Based on this framework, upper and lower a posteriori error estimates in weighted L^2-norms are derived. Furthermore, the performance of the proposed error estimators is illustrated with a series of numerical experiments.

Key Words. Finite element methods, a posteriori error analysis, L^2-norm error estimation, non-convex polygonal domains.

1. Introduction

Given a (possibly non-convex) bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with (Lipschitz) boundary $\Gamma = \partial\Omega$, and a function $f \in L^2(\Omega)$, we consider the elliptic model problem

$$-\Delta u = f \qquad \text{in } \Omega \tag{1}$$

$$u = 0 \qquad \text{on } \Gamma. \tag{2}$$

The standard weak formulation of (1)–(2) reads: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, d\boldsymbol{x} = \int_\Omega f v \, d\boldsymbol{x} \tag{3}$$

for all $v \in H_0^1(\Omega)$. Here and it what follows, we use the following notation: For a domain $D \subset \mathbb{R}^n$ ($n = 1$ or $n = 2$) and an integer $k \in \mathbb{N}_0$, we denote by $H^k(D)$ the usual Sobolev space of order k on D, with norm $\|\cdot\|_{k,D}$ and semi-norm $|\cdot|_{k,D}$. The space $H_0^1(\Omega)$ is defined as the subspace of $H^1(\Omega)$ with zero trace on $\partial\Omega$. Furthermore, $H^{-1}(D)$ denotes the dual space of $H_0^1(D)$, and $L^2(D) = H^0(D)$.

In order to discretize the variational formulation (3) by a finite element method, we consider a regular subdivision $\mathcal{T}_{FE}$ (finite element mesh) of Ω into disjoint open triangles K (elements), i.e. $\mathcal{T}_{FE} = \{K\}$, $\bigcup_{K \in \mathcal{T}_{FE}} \overline{K} = \overline{\Omega}$. By h_K, we denote the diameter of an element $K \in \mathcal{T}_{FE}$. We assume that the subdivision $\mathcal{T}_{FE}$ is shape-regular (see, e.g., [6]) and of local bounded variation. The latter assumption means

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that there exists a constant $\sigma > 1$ such that $\sigma^{-1} < h_{K_\sharp}/h_{K_\flat} < \sigma$, for any two neighboring elements $K_\sharp$ and $K_\flat$. Moreover, we introduce the finite element space

$$\mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE}) = \{\phi \in H_0^1(\Omega) : \phi|_K \in \mathcal{P}_1(K), K \in \mathcal{T}_{FE}\},$$

where, for $k \in \mathbb{N}_0$ and $K \in \mathcal{T}_{FE}$, $\mathcal{P}_k(K)$ is defined as the set of all polynomials of total degree (at most) k on K.

A finite element approximation of the exact solution $u \in H_0^1(\Omega)$ of (1)–(2) can now be obtained in the usual way by finding the unique solution $u_{FE} \in \mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE})$ of the discrete variational formulation

$$a(u_{FE}, v) = \ell(v) \qquad \forall v \in \mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE}), \tag{4}$$

where

$$a(w, v) = \int_\Omega \nabla w \cdot \nabla v \, d\boldsymbol{x}, \qquad \ell(v) = \int_\Omega f v \, d\boldsymbol{x}.$$

Clearly, there holds the Galerkin orthogonality

$$a(e_{FE}, v) = 0 \qquad \forall v \in \mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE}). \tag{5}$$

Here, e_{FE} is the finite element error given by

$$e_{FE} = u - u_{FE}, \tag{6}$$

where $u \in H_0^1(\Omega)$ is again the exact solution of (1)–(2), and $u_{FE} \in \mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE})$ is its numerical approximation from (4).

Standard techniques for the a posteriori error estimation of the L^2-norm of e_{FE}, consist usually of the following two main steps (Aubin-Nitsche trick; see, e.g., [1, 5, 12], and the references therein): Firstly, a suitable dual problem is formulated; this makes it possible to relate the L^2-norm of e_{FE} to the finite element method (4). Secondly, using the Galerkin orthogonality (5), the L^2-error $\|e_{FE}\|_{L^2(\Omega)}$ is bounded in terms of some approximation errors between the solution of the dual problem and an appropriate interpolant in the finite element space $\mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE})$. Here, standard L^2-norm a posteriori analyses are typically based on approximation results that require the global H^2-regularity of the dual problem, which is in fact available if the polygonal domain Ω is convex; see [2, 7, 8], for example. In non-convex polygons, however, this assumption does generally not hold; here, due to the presence of corner singularities, the regularity of the dual problem is typically reduced to $H^{1+\varepsilon}$, for $\varepsilon < 1$. Consequently, standard H^2-approximation results cannot be applied.

The goal of this paper is to generalize the above-mentioned approach for the L^2-norm a posteriori error estimation of the finite element error e_{FE} to the case where the domain Ω is a possibly non-convex polygon. To this end, we describe the regularity of the dual problem in terms of weighted Sobolev spaces using the results in [2], and apply some appropriate interpolation results (see, e.g., [11], and the references therein) to approximate its solution in the finite element space $\mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE})$. We will then be able to derive upper a posteriori bounds for some weighted L^2-norms of the error e_{FE}; see Theorem 3.1. More precisely, we will show that there holds an estimate of the form

$$\|\Phi^{-1} e_{FE}\|_{L^2(\Omega)}^2 \leq C \sum_{K \in \mathcal{T}_{FE}} \eta_K(u_{FE}, f)^2, \tag{7}$$

where Φ is a certain weight function (associated to Ω), and η_K, $K \in \mathcal{T}_{FE}$, are local error indicators depending on the mesh $\mathcal{T}_{FE}$, the finite element solution u_{FE} from (4) and the right-hand side f in (1). In addition, using suitable cut-off functions (see [9]), we will also prove some (weighted) local lower bounds; see Theorem 3.4.

Throughout the paper, we shall use the following notation: By $\mathcal{E}_I$ we denote the set of all element edges in the finite element mesh $\mathcal{T}_{FE}$ which do not belong to $\partial\Omega$. Note that, for $E \in \mathcal{E}_{\mathcal{I}}$, there exist two neighboring elements $K_\sharp, K_\flat \in \mathcal{T}_{FE}$ such that $E = \partial K_\sharp \cap \partial K_\flat$; we let $\omega_E = \{K_\sharp, K_\flat\}$, and define $\Omega_E = (\overline{K}_\sharp \cup \overline{K}_\flat)^\circ$. Furthermore, for $E \in \mathcal{E}_{\mathcal{I}}$ and a vector function $\boldsymbol{v} \in \{\boldsymbol{\phi} \in L^2(\Omega_E)^2 : \boldsymbol{\phi}|_K \in H^1(K)^2, K \in \omega_E\}$, we define the jump

$$\llbracket \boldsymbol{v} \rrbracket(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x})|_{K_\sharp} \cdot \boldsymbol{n}_{K_\sharp} + \boldsymbol{v}(\boldsymbol{x})|_{K_\flat} \cdot \boldsymbol{n}_{K_\flat}, \qquad \boldsymbol{x} \in E. \tag{8}$$

Here, for an element $K \in \mathcal{T}_{FE}$, $\boldsymbol{n}_K$ represents the outward normal unit vector to ∂K. Finally, for an edge $E \in \mathcal{E}_{\mathcal{I}}$, we denote by h_E the length of E; note that, due to the shape regularity of $\mathcal{T}_{FE}$, there holds that

$$\mu^{-1} h_K \leq h_E \leq \mu h_K, \qquad \forall K \in \omega_E, \quad \forall E \in \mathcal{E}_{\mathcal{I}}, \tag{9}$$

where the constant $\mu \geq 1$ is independent of the element diameters.

The remaining part of this article is organized as follows: In Section 2 we introduce the dual problem, which the a posteriori error analysis in this paper is based upon, and discuss its regularity. Section 3 contains the upper and lower a posteriori error estimates and their proofs. In Section 4, we illustrate the performance of the error estimators presented in Section 3 with some numerical experiments.

2. Dual Problem

In order to prove the a posteriori error estimate (7), we shall study a dual problem of the form

$$-\Delta\psi = \chi(e_{FE}) \qquad \text{in } \Omega \tag{10}$$

$$\psi = 0 \qquad \text{on } \Gamma, \tag{11}$$

with a (weak) solution $\psi \in H_0^1(\Omega)$. Here, the right-hand side $\chi(e_{FE}) \in H^{-1}(\Omega)$ depends on the finite element error (6) and will be specified later.

An important factor in our analysis will be the approximation of the dual solution ψ by functions in the finite element space $\mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE})$. Here, the regularity of ψ plays an essential role. We note that, since the polygonal domain Ω is possibly non-convex, the solution ψ of (10)–(11) does not necessarily belong to $H^2(\Omega)$ (even if the right-hand side $\chi(e_{FE})$ was smooth) due to the presence of corner singularities. A possible way to describe this low regularity of ψ is given by the use of weighted Sobolev spaces; cf. [2, 3], for example.

2.1. Weighted Sobolev Spaces.

Let $\mathcal{A} = \{A_i\}_{i=1}^M$ be the set of all corners of the polygonal domain Ω. To each of these points A_i, $i = 1, 2, \ldots, M$, we associate a weight

$$\beta_i \in [0, 1). \tag{12}$$

These numbers are stored in a weight vector

$$\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_M). \tag{13}$$

Next, we introduce the following weight function on Ω:

$$\Phi_{\boldsymbol{\beta}}(\boldsymbol{x}) = \prod_{i=1}^M r_i(\boldsymbol{x})^{\beta_i}, \qquad r_i(\boldsymbol{x}) = |\boldsymbol{x} - A_i|.$$

Then, for any integers $m \geq l \geq 0$, the weighted Sobolev spaces $H^{m,l}_{\boldsymbol{\beta}}(\Omega)$ are defined as the completion of the space $C^\infty(\overline{\Omega})$ with respect to the weighted Sobolev norms

$$\|u\|^2_{H^{m,l}_{\boldsymbol{\beta}}(\Omega)} = \|u\|^2_{l-1,\Omega} + \sum_{k=l}^{m} \sum_{|\alpha|=k} \|\Phi_{\boldsymbol{\beta}+k-l} |D^{\boldsymbol{\alpha}} u|\|^2_{0,\Omega}, \qquad l \geq 1,$$

$$\|u\|^2_{H^{m,0}_{\boldsymbol{\beta}}(\Omega)} = \sum_{k=0}^{m} \sum_{|\alpha|=k} \|\Phi_{\boldsymbol{\beta}+k-l} |D^{\boldsymbol{\alpha}} u|\|^2_{0,\Omega}.$$

Here,

$$D^{\boldsymbol{\alpha}} u = \frac{\partial^{|\boldsymbol{\alpha}|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} = u_{x_1^{\alpha_1}, x_2^{\alpha_2}},$$

with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ and $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2$.

Before discussing the regularity of the dual problem (10)–(11), we prove the following auxiliary result:

Lemma 2.1. *Let $\boldsymbol{\beta}$ be a weight vector as in* (12)–(13). *Then, for a function $v \in H^1(\Omega)$, there holds that $\Phi^{-2}_{\boldsymbol{\beta}} v \in H^{0,0}_{\boldsymbol{\beta}}(\Omega)$.*

Proof. For each corner $A_i \in \mathcal{A}$, $1 \leq i \leq M$, of the polygon Ω, we define a sector $S_i = \Omega \cap U_\epsilon(A_i)$, where $U_\epsilon(A_i) = \{\boldsymbol{x} : |\boldsymbol{x} - A_i| < \epsilon\}$. Here, we assume that $\epsilon > 0$ is chosen sufficiently small, so that the sectors S_i, $1 \leq i \leq M$, are disjoint. Furthermore, we let $\Omega_\epsilon = \Omega \setminus (\bigcup_{i=1}^M S_i)$.

Recalling the definition of the weighted Sobolev norm $\|\cdot\|_{H^{0,0}_{\boldsymbol{\beta}}(\Omega)}$, we have

$$\|\Phi^{-2}_{\boldsymbol{\beta}} v\|^2_{H^{0,0}_{\boldsymbol{\beta}}(\Omega)} = \int_\Omega \Phi^{-2}_{\boldsymbol{\beta}} v^2 \, d\boldsymbol{x}.$$

Hence, we obtain

$$\begin{aligned}
\|\Phi^{-2}_{\boldsymbol{\beta}} v\|^2_{H^{0,0}_{\boldsymbol{\beta}}(\Omega)} &= \sum_{i=1}^{M} \int_{S_i} \Phi^{-2}_{\boldsymbol{\beta}} v^2 \, d\boldsymbol{x} + \int_{\Omega_\epsilon} \Phi^{-2}_{\boldsymbol{\beta}} v^2 \, d\boldsymbol{x} \\
&\leq \sum_{i=1}^{M} \sup_{\boldsymbol{x} \in S_i} \Bigg(\prod_{\substack{j=1 \\ j \neq i}}^{M} r_j(\boldsymbol{x})^{-2\beta_j} \Bigg) \int_{S_i} r_i^{-2\beta_i} v^2 \, d\boldsymbol{x} \\
&\quad + \sup_{\boldsymbol{x} \in \Omega_\epsilon} \Phi_{\boldsymbol{\beta}}(\boldsymbol{x})^{-2} \int_{\Omega_\epsilon} v^2 \, d\boldsymbol{x} \\
&\leq C \sum_{i=1}^{M} \int_{S_i} r_i^{-2\beta_i} v^2 \, d\boldsymbol{x} + C \int_{\Omega_\epsilon} v^2 \, d\boldsymbol{x}.
\end{aligned}$$

Furthermore, there holds (cf. the proof of [4, Lemma 4.3], see also the proof of [10, Proposition 25]) that

$$\int_{S_i} r_i^{-2\beta_i} v^2 \, d\boldsymbol{x} \leq C \|v\|^2_{H^{1,1}_{1-\beta_i}(S_i)} \leq C \|v\|^2_{1,S_i}$$

for all $1 \leq i \leq M$. Thus, it follows that

$$\|\Phi^{-2}_{\boldsymbol{\beta}} v\|^2_{H^{0,0}_{\boldsymbol{\beta}}(\Omega)} \leq C \|v\|^2_{1,\Omega},$$

which completes the proof. □

2.2. Regularity of the Dual Problem. Let $\boldsymbol{\beta}$ be a weight vector as in Section 2.1. Then, we define $\chi(e_{FE})$ in (10) by

$$\chi(e_{FE}) = \Phi_{\boldsymbol{\beta}}^{-2} e_{FE}. \tag{14}$$

By the definition of the finite element space $\mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE})$, there holds $e_{FE} \in H_0^1(\Omega)$, and thus, the previous Lemma 2.1 implies that $\chi(e_{FE}) \in H_{\boldsymbol{\beta}}^{0,0}(\Omega)$. Therefore, applying [2, Theorem 2.1 and Remark 3], we obtain the following regularity result for the dual problem (10)–(11):

Proposition 2.2. *Let $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_M)$ be a weight vector on Ω with*

$$1 > \beta_i > 1 - \frac{\pi}{\omega_i}, \tag{15}$$

where ω_i denotes the interior angle of Ω at the corner A_i, $i = 1, 2, \dots, M$. Then, the weak solution $\psi \in H_0^1(\Omega)$ of the dual problem (10)–(11) *exists (and is unique) and belongs to $H_{\boldsymbol{\beta}}^{2,2}(\Omega)$. Furthermore, the bound*

$$\|\psi\|_{H_{\boldsymbol{\beta}}^{2,2}(\Omega)} \le C \|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,\Omega} \tag{16}$$

holds true.

2.3. Approximation of $H_{\boldsymbol{\beta}}^{2,2}(\Omega)$ in $\mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE})$. In order to approximate the solution ψ of the dual problem (10)–(11) by functions in the finite element space $\mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE})$, an interpolation estimate for the space $H_{\boldsymbol{\beta}}^{2,2}(\Omega)$ (when approximated by the finite element space $\mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE})$) is required.

Here and it what follows, we define for an element $K \in \mathcal{T}_{FE}$ the local quantity

$$\mathtt{H}_{\boldsymbol{\beta},K} = \sup_{\boldsymbol{x} \in K} \Phi_{\boldsymbol{\beta}}(\boldsymbol{x}). \tag{17}$$

Furthermore, by $\mathcal{K}_{\mathcal{A}}$ we denote the set of all elements in $\mathcal{T}_{FE}$ that contain a corner of the polygon Ω on their boundary, i.e.

$$\mathcal{K}_{\mathcal{A}} = \left\{ K \in \mathcal{T}_{FE} : A \in \overline{K} \text{ for a certain } A \in \mathcal{A} \right\}.$$

For simplicity, we assume that the mesh $\mathcal{T}_{FE}$ is sufficiently refined, so that each $K \in \mathcal{K}_{\mathcal{A}}$ contains exactly one corner of Ω. Moreover, for $K \in \mathcal{K}_{\mathcal{A}}$, we let $\beta_K = \beta_i$, where β_i is the weight associated with the corner of Ω that is contained in $\overline{K}$.

Due to our assumptions on the mesh $\mathcal{T}_{FE}$, there holds:

Lemma 2.3. *There exists a constant $C > 0$ independent of the element sizes such that*

$$\mathtt{H}_{\boldsymbol{\beta},K} \le C \inf_{\boldsymbol{x} \in K} \Phi_{\boldsymbol{\beta}}(\boldsymbol{x}) \tag{18}$$

for all $K \in \mathcal{T}_{FE} \setminus \mathcal{K}_{\mathcal{A}}$.

Proof. Let $K \in \mathcal{T}_{FE} \setminus \mathcal{K}_{\mathcal{A}}$. Then,

(19)

$$\begin{aligned}
\mathtt{H}_{\boldsymbol{\beta},K} &= \sup_{\boldsymbol{x} \in K} \Phi_{\boldsymbol{\beta}}(\boldsymbol{x}) \le \prod_{i=1}^{M} \sup_{\boldsymbol{x} \in K} r_i(\boldsymbol{x})^{\beta_i} = \prod_{i=1}^{M} \inf_{\boldsymbol{x} \in K} r_i(\boldsymbol{x})^{\beta_i} \prod_{i=1}^{M} \frac{\sup_{\boldsymbol{x} \in K} r_i(\boldsymbol{x})^{\beta_i}}{\inf_{\boldsymbol{x} \in K} r_i(\boldsymbol{x})^{\beta_i}} \\
&\le \inf_{\boldsymbol{x} \in K} \left(\prod_{i=1}^{M} r_i(\boldsymbol{x})^{\beta_i} \right) \prod_{i=1}^{M} \frac{\sup_{\boldsymbol{x} \in K} r_i(\boldsymbol{x})^{\beta_i}}{\inf_{\boldsymbol{x} \in K} r_i(\boldsymbol{x})^{\beta_i}} = \inf_{\boldsymbol{x} \in K} \Phi_{\boldsymbol{\beta}}(\boldsymbol{x}) \prod_{i=1}^{M} \frac{\sup_{\boldsymbol{x} \in K} r_i(\boldsymbol{x})^{\beta_i}}{\inf_{\boldsymbol{x} \in K} r_i(\boldsymbol{x})^{\beta_i}}.
\end{aligned}$$

It remains to show that the expression

$$\prod_{i=1}^{M} \frac{\sup_{\boldsymbol{x}\in K} r_i(\boldsymbol{x})^{\beta_i}}{\inf_{\boldsymbol{x}\in K} r_i(\boldsymbol{x})^{\beta_i}}$$

is bounded independently of the element sizes. To this end, we notice that

$$\begin{aligned}\prod_{i=1}^{M} \frac{\sup_{\boldsymbol{x}\in K} r_i(\boldsymbol{x})^{\beta_i}}{\inf_{\boldsymbol{x}\in K} r_i(\boldsymbol{x})^{\beta_i}} &= \prod_{i=1}^{M} \left(\frac{\sup_{\boldsymbol{x}\in K} r_i(\boldsymbol{x})}{\inf_{\boldsymbol{x}\in K} r_i(\boldsymbol{x})}\right)^{\beta_i} \leq C \prod_{i=1}^{M} \left(\frac{\inf_{\boldsymbol{x}\in K} r_i(\boldsymbol{x}) + h_K}{\inf_{\boldsymbol{x}\in K} r_i(\boldsymbol{x})}\right)^{\beta_i} \\ &\leq C \prod_{i=1}^{M} \left(1 + \frac{h_K}{\inf_{\boldsymbol{x}\in K} r_i(\boldsymbol{x})}\right)^{\beta_i}.\end{aligned}$$

Now, the bounded variation property of $\mathcal{T}_{FE}$ implies that $h_K(\inf_{\boldsymbol{x}\in K} r_i(\boldsymbol{x}))^{-1}$ is bounded independently of h_K and of r_i, for any $i \in \{1, \ldots, M\}$. $\square$

There holds the following approximation result:

Proposition 2.4. *Let $\mathcal{T}_{FE}$ be a given finite element mesh and φ a function in $H^{2,2}_{\boldsymbol{\beta}}(\Omega)$, where $\boldsymbol{\beta}$ is a weight vector as in Section 2.1. Then, there exists an interpolant $\widehat{\varphi} \in \mathcal{S}^{1,1}_0(\Omega, \mathcal{T}_{FE})$ such that*

$$\sum_{K\in\mathcal{T}_{FE}} \mathtt{H}^2_{\boldsymbol{\beta},K}\Big(h_K^{-4}\|\varphi - \widehat{\varphi}\|^2_{0,K} + h_K^{-2}\|\nabla(\varphi - \widehat{\varphi})\|^2_{0,K} + h_K^{-3}\|\varphi - \widehat{\varphi}\|^2_{0,\partial K}\Big) \leq C|\varphi|^2_{H^{2,2}_{\boldsymbol{\beta}}(\Omega)}.$$

Here, the constant $C > 0$ is independent of φ and of the element sizes.

Proof. Due to the standard trace inequality

$$\|v\|_{0,\partial K} \leq C\Big(h_K^{-\frac{1}{2}}\|v\|_{0,K} + h_K^{\frac{1}{2}}\|\nabla v\|_{0,K}\Big) \qquad \forall K \in \mathcal{T}_{FE},$$

it is sufficient to prove that

$$\sum_{K\in\mathcal{T}_{FE}} \mathtt{H}^2_{\boldsymbol{\beta},K}\Big(h_K^{-4}\|\varphi - \widehat{\varphi}\|^2_{0,K} + h_K^{-2}\|\nabla(\varphi - \widehat{\varphi})\|^2_{0,K}\Big) \leq C|\varphi|^2_{H^{2,2}_{\boldsymbol{\beta}}(\Omega)}. \tag{20}$$

We choose $\widehat{\varphi} \in \mathcal{S}^{1,1}_0(\Omega, \mathcal{T}_{FE})$ to be the elementwise linear interpolant of φ in the vertices of each element $K \in \mathcal{T}_{FE}$. Then, by standard approximation results, there holds

$$h_K^{-4}\|\varphi - \widehat{\varphi}\|^2_{0,K} + h_K^{-2}\|\nabla(\varphi - \widehat{\varphi})\|^2_{0,K} \leq C|\varphi|_{2,K} \tag{21}$$

for all $K \in \mathcal{T}_{FE} \setminus \mathcal{K}_{\mathcal{A}}$. Here, we have used, by the definition of the space $H^{2,2}_{\boldsymbol{\beta}}(\Omega)$, that φ is H^2-regular away from the corners of Ω.

Furthermore, for $K \in \mathcal{K}_{\mathcal{A}}$, we have

$$h_K^{-4}\|\varphi - \widehat{\varphi}\|^2_{0,K} + h_K^{-2}\|\nabla(\varphi - \widehat{\varphi})\|^2_{0,K} \leq C h_K^{-2\beta_K}|\varphi|^2_{H^{2,2}_{\beta_K}(K)}; \tag{22}$$

see [11], for example. Combining (21) and (22), recalling the definition of $\mathtt{H}_{\boldsymbol{\beta},K}$, $K \in \mathcal{T}_{FE}$, and applying (18), yields

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_{FE}} \mathtt{H}^2_{\boldsymbol{\beta},K}\Big(h_K^{-4}\|\varphi-\widehat{\varphi}\|^2_{0,K} + h_K^{-2}\|\nabla(\varphi-\widehat{\varphi})\|^2_{0,K}\Big)\\
&\quad\le C\bigg(\sum_{K\in\mathcal{K}_{\mathcal{A}}} h_K^{-2\beta_K}\mathtt{H}^2_{\boldsymbol{\beta},K}\,|\varphi|^2_{H^{2,2}_{\beta_K}(K)} + \sum_{K\in\mathcal{T}_{FE}\setminus\mathcal{K}_{\mathcal{A}}} \mathtt{H}^2_{\boldsymbol{\beta},K}|\varphi|^2_{2,K}\bigg)\\
&\quad\le C\bigg(\sum_{K\in\mathcal{K}_{\mathcal{A}}} h_K^{-2\beta_K}\sup_{\boldsymbol{x}\in K}\Phi^2_{\boldsymbol{\beta}}(\boldsymbol{x})\,|\varphi|^2_{H^{2,2}_{\beta_K}(K)} + \sum_{K\in\mathcal{T}_{FE}\setminus\mathcal{K}_{\mathcal{A}}} \inf_{\boldsymbol{x}\in K}\Phi^2_{\boldsymbol{\beta}}(\boldsymbol{x})\int_K |D^2\varphi|^2\,d\boldsymbol{x}\bigg)\\
&\quad\le C\bigg(\sum_{K\in\mathcal{K}_{\mathcal{A}}} |\varphi|^2_{H^{2,2}_{\beta_K}(K)} + \sum_{K\in\mathcal{T}_{FE}\setminus\mathcal{K}_{\mathcal{A}}} \int_K \Phi^2_{\boldsymbol{\beta}}|D^2\varphi|^2\,d\boldsymbol{x}\bigg)\\
&\quad\le C|\varphi|^2_{H^{2,2}_{\boldsymbol{\beta}}(\Omega)}.
\end{aligned}
$$

This shows (20) and thereby completes the proof. $\square$

3. A Posteriori Error Analysis

The aim of this section is to establish an a posteriori error analysis for the (weighted) L^2-norm of the finite element error e_{FE}. The upper bound (7) shall be proved first; see the following Theorem 3.1. The local lower bounds will be presented in Theorem 3.4 later on in this section.

We shall need some additional notation: For an element $K \in \mathcal{T}_{FE}$ and a function $v \in L^2(K)$, we denote by $\overline{v}_K$ the L^2-projection of v onto the space of all constant functions on K, i.e.

$$\overline{v}_K = \frac{1}{|K|}\int_K v\,d\boldsymbol{x},$$

where $|K|$ is the area of K. Furthermore, we define the following data oscillation term:

$$\mathcal{O}_K(v) = h_K^2\|v-\overline{v}_K\|_{0,K}. \tag{23}$$

Theorem 3.1 (Upper Bound)**.** *Let $\boldsymbol{\beta}$ be a weight vector as in Proposition 2.2. Then, the finite element error e_{FE} from* (6) *satisfies the a posteriori error estimate*

$$\|\Phi^{-1}_{\boldsymbol{\beta}} e_{FE}\|_{0,\Omega} \le C\bigg(\sum_{K\in\mathcal{T}_{FE}} \eta^2_{\boldsymbol{\beta},K}\bigg)^{\frac12}, \tag{24}$$

where the constant $C > 0$ is independent of the finite element solution u_{FE}, of the right-hand side f, and of the element sizes. The local error indicators $\eta_{\boldsymbol{\beta},K}$ are given by

$$\eta^2_{\boldsymbol{\beta},K} = \mathtt{H}^{-2}_{\boldsymbol{\beta},K}\Big(h_K^4\|\overline{f}_K\|^2_{0,K} + h_K^3\|[\![\nabla u_{FE}]\!]\|^2_{0,\partial K\setminus\partial\Omega} + \mathcal{O}_K(f)^2\Big), \tag{25}$$

where $\mathtt{H}_{\boldsymbol{\beta},K}$ is the elementwise constant from (17)*, and the jump $[\![\cdot]\!]$ is defined in* (8)*.*

Remark 3.2. In convex polygonal domains, the opening angles at the corners satisfy $\omega_i < \pi$. Hence, according to (15), the weight vector $\boldsymbol{\beta}$ can be chosen to be the zero vector; in particular, we note that, by Proposition 2.2 (see also [2, 7, 8]), the solution of the dual problem (10)–(11) belongs to $H^2(\Omega)$ in this case. Furthermore, for $\boldsymbol{\beta} = (0, 0, \dots, 0)$, the bound (24) is the well-known a posteriori error estimate for the L^2-norm of the finite element error e_{FE} in convex polygons; cf. [1, 5, 12], for example.

Remark 3.3. In order to evaluate the quantity $\mathtt{H}_{\boldsymbol{\beta},K}$, $K \in \mathcal{T}_{FE}$, arising in the local error estimators (25) in practice, we compute the set $\mathcal{H}_{\boldsymbol{\beta},K} = \{\mathtt{H}_{\boldsymbol{\beta},K}(C_i)\}_{i=1}^3$, where C_1, C_2, C_3 denote the corners of K, and set $\mathtt{H}_{\boldsymbol{\beta},K} = \max \mathcal{H}_{\boldsymbol{\beta},K}$.

Proof of Theorem 3.1. We start by multiplying the equation (10) by e_{FE} and integrating both sides over Ω, and by noting that $\chi(e_{FE}) = \Phi_{\boldsymbol{\beta}}^{-2} e_{FE}$ (cf. (14)). This leads to the following identities:

$$\|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,\Omega}^2 = \int_\Omega \Phi_{\boldsymbol{\beta}}^{-2} e_{FE}^2 \, d\boldsymbol{x} = -\int_\Omega e_{FE} \Delta\psi \, d\boldsymbol{x}.$$

Moreover, integration by parts yields

$$\|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,\Omega}^2 = \int_\Omega \nabla e_{FE} \cdot \nabla \psi \, d\boldsymbol{x} = a(e_{FE}, \psi).$$

In addition, recalling the Galerkin orthogonality (5), we have that

$$\|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,\Omega}^2 = a(e_{FE}, \psi - \widehat{\psi}),$$

where $\widehat{\psi} \in \mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE})$ is the interpolant from Proposition 2.4. Hence, it follows

$$\begin{aligned} \|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,\Omega}^2 &= \int_\Omega \nabla e_{FE} \cdot \nabla(\psi - \widehat{\psi}) \, d\boldsymbol{x} \\ &= \int_\Omega \nabla u \cdot \nabla(\psi - \widehat{\psi}) \, d\boldsymbol{x} - \int_\Omega \nabla u_{FE} \cdot \nabla(\psi - \widehat{\psi}) \, d\boldsymbol{x} \\ &= \int_\Omega f(\psi - \widehat{\psi}) \, d\boldsymbol{x} - \int_\Omega \nabla u_{FE} \cdot \nabla(\psi - \widehat{\psi}) \, d\boldsymbol{x}. \end{aligned} \tag{26}$$

The second term can be manipulated by an elementwise integration by parts and by using the fact, since u_{FE} is elementwise linear, that $\Delta u_{FE} \equiv 0$:

$$\begin{aligned} -\int_\Omega \nabla u_{FE} \cdot \nabla(\psi - \widehat{\psi}) \, d\boldsymbol{x} &= -\sum_{K \in \mathcal{T}_{FE}} \int_K \nabla u_{FE} \cdot \nabla(\psi - \widehat{\psi}) \, d\boldsymbol{x} \\ &= -\sum_{K \in \mathcal{T}_{FE}} \int_{\partial K} (\nabla u_{FE} \cdot \boldsymbol{n}_K)(\psi - \widehat{\psi}) \, ds \\ &= -\int_{\mathcal{E}_\mathcal{I}} [\![\nabla u_{FE}]\!](\psi - \widehat{\psi}) \, ds. \end{aligned} \tag{27}$$

Then, combining (26) and (27), results in

$$\begin{aligned} &\|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,\Omega}^2 \\ &\quad = \int_\Omega \overline{f}_K(\psi - \widehat{\psi}) \, d\boldsymbol{x} - \int_{\mathcal{E}_\mathcal{I}} [\![\nabla u_{FE}]\!](\psi - \widehat{\psi}) \, ds + \int_\Omega (f - \overline{f}_K)(\psi - \widehat{\psi}) \, d\boldsymbol{x} \\ &\quad \le \sum_{K \in \mathcal{T}_{FE}} \|\overline{f}_K\|_{0,K} \|\psi - \widehat{\psi}\|_{0,K} + \sum_{E \in \mathcal{E}_\mathcal{I}} \|[\![\nabla u_{FE}]\!]\|_{0,E} \|\psi - \widehat{\psi}\|_{0,E} \\ &\qquad + \sum_{K \in \mathcal{T}_{FE}} \|f - \overline{f}_K\|_{0,K} \|\psi - \widehat{\psi}\|_{0,K} \\ &\quad \le \sum_{K \in \mathcal{T}_{FE}} \|\overline{f}_K\|_{0,K} \|\psi - \widehat{\psi}\|_{0,K} + \sum_{K \in \mathcal{T}_{FE}} \|[\![\nabla u_{FE}]\!]\|_{0,\partial K \setminus \partial\Omega} \|\psi - \widehat{\psi}\|_{0,\partial K \setminus \partial\Omega} \\ &\qquad + \sum_{K \in \mathcal{T}_{FE}} h_K^{-2} \mathcal{O}_K(f) \|\psi - \widehat{\psi}\|_{0,K}. \end{aligned}$$

Therefore, applying a weighted Cauchy-Schwarz inequality and recalling the approximation properties from Proposition 2.4, we obtain

$$\begin{aligned}
&\|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,\Omega}^2 \\
&\le \Bigg(\sum_{K\in\mathcal{T}_{FE}} \mathtt{H}_{\boldsymbol{\beta},K}^2\Big(h_K^{-4}\|\psi-\widehat{\psi}\|_{0,K}^2 + h_K^{-3}\|\psi-\widehat{\psi}\|_{0,\partial K\setminus\partial\Omega}^2\Big)\Bigg)^{\frac12}\Bigg(\sum_{K\in\mathcal{T}_{FE}} \eta_{\boldsymbol{\beta},K}^2\Bigg)^{\frac12} \\
&\le C|\varphi|_{H^{2,2}_{\boldsymbol{\beta}}(\Omega)}\Bigg(\sum_{K\in\mathcal{T}_{FE}} \eta_{\boldsymbol{\beta},K}^2\Bigg)^{\frac12}.
\end{aligned}$$

Furthermore, making use of the regularity estimate (16), implies

$$\|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,\Omega}^2 \le C\|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,\Omega}\Bigg(\sum_{K\in\mathcal{T}_{FE}} \eta_{\boldsymbol{\beta},K}^2\Bigg)^{\frac12}.$$

Dividing both sides of the above inequality by $\|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,\Omega}$ completes the proof. $\square$

The following result implies the efficiency of the a posteriori error bound (24).

Theorem 3.4 (Local Lower Bounds)**.** *Let the assumptions of the previous Theorem 3.1 be satisfied. Furthermore, consider an element* $K \in \mathcal{T}_{FE}$ *and an edge* $E \in \mathcal{E}_{\mathcal{I}}$*. Then, the two estimates*

$$\|\overline{f}_K\|_{0,K} \le Ch_K^{-2}\big(\mathtt{H}_{\boldsymbol{\beta},K}\|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,K} + \mathcal{O}_K(f)\big), \tag{28}$$

and

$$\|[\![\nabla u_{FE}]\!]\|_{0,E} \le C\sum_{K\in\omega_E} h_K^{-\frac32}\big(\mathtt{H}_{\boldsymbol{\beta},K}\|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,K} + \mathcal{O}_K(f)\big), \tag{29}$$

hold true, where the constant $C > 0$ *is independent of* u_{FE}*,* f*, and of the element sizes.*

Proof. We prove (28) and (29) separately.

Proof of (28)*:* For $K \in \mathcal{T}_{FE}$, [9, Lemma 2 and Lemma 3] imply the existence of a polynomial cut-off function $B_K \in \mathcal{P}_6(K)$ satisfying

$$B_K|_{\partial K} = 0, \qquad \nabla B_K|_{\partial K} = \mathbf{0}, \tag{30}$$

and

$$\|B_K\|_{0,K} \le Ch_K, \qquad h_K^2 \le C\int_K B_K\,d\boldsymbol{x}, \tag{31}$$

with a constant $C > 0$ independent of h_K.

Then, since $\overline{f}_K$ is constant on K, we have that

$$|\overline{f}_K| = \left|\frac{1}{\int_K B_K\,d\boldsymbol{x}}\int_K B_K\overline{f}_K\,d\boldsymbol{x}\right| \le Ch_K^{-2}\left|\int_K B_K f\,d\boldsymbol{x} + \int_K B_K(\overline{f}_K - f)\,d\boldsymbol{x}\right|.$$

Noticing that $\Delta u_{FE} \equiv 0$ and integrating by parts twice, results in

$$\int_K B_K f\,d\boldsymbol{x} = -\int_K B_K\Delta e_{FE}\,d\boldsymbol{x} = \int_K \nabla B_K\cdot\nabla e_{FE}\,d\boldsymbol{x} = -\int_K \Delta B_K\,e_{FE}\,d\boldsymbol{x},$$

where we have used the properties from (30). Therefore, it holds that

$$\begin{aligned}
|\overline{f}_K| &\le Ch_K^{-2}\left|-\int_K \Delta B_K\,e_{FE}\,d\boldsymbol{x} + \int_K B_K(\overline{f}_K - f)\,d\boldsymbol{x}\right| \\
&\le Ch_K^{-2}\big(\|\Phi_{\boldsymbol{\beta}}\Delta B_K\|_{0,K}\|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,K} + \|B_K\|_{0,K}\|f - \overline{f}_K\|_{0,K}\big).
\end{aligned}$$

Recalling the definition of $\mathtt{H}_{\boldsymbol{\beta},K}$ and applying a standard inverse estimate for polynomials (cf. [11], for example), it follows that

$$\|\Phi_{\boldsymbol{\beta}}\Delta B_K\|_{0,K} \le \sup_{\boldsymbol{x}\in K}\Phi_{\boldsymbol{\beta}}(\boldsymbol{x})\|\Delta B_K\|_{0,K} \le Ch_K^{-2}\mathtt{H}_{\boldsymbol{\beta},K}\|B_K\|_{0,K}. \tag{32}$$

Therefore, using the first inequality from (31), gives

$$\begin{aligned}|\overline{f}_K| &\le Ch_K^{-2}\|B_K\|_{0,K}\big(h_K^{-2}\mathtt{H}_{\boldsymbol{\beta},K}\|\Phi_{\boldsymbol{\beta}}^{-1}e_{FE}\|_{0,K} + \|f-\overline{f}_K\|_{0,K}\big)\\ &\le Ch_K^{-1}\big(h_K^{-2}\mathtt{H}_{\boldsymbol{\beta},K}\|\Phi_{\boldsymbol{\beta}}^{-1}e_{FE}\|_{0,K} + \|f-\overline{f}_K\|_{0,K}\big)\\ &\le Ch_K^{-3}\big(\mathtt{H}_{\boldsymbol{\beta},K}\|\Phi_{\boldsymbol{\beta}}^{-1}e_{FE}\|_{0,K} + \mathcal{O}_K(f)\big),\end{aligned}$$

which leads to

$$\|\overline{f}_K\|_{0,K} \le Ch_K\left|\overline{f}_K\right| \le Ch_K^{-2}\big(\mathtt{H}_{\boldsymbol{\beta},K}\|\Phi_{\boldsymbol{\beta}}^{-1}e_{FE}\|_{0,K} + \mathcal{O}_K(f)\big).$$

Proof of (29)*:* For $E\in\mathcal{E}_{\mathcal{I}}$, there is a cut-off function $B_E\in C^1(\overline{\Omega}_E)$ with the following properties (cf. [9, Lemma 2 and Lemma 3] and (9)):

$$B_E|_{\partial\Omega_E} = 0, \qquad \nabla B_E|_{\partial\Omega_E} = \mathbf{0}, \qquad B_E|_K \in \mathcal{P}_8(K), K\in\omega_E, \tag{33}$$

and

$$\|B_E\|_{0,\Omega_E} \le Ch_E, \qquad h_E \le C\int_E B_E\,ds, \tag{34}$$

with a constant $C>0$ independent of h_E.

Since $\Delta u\in L^2(\Omega)$, we have that $[\![\nabla u]\!]|_E = 0$. Therefore, and due to the fact that ∇u_{FE} is constant along E, there holds

$$|[\![\nabla u_{FE}]\!]|_E| = \left|\frac{1}{\int_E B_E\,ds}\int_E -B_E[\![\nabla e_{FE}]\!]\,ds\right| \le Ch_E^{-1}\left|\int_E -B_E[\![\nabla e_{FE}]\!]\,ds\right|.$$

Twofold integration by parts and making use of the properties (33), yields

$$\begin{aligned}\int_E -B_E[\![\nabla e_{FE}]\!]\,ds &= -\sum_{K\in\omega_E}\int_{\partial K} B_E(\nabla e_{FE}\cdot\boldsymbol{n}_K)\,ds\\ &= \sum_{K\in\omega_E}\int_K (e_{FE}\Delta B_E - B_E\Delta e_{FE})\,d\boldsymbol{x}.\end{aligned}$$

Therefore, recalling that $-\Delta e_{FE} = -\Delta u = f$, results in

$$\begin{aligned}&|[\![\nabla u_{FE}]\!]|_E|\\ &\quad\le Ch_E^{-1}\left|\sum_{K\in\omega_E}\int_K (e_{FE}\Delta B_E + B_E f)\,d\boldsymbol{x}\right|\\ &\quad\le Ch_E^{-1}\sum_{K\in\omega_E}\left(\int_K |\Phi_{\boldsymbol{\beta}}^{-1}e_{FE}||\Phi_{\boldsymbol{\beta}}\Delta B_E|\,d\boldsymbol{x} + \int_K |B_E|(|\overline{f}_K| + |f-\overline{f}_K|)\,d\boldsymbol{x}\right)\\ &\quad\le Ch_E^{-1}\sum_{K\in\omega_E}\|\Phi_{\boldsymbol{\beta}}^{-1}e_{FE}\|_{0,K}\|\Phi_{\boldsymbol{\beta}}\Delta B_E\|_{0,K}\\ &\qquad + Ch_E^{-1}\sum_{K\in\omega_E}\|B_E\|_{0,K}\big(\|\overline{f}_K\|_{0,K} + \|f-\overline{f}_K\|_{0,K}\big).\end{aligned}$$

Proceeding as in (32), we obtain

$$\|\Phi_{\boldsymbol{\beta}}\Delta B_E\|_{0,K} \le Ch_K^{-2}\mathtt{H}_{\boldsymbol{\beta},K}\|B_E\|_{0,K}, \qquad K\in\omega_E.$$

Hence, we have

$$
\begin{aligned}
&|[\![\nabla u_{FE}]\!]|_E| \\
&\quad \leq C h_E^{-1} \sum_{K\in\omega_E} \|B_E\|_{0,K} \big(h_K^{-2} \mathtt{H}_{\boldsymbol{\beta},K} \|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,K} + \|\overline{f}_K\|_{0,K} + \|f-\overline{f}_K\|_{0,K} \big) \\
&\quad \leq C h_E^{-1} \|B_E\|_{0,\Omega_E} \sum_{K\in\omega_E} \big(h_K^{-2} \mathtt{H}_{\boldsymbol{\beta},K} \|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,K} + \|\overline{f}_K\|_{0,K} + h_K^{-2} \mathcal{O}_K(f) \big),
\end{aligned}
$$

which, by applying (34), leads to

$$
|[\![\nabla u_{FE}]\!]|_E| \leq C \sum_{K\in\omega_E} \big(h_K^{-2} \mathtt{H}_{\boldsymbol{\beta},K} \|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,K} + \|\overline{f}_K\|_{0,K} + h_K^{-2} \mathcal{O}_K(f) \big).
$$

Thus, it follows that

$$
\||[\![\nabla u_{FE}]\!]|_E\|_{0,E} \leq C h_E^{\frac{1}{2}} \sum_{K\in\omega_E} \big(h_K^{-2} \mathtt{H}_{\boldsymbol{\beta},K} \|\Phi_{\boldsymbol{\beta}}^{-1} e_{FE}\|_{0,K} + \|\overline{f}_K\|_{0,K} + h_K^{-2} \mathcal{O}_K(f) \big).
$$

Finally, using (9) and inserting the estimate (28), completes the proof. □

4. Numerical Experiments

In this section we shall illustrate the practical performance of the local error estimators $\eta_{\boldsymbol{\beta},K}$, $K \in \mathcal{T}_{FE}$, from (25) with a series of numerical experiments. All of our computations are based on the following widely-used adaptive mesh refinement algorithm (see, e.g., [12]):

(1) Set $k = 0$, and consider an initial mesh $\mathcal{T}_{FE}^{(0)}$ on Ω.
(2) Compute the numerical solution $u_{FE}^{(k)}$ from (4) on $\mathcal{T}_{FE}^{(k)}$.
(3) Compute the local error indicators $\eta_{\boldsymbol{\beta},K}$ from (25).
(4) If $\sum_{K\in\mathcal{T}_{FE}} \eta_{\boldsymbol{\beta},K}^2$ is sufficiently small then stop. Otherwise, find $\eta_{\max} = \max_{K\in\mathcal{T}_{FE}^{(k)}} \eta_{\boldsymbol{\beta},K}$, and refine those elements $K \in \mathcal{T}_{FE}$ for which $\eta_{\boldsymbol{\beta},K} > \tau\eta_{\max}$. Set $k = k+1$ and go to (2).

Here, $0 < \tau < 1$ is a fixed threshold which, in our numerical experiments, is set to be 0.5.

4.1. Example 1: Smooth Solution in Convex Polygon.

On the unit square

$$\Omega = (0,1)^2,$$

with corners

$$A_1 = (0,0), \quad A_2 = (1,0), \quad A_3 = (1,1), \quad A_4 = (0,1),$$

we consider the model problem

$$
\begin{aligned}
-\Delta u &= 2(-x^2 + x - y^2 + y) && \text{in } \Omega \\
u &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

The exact solution is given by $u(x,y) = xy(x-1)(y-1)$ and is analytic in $\overline{\Omega}$. We note that, since Ω is convex, the weights β_i associated to the corners A_i, $i = 1,2,3,4$, can be chosen arbitrarily in $[0,1)$; cf. Proposition 2.2 and Remark 3.2. In our numerical experiments we focus on two particular cases. For simplicity, the data oscillation terms shall be neglected.

Firstly, we consider weight vectors of the form $\boldsymbol{\beta} = (\beta_1, 0, 0, 0)$, with $\beta_1 \in [0,1)$. The corresponding weighted L^2-norms are then given by $\||\boldsymbol{x}|^{-\beta_1} \cdot \|_{0,\Omega}$. In Figure 1, we present the weighted L^2-errors for $\beta \in \{0.00, 0.25, 0.50, 0.75, 0.90, 0.99\}$, as well

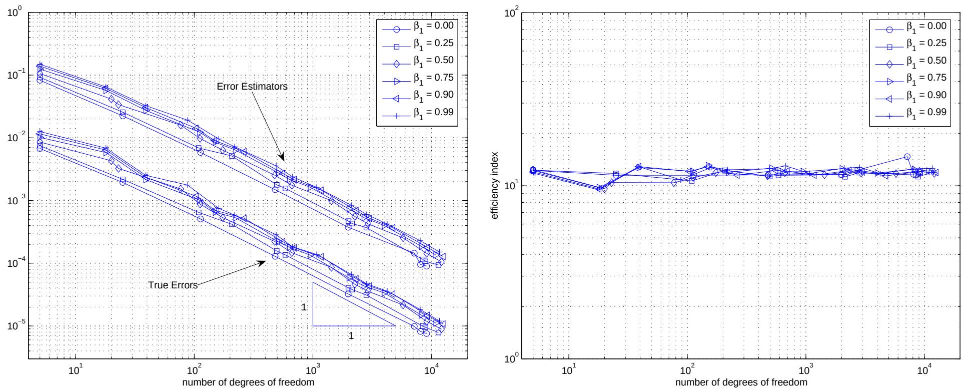

FIGURE 1. Example 1: $\beta_2 = \beta_3 = \beta_4 = 0.0$. Left: True (weighted) L^2-errors vs. global error estimators η_{global}. Right: Efficiency indices i_{eff}.

as the corresponding (global) error estimators (neglecting the data oscillation term) given by

$$\eta_{\text{global}} = \left(\sum_{K \in \mathcal{T}_{FE}} \eta^2_{\boldsymbol{\beta},K} \right)^{\frac{1}{2}}.$$

We observe that the (global) error estimators over-estimate the true (weighted) L^2-error by an approximately consistent factor. More precisely, the efficiency index, given by

$$i_{\text{eff}} = \frac{\left(\sum_{K \in \mathcal{T}_{FE}} \eta^2_{\boldsymbol{\beta},K} \right)^{\frac{1}{2}}}{\|\Phi^{-1}_{\boldsymbol{\beta}} e_{FE}\|_{0,\Omega}},$$

is around 10. This number results from the presence of the (unknown) constants in the lower and upper a posteriori error bounds from Theorems 3.1 and 3.4, respectively. Furthermore, we note that the decay of the global error estimators and of the true errors is of order 1 with respect to the number of degrees of freedom, and therefore optimal. Figure 3 shows the adaptively refined meshes for $\beta_1 = 0.50$ and $\beta_1 = 0.99$; as expected, they are strongly refined near the corner A_1, thereby resolving the singularity of the weight $\Phi^{-1}_{\boldsymbol{\beta}}(\boldsymbol{x})$ at the origin.

Secondly, we consider weight vectors of the form $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4)$, with $\beta_1 = \beta_2 = \beta_3 = \beta_4 \in \{0.00, 0.25, 0.50, 0.75, 0.90, 0.99\}$. The results are very similar to the previous case. In particular, the decay of the global error indicators η_{global} and of the weighted L^2-errors is again of optimal order, and the efficiency indices i_{eff} have an approximately constant value of 10; cf. Figure 2. Furthermore, from Figure 4, we see that the adaptive meshes are now refined at all of the four corners of Ω, thereby again resolving the singularities of the weight $\Phi^{-1}_{\boldsymbol{\beta}}$.

4.2. Example 2: Singular Solution in Non-Convex Polygon. The second series of our numerical experiments is based on a polygon with a re-entrant corner at the origin. More precisely, on the L-shaped domain

$$\Omega = (-1, 1)^2 \setminus ([0, 1] \times [-1, 0]),$$

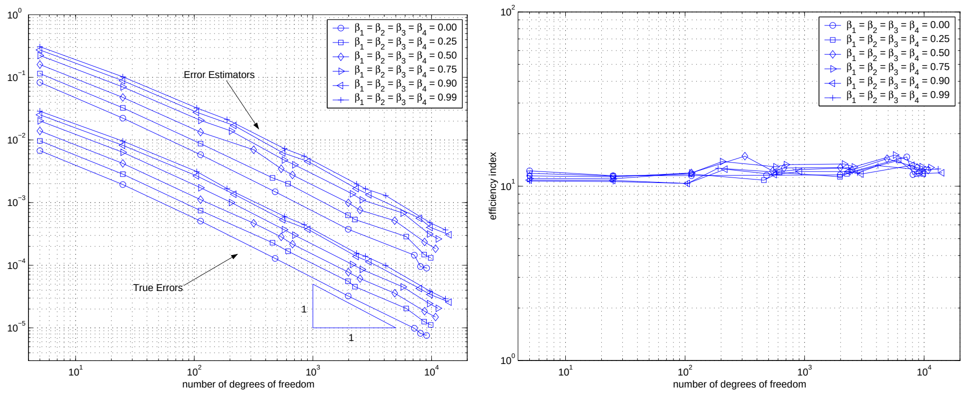

FIGURE 2. Example 1: Left: True (weighted) L^2-errors vs. global error estimators η_{global}. Right: Efficiency indices i_{eff}.

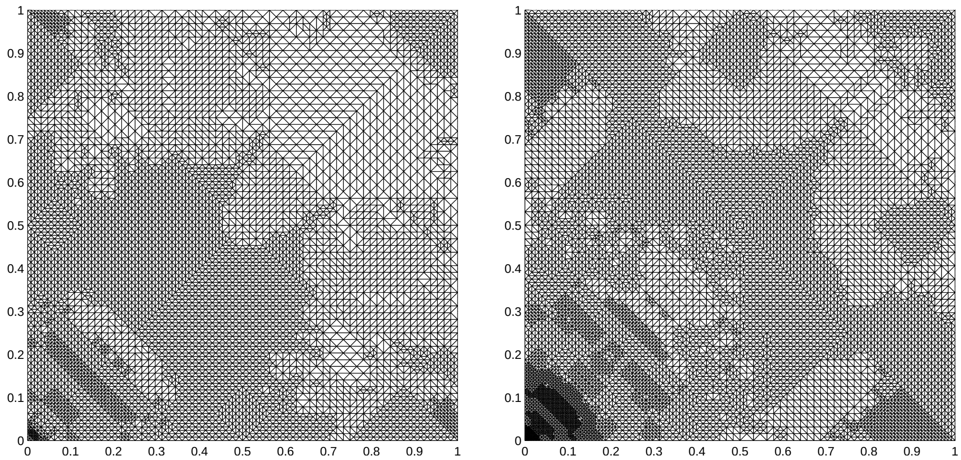

FIGURE 3. Example 1: Adaptively refined meshes. Left: $\beta_1 = 0.5$, $\beta_2 = \beta_3 = \beta_4 = 0.0$ (12 refinement steps, 11768 elements). Right: $\beta_1 = 0.99$, $\beta_2 = \beta_3 = \beta_4 = 0.0$ (13 refinement steps, 16362 elements).

with corners $A_1 = (0, 0), A_2 = (1, 0), \ldots, A_6 = (0, -1)$, we consider the problem

$$
\begin{aligned}
-\Delta u &= 0 \qquad \text{in } \Omega \\
u &= g \qquad \text{on } \partial\Omega.
\end{aligned}
$$

The exact solution is given by $u(r, \theta) = r^{\frac{2}{3}} \sin(\frac{2}{3}\theta)$, where (r, θ) denote polar coordinates in $\mathbb{R}^2$. The Dirichlet boundary data g is suitably chosen, and incorporated in the finite element method in the usual way. We remark that the function u represents the typical solution behavior of linear elliptic problems near the re-entrant corner A_1 of Ω; in particular, we note that $u \notin H^2(\Omega)$.

In accordance with Proposition 2.2, the non-convexity of the polygon Ω at the corner A_1 implies a restriction on the range of the associated weight $\beta_1 \in [0, 1)$: $\beta_1 > \frac{1}{3}$. Consequently, in our numerical experiments, we consider weight vectors $\boldsymbol{\beta} = (\beta_1, 0, \ldots, 0)$ with $\beta_1 \in \{0.34, 0.4, 0.5, 0.75, 0.9, 0.99\}$. From Figure 5, we see

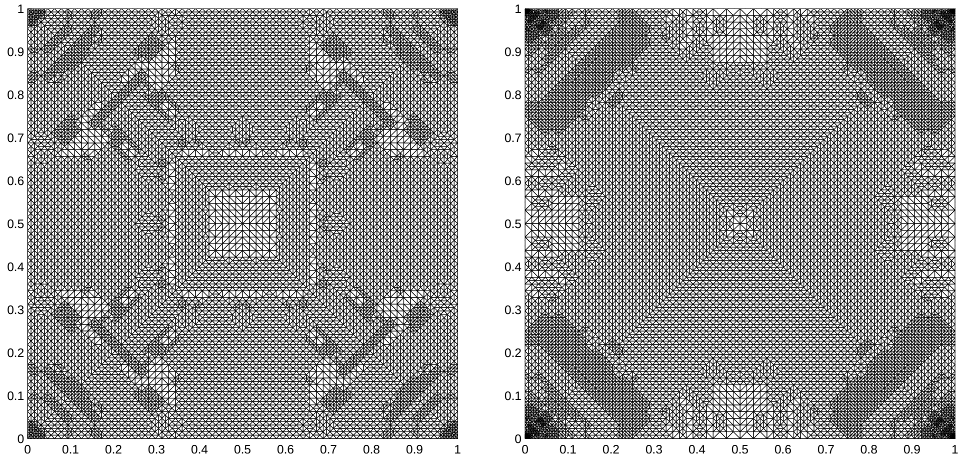

FIGURE 4. Example 1: Adaptively refined meshes. Left: $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0.5$ (10 refinement steps, 17760 elements). Right: $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0.99$ (10 refinement steps, 19744 elements).

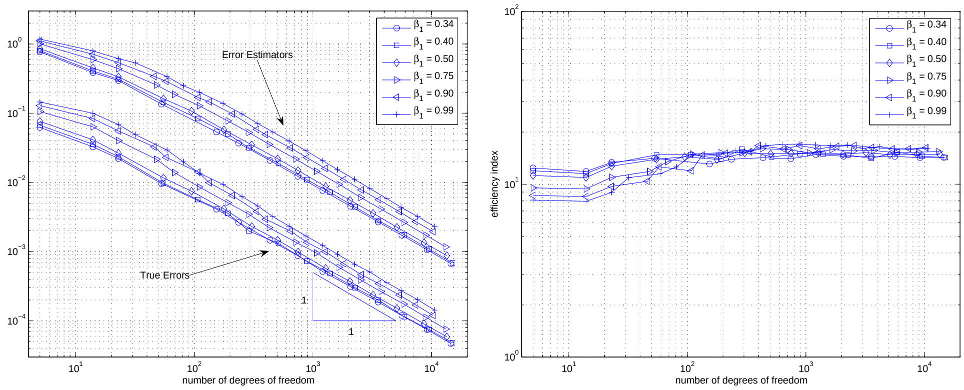

FIGURE 5. Example 2: $\beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = 0.0$. Left: True (weighted) L^2-errors vs. global error estimators η_{global}. Right: Efficiency indices i_{eff}.

that the numerical results resemble those obtained in Example 1, and that the rates of decay (for the global error estimators and for the weighted L^2-errors) are again optimal. In addition, the adaptively refined meshes in Figures 6–7 show a strong refinement at A_1; the method hereby resolves both, the low regularity of the exact solution u at this point as well as the singularity of the weight occurring in the weighted L^2-norm of the error.

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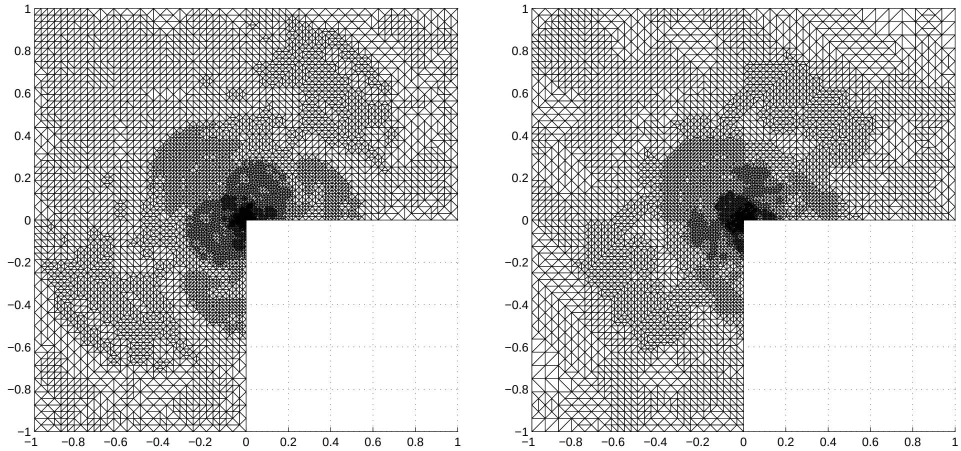

Figure 6. Example 2: Adaptively refined meshes. Left: $\beta_1 = 0.34$ (12 refinement steps, 11458 elements). Right: $\beta_1 = 0.5$ (13 refinement steps, 10454 elements).

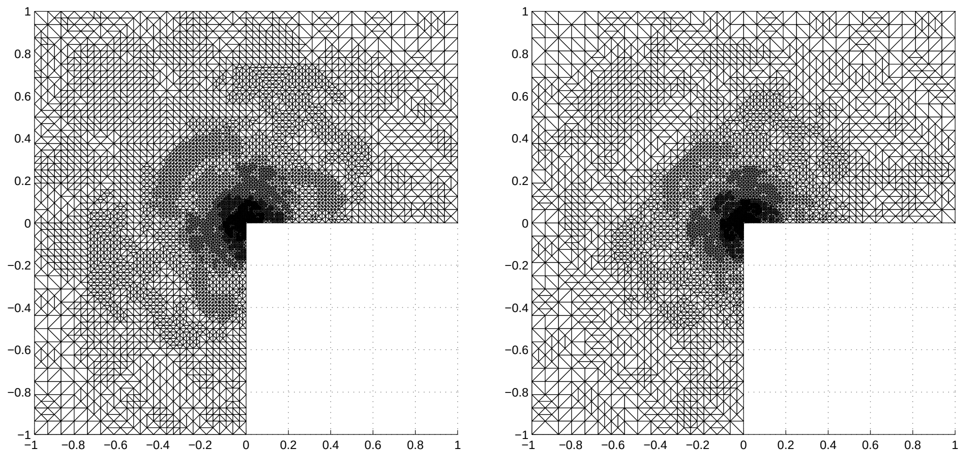

Figure 7. Example 2: Adaptively refined meshes. Left: $\beta_1 = 0.75$ (15 refinement steps, 11926 elements). Right: $\beta_1 = 0.99$ (19 refinement steps, 11336 elements).

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