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ERROR ANALYSIS FOR A FAST NUMERICAL METHOD TO A BOUNDARY INTEGRAL EQUATION OF THE FIRST KIND*

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Abstract

For two-dimensional boundary integral equations of the first kind with logarithmic kernels, the use of the conventional boundary element methods gives linear systems with dense matrix. In a recent work [*J. Comput. Math.*, 22 (2004), pp. 287-298], it is demonstrated that the dense matrix can be replaced by a sparse one if appropriate graded meshes are used in the quadrature rules. The numerical experiments also indicate that the proposed numerical methods require less computational time than the conventional ones while the formal rate of convergence can be preserved. The purpose of this work is to establish a stability and convergence theory for this fast numerical method. The stability analysis depends on a decomposition of the coefficient matrix for the collocation equation. The formal orders of convergence observed in the numerical experiments are proved rigorously.

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1. Introduction

Consider the first-kind boundary integral equation of the form

$$-\int_\Gamma \log|\mathbf{x}-\mathbf{y}|u(\mathbf{y})\,ds_{\mathbf{y}} = f(\mathbf{x}), \qquad \mathbf{x} := (x_1,x_2)\in\Gamma, \tag{1.1}$$

where $\Gamma \subset \mathbb{R}^2$ is a smooth and closed curve in the plane, u is a unknown function, f is a given function, $|\mathbf{x}-\mathbf{y}|$ denotes the Euclidean distance between $\mathbf{x}$ and $\mathbf{y}$, and $ds_{\mathbf{y}}$ is the measure of arclength. The boundary integral equation (1.1) arises in connection with the single layer potential method for

$$\Delta v(\mathbf{x}) = 0, \quad \mathbf{x}\in\Omega; \quad v(\mathbf{x}) = u(\mathbf{x}), \quad \mathbf{x}\in\Gamma, \tag{1.2}$$

whose solution can be represented by

$$v(\mathbf{x}) = -\int_\Gamma \log|\mathbf{x}-\mathbf{y}|u(\mathbf{y})\,ds_{\mathbf{y}}, \qquad \mathbf{x}\in\Omega. \tag{1.3}$$

Thus sampling of (1.3) on the boundary leads to the boundary integral equation (1.1). If the boundary Γ is sufficiently smooth, then the solution $v(\mathbf{x})$ can be very smooth due to the

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connection of the solutions of (1.2) and (1.3). The applications and some numerical aspects of the boundary integral equation (1.1) can be found in Sloan [14]. A more relevant paper by Bialecki and Yan [3] introduced a rectangular quadrature method for (1.1). More recently, Cheng et al. [6] proposed a new quadrature method for (1.1) based on a *graded mesh* approach. Unlike the quadrature method in [3] and other traditional numerical methods, the resulting system of equations in [6] contains a sparse coefficient matrix. It was demonstrated numerically that the proposed approach can not only preserve the formal rate of convergence but also save a significant amount of computational time.

The purpose of this paper is to provide a convergence theory for the method proposed in [6]. To begin with, let Γ be parameterized by the arclength:

$$\nu: \ [-L/2, L/2] \to \Gamma,$$

where L is the length of Γ,

$$|d\nu/ds| = 1 \text{ and } \nu(\sigma) \text{ is a periodic function with period of } L. \tag{1.4}$$

Then the integral equation (1.1) is equivalent to

$$-\int_{-L/2}^{L/2} \log|\nu(s)-\nu(\sigma)|u\big(\nu(\sigma)\big)\, d\sigma = f\big(\nu(s)\big), \qquad s \in [-L/2, L/2]. \tag{1.5}$$

The conventional way in solving Eq. (1.5) is to obtain n collocation equations by using n collocation points. Then for each fixed s the integral in (1.5) is approximated by an appropriate quadrature rule using the information on the n collocation points. This approach will lead to a linear system with a full matrix. In [6], the integral term in (1.5) is approximated by using a subset of the n collocation points. More precisely, let us consider the case when the unknown function u is reasonably smooth and the curve Γ is smooth and closed. In this case, some suitable graded-meshes can be used as the quadrature points to handle the logarithmic kernel, which yields a linear system with a sparse matrix. The graded-mesh concept was proposed by Rice [12]. It was then used to improve the formal order of convergence when solutions have weak singularity, see, e.g., [7, 19] for boundary integral equations and [4, 5, 15, 16] for weakly *singular* Volterra equations. However, with a smooth solution we just need to use a uniform mesh for the collocation points; while the graded mesh which is a subset of the uniform mesh is employed to evaluate the integrals.

To be more specific of numerical techniques, let us first introduce some notations. Set the *uniform mesh* with the mesh points

$$A := \{\alpha_i\}, \qquad \alpha_i = \frac{2i}{n-1}\cdot\frac{L}{2} \quad (i = -(n-1)/2, \cdots, (n-1)/2), \tag{1.6}$$

where n is supposed to be odd; and set the *graded mesh* with the mesh points

$$B := \{\beta_j\}, \qquad \beta_j = \mathrm{sgn}(j)\left(\frac{2|j|}{m}\right)^q \cdot \frac{L}{2} \quad (j = -m/2, \cdots, -1, 1, \cdots, m/2), \tag{1.7}$$

where $q \geq 1$ is the grading exponent. For ease of finding the mesh point set $B \subset A$, the value of q is usually taken as even. In this paper, we analyze the result for $q = 2$ and $q = 4$. For $q = 2$, it is assumed that $m = \sqrt{n-1}$. It can be verified that $B \subset A$. Transforming the negative index in (1.6) and (1.7) to positive one, we obtain the equivalent mesh-point sets:

$$\bar{A} := \{\bar{\alpha}_i\}, \qquad \bar{\alpha}_i = \alpha_{(i-1)-(n-1)/2} \quad (i = 1, \cdots, n), \tag{1.8}$$

and

$$\bar{B} := \{\bar{\beta}_j\}, \qquad \bar{\beta}_j = \begin{cases} \beta_{(j-1)-m/2}, & j = 1, \cdots, m/2, \\ \beta_{j-m/2}, & j = m/2+1, \cdots, m. \end{cases} \tag{1.9}$$

Rewriting Eq. (1.5) using the variable substitution $\rho = \sigma - s$ and the periodicity property of ν gives

$$-\int_{-L/2}^{L/2} \log|\nu(s) - \nu(\sigma+s)| u\big(\nu(\sigma+s)\big)\, d\sigma = f\big(\nu(s)\big), \qquad s \in [-L/2, L/2]. \tag{1.10}$$

Applying the trapezoidal rule with the point set $\bar{B}$ to the integral involved in (1.10) and collocating the resulting equation with respect to the point set $\bar{A}$, we obtain the following system of equations:

$$\sum_{j=1}^{m} \mu_{i,j} u_n(\nu(\bar{\beta}_j + \bar{\alpha}_i)) = f(\nu(\bar{\alpha}_i)), \qquad i = 1, \cdots, n, \tag{1.11}$$

where $u_n(\nu(s))$ is the numerical solution to Eq. (1.5) (or to its equivalent form (1.10)) for $s \in [-L/2, L/2]$, and the values of $\mu_{i,j}$ ($i = 1, \cdots, n$ and $j = 1, \cdots, m$) are given by

$$\begin{aligned} \mu_{i,1} &= -\frac{1}{2}\log|\nu(\bar{\beta}_1 + \bar{\alpha}_i) - \nu(\bar{\alpha}_i)| \cdot (\bar{\beta}_2 - \bar{\beta}_1), \\ \mu_{i,m} &= -\frac{1}{2}\log|\nu(\bar{\beta}_m + \bar{\alpha}_i) - \nu(\bar{\alpha}_i)| \cdot (\bar{\beta}_m - \bar{\beta}_{m-1}), \\ \mu_{i,j} &= -\frac{1}{2}\log|\nu(\bar{\beta}_j + \bar{\alpha}_i) - \nu(\bar{\alpha}_i)| \cdot (\bar{\beta}_{j+1} - \bar{\beta}_{j-1}) \quad (2 \le j \le m-1). \end{aligned}$$

We find that the number of nonzero elements of the coefficient matrix in the (1.11) is equal to $\mathrm{Card}(\bar{B}) \cdot \mathrm{Card}(\bar{A}) = m \cdot n = n \cdot \sqrt{n-1}$.

We finish the introduction by outlining the rest of the paper. In the next section, we will study the stability properties of the numerical method (1.11), which is done by using the kernel-splitting ideas. The convergence results will be established in Section 3. Some concluding remarks will be given in the final section.

2. Stability

In this section, we will employ the splitting kernels technique to study the stability for (1.11). This technique has been used in many cases (see, e.g., [1, 2, 3, 10, 17]). Let us split the kernel in (1.5) into the form

$$-\log|\nu(s) - \nu(\sigma)| = k^{[1]}(s-\sigma) + k^{[2]}(s,\sigma), \tag{2.1}$$

where

$$k^{[1]}(s-\sigma) = -\log|\sin[\pi(s-\sigma)/2L]|, \tag{2.2}$$

$$k^{[2]}(s,\sigma) = \begin{cases} -\log(2L/\pi), & \text{if } s-\sigma = 2jL, \quad j = 0, \pm 1, \cdots \\ -\log|[\nu(s) - \nu(\sigma)]/\sin[\pi(s-\sigma)/2L]|, & \text{otherwise.} \end{cases} \tag{2.3}$$

Note that the kernel $k^{[1]}$ is convolutional and the kernel $k^{[2]}$ is symmetric: $k^{[2]}(s,\sigma) = k^{[2]}(\sigma, s)$. Inserting (2.1) into (1.10) yields

$$\int_{-L/2}^{L/2} [k^{[1]}(\sigma) + k^{[2]}(s, \sigma+s)] u\big(\nu(\sigma+s)\big)\, d\sigma = f\big(\nu(s)\big), \qquad s \in [-L/2, L/2]. \tag{2.4}$$

Applying the same process for deriving (1.11) to Eq. (2.4) gives

$$\sum_{j=1}^{m}\left(\mu_{i,j}^{[1]}+\mu_{i,j}^{[2]}\right)u_n(\nu(\bar{\beta}_j+\bar{\alpha}_i))=f(\nu(\bar{\alpha}_i)),\qquad i=1,\cdots,n, \tag{2.5}$$

where the values of $\mu_{i,j}^{[1]}$ and $\mu_{i,j}^{[2]}$ ($i=1,\cdots,n$ and $j=1,\cdots,m$) are given by

$$\begin{aligned}
\mu_{i,1}^{[1]}&=\frac{1}{2}k^{[1]}(\bar{\beta}_1)\cdot(\bar{\beta}_2-\bar{\beta}_1)=-\frac{1}{2}\log|\sin(\pi\bar{\beta}_1/2L)|\cdot(\bar{\beta}_2-\bar{\beta}_1),\\
\mu_{i,m}^{[1]}&=\frac{1}{2}k^{[1]}(\bar{\beta}_m)\cdot(\bar{\beta}_m-\bar{\beta}_{m-1})=-\frac{1}{2}\log|\sin(\pi\bar{\beta}_m/2L)|\cdot(\bar{\beta}_m-\bar{\beta}_{m-1}),\\
\mu_{i,j}^{[1]}&=\frac{1}{2}k^{[1]}(\bar{\beta}_j)\cdot(\bar{\beta}_{j+1}-\bar{\beta}_{j-1})=-\frac{1}{2}\log|\sin(\pi\bar{\beta}_j/2L)|\cdot(\bar{\beta}_{j+1}-\bar{\beta}_{j-1})\quad(2\le j\le m-1),
\end{aligned}$$

and

$$\begin{aligned}
\mu_{i,1}^{[2]}&=\frac{1}{2}k^{[2]}\left(\bar{\alpha}_i,\bar{\beta}_1+\bar{\alpha}_i\right)\cdot(\bar{\beta}_2-\bar{\beta}_1),\\
\mu_{i,m}^{[2]}&=\frac{1}{2}k^{[2]}\left(\bar{\alpha}_i,\bar{\beta}_m+\bar{\alpha}_i\right)\cdot(\bar{\beta}_m-\bar{\beta}_{m-1}),\\
\mu_{i,j}^{[2]}&=\frac{1}{2}k^{[2]}\left(\bar{\alpha}_i,\bar{\beta}_j+\bar{\alpha}_i\right)\cdot(\bar{\beta}_{j+1}-\bar{\beta}_{j-1})\quad(2\le j\le m-1).
\end{aligned}$$

Write (1.11) and (2.5), respectively, into the matrix forms:

$$\mathbf{DU}=\mathbf{F}, \tag{2.6}$$

and

$$\left(\mathbf{D}^{[1]}+\mathbf{D}^{[2]}\right)\mathbf{U}=\mathbf{F}, \tag{2.7}$$

where $\mathbf{U}=(u_n(\nu(\bar{\alpha}_1)),\cdots,u_n(\nu(\bar{\alpha}_n))^\top$ and $\mathbf{F}=(f(\nu(\bar{\alpha}_1)),\cdots,f(\nu(\bar{\alpha}_n)))^\top$. The matrices $\mathbf{D}$, $\mathbf{D}^{[1]}$ and $\mathbf{D}^{[2]}$ are sparse with non-zero elements:

$$\begin{aligned}
&\bar{d}_{1,j}\neq 0\quad\text{for } j=\begin{cases} n-2[(k-1)-m/2]^2, & k=1,\cdots,m/2,\\ (n+1)/2+2(k-m/2)^2, & k=m/2+1,\cdots,m;\end{cases}\\
&\text{if } \bar{d}_{i,j}\neq 0,\ \text{then } \bar{d}_{i+1,j+1\ (\text{mod } n)}\neq 0.
\end{aligned}$$

Moreover, the matrix $\mathbf{D}^{[1]}:=(d_{i,j}^{[1]})_{i,j=1}^n$ is a circulant matrix (see e.g., [8]) with the elements described by the following expressions:

- (1): In the first row, $d_{1,j}^{[1]}=\mu_{1,k}^{[1]}$ for

  $$j=\begin{cases} n-2[(k-1)-m/2]^2, & k=1,\cdots,m/2,\\ (n+1)/2+2(k-m/2)^2, & k=m/2+1,\cdots,m,\end{cases}$$

  and $d_{1,j}^{[1]}=0$ otherwise;

- (2): $d_{i,j}^{[1]}=d_{i+1,j+1\ (\text{mod } n)}^{[1]}$.

It can be verified that

$$\mathbf{D}=\mathbf{D}^{[1]}+\mathbf{D}^{[2]}. \tag{2.8}$$

As will be shown below, the circulant property helps us to verify that $\mathbf{D}^{[1]}$ is invertible. It also allows us to derive the upper bound of its condition number. As a result, we can rewrite (2.8) into the form

$$\mathbf{D} = \mathbf{D}^{[1]}\Big(\mathbf{I} + \big(\mathbf{D}^{[1]}\big)^{-1}\mathbf{D}^{[2]}\Big). \tag{2.9}$$

Therefore, the stability of (1.11) is then proved by verifying that the matrix $\mathbf{I} + \big(\mathbf{D}^{[1]}\big)^{-1}\mathbf{D}^{[2]}$ is invertible (cf. Lemma 2.2).

Lemma 2.1. *For the matrix $\mathbf{D}^{[1]}$, we have the following estimation for its inverse:*

$$\|\big(\boldsymbol{D}^{[1]}\big)^{-1}\|_{\mathcal{F}} \le Cm,$$

where $\|\cdot\|_{\mathcal{F}}$ is the Frobenius norm and the positive constant C is independent of m and n.

Proof. Since $\mathbf{D}^{[1]}$ is a circulant matrix, it follows from [8, Theorem 3.2.2] that the eigenvalues λ_j are given by

$$\lambda_j = \sum_{\ell=1}^{n} e^{\mathbf{i}(j-1)(\ell-1)2\pi/n} d_{1,\ell}^{[1]} \quad (j = 1, \cdots, n),$$

where $\mathbf{i}^2 := -1$. Using the expression of $d_{1,\ell}^{[1]}$, we can formulate λ_j as

$$\begin{aligned}
\lambda_j &= \sum_{k=1}^{m/2} e^{\mathbf{i}(j-1)\left(n-2[(k-1)-m/2]^2\right)2\pi/n}\mu_{1,k}^{[1]} + \sum_{k=m/2+1}^{m} e^{\mathbf{i}(j-1)\left((n+1)/2+2(k-m/2)^2\right)2\pi/n}\mu_{1,k}^{[1]} \\
&= \frac{1}{2}\sum_{k=1}^{m/2} e^{\mathbf{i}(j-1)\left(n-2[(k-1)-m/2]^2\right)2\pi/n}(-\log|\sin(\pi\bar{\beta}_k/2L)|\cdot(\bar{\beta}_{k+1}-\bar{\beta}_{k-1})) \qquad (2.10)\\
&\quad + \frac{1}{2}\sum_{k=m/2+1}^{m} e^{\mathbf{i}(j-1)\left((n+1)/2+2(k-m/2)^2\right)2\pi/n}(-\log|\sin(\pi\bar{\beta}_k/2L)|\cdot(\bar{\beta}_{k+1}-\bar{\beta}_{k-1})),
\end{aligned}$$

where $\bar{\beta}_{-1} := \bar{\beta}_1$ and $\bar{\beta}_{m+1} := \bar{\beta}_m$. The modulus of λ_j can be bounded from below by

$$\begin{aligned}
|\lambda_j| \ge \frac{1}{2}\min_{k=1,\cdots,m}(\bar{\beta}_{k+1}-\bar{\beta}_{k-1})\Bigg(&\sum_{k=1}^{m/2} e^{\mathbf{i}(j-1)\left(n-2[(k-1)-m/2]^2\right)2\pi/n}(-\log|\sin(\pi\bar{\beta}_k/2L)|) \\
&+ \sum_{k=m/2+1}^{m} e^{\mathbf{i}(j-1)\left((n+1)/2+2(k-m/2)^2\right)2\pi/n}(-\log|\sin(\pi\bar{\beta}_k/2L)|)\Bigg). \qquad (2.11)
\end{aligned}$$

Using a formula from [9, 1.441.2], we have

$$\begin{aligned}
\log|\sin(\pi\bar{\beta}_k/2L)| &= -\log 2 - \sum_{\ell=1}^{\infty}\frac{\cos(\ell\pi\bar{\beta}_k/L)}{\ell} \\
&= -\log 2 - \sum_{\ell=1}^{n-1}\frac{\cos(\ell\pi\bar{\beta}_k/L)}{\ell} - \sum_{p=1}^{\infty}\sum_{\ell=1}^{n-1}\frac{\cos(\ell\pi\bar{\beta}_k/L)}{p(n-1)+\ell}. \qquad (2.12)
\end{aligned}$$

Inserting (2.12) into (2.10) and noting that

$$\min_{1\le k\le m}(\bar{\beta}_{k+1}-\bar{\beta}_{k-1}) = \frac{16L}{m^2}$$

give

$$|\lambda_j| \geq \left| \frac{8L\log 2}{m^2} \left(\sum_{k=1}^{m/2} e^{\mathbf{i}(j-1)\left(n-2[(k-1)-m/2]^2\right)2\pi/n} + \sum_{k=m/2+1}^{m} e^{\mathbf{i}(j-1)\left((n+1)/2+2(k-m/2)^2\right)2\pi/n} \right) \right.$$
$$\left. + \frac{8L}{m^2} \left(\sum_{\ell=1}^{n-1} \frac{\gamma_{j,\ell}}{\ell} + \sum_{p=1}^{\infty}\sum_{\ell=1}^{n-1} \frac{\gamma_{j,\ell}}{p(n-1)+\ell} \right) \right|, \tag{2.13}$$

where

$$\begin{aligned}
\gamma_{j,\ell} &= \sum_{k=1}^{m/2} e^{\mathbf{i}(j-1)\left(n-2[(k-1)-m/2]^2\right)2\pi/n} \left(e^{\mathbf{i}\ell\pi\bar{\beta}_k/L} + e^{-\mathbf{i}\ell\pi\bar{\beta}_k/L} \right) \\
&\quad + \sum_{k=m/2+1}^{m} e^{\mathbf{i}(j-1)\left((n+1)/2+2(k-m/2)^2\right)2\pi/n} \left(e^{\mathbf{i}\ell\pi\bar{\beta}_k/L} + e^{-\mathbf{i}\ell\pi\bar{\beta}_k/L} \right) \\
&= \sum_{k=1}^{m/2} e^{\mathbf{i}(j-1)\left(n-2[(k-1)-m/2]^2\right)2\pi/n} \left(e^{\mathbf{i}\ell\pi 2[(k-1)-m/2]^2/(n-1)} + e^{-\mathbf{i}\ell\pi 2[(k-1)-m/2]^2/(n-1)} \right) \\
&\quad + \sum_{k=m/2+1}^{m} e^{\mathbf{i}(j-1)\left((n+1)/2+2(k-m/2)^2\right)2\pi/n} \left(e^{\mathbf{i}\ell\pi 2(k-m/2)^2/(n-1)} + e^{-\mathbf{i}\ell\pi 2(k-m/2)^2/(n-1)} \right).
\end{aligned}$$

It is straightforward to verify that

$$\begin{aligned}
&\sum_{k=1}^{m/2} e^{\mathbf{i}(j-1)\left(n-2[(k-1)-m/2]^2\right)2\pi/n} + \sum_{k=m/2+1}^{m} e^{\mathbf{i}(j-1)\left((n+1)/2+2(k-m/2)^2\right)2\pi/n} \\
&\geq \sum_{k=1}^{m} e^{\mathbf{i}(j-1)(k-1)2\pi/mn} = \begin{cases} m, & \text{if } j = 0 \bmod m, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{2.14}$$

By a simple calculation, (2.14) and (2.13) lead to

$$|\lambda_j| \geq C\left(\frac{1}{m} + \frac{1}{j} + \frac{1}{m-j}\right) \geq \frac{C}{m}, \tag{2.15}$$

where in the last step we have used

$$\frac{1}{j} + \frac{1}{m-j} \geq \frac{4}{m}.$$

Thus the proof of Lemma 2.1 is complete. □

Furthermore, we derive an upper bound of the inverse of the matrix $\mathbf{I} + \left(\mathbf{D}^{[1]}\right)^{-1}\mathbf{D}^{[2]}$ in the following lemma.

Lemma 2.2. *Let* $\mathbf{D}^{[1]}$ *and* $\mathbf{D}^{[2]}$ *be the matrices involved in* (2.9). *Then for sufficiently large* n *and* m, *we have*

$$\left\| \left(\boldsymbol{I} + \left(\boldsymbol{D}^{[1]}\right)^{-1} \boldsymbol{D}^{[2]} \right)^{-1} \right\|_{\mathcal{F}} \leq C,$$

where the positive constant C *is independent of* n *and* m.

Proof. Recall the form of the splitting kernel (2.1):

$$-\log|\nu(s)-\nu(\sigma)| = k^{[1]}(s-\sigma) + k^{[2]}(s,\sigma),$$

and define the restriction operators $H^{[1]}$ and $H^{[2]}$ by

$$H^{[1]}w(s) := \int_{-L/2}^{L/2} k^{[1]}(s-\sigma)w(\sigma)\,d\sigma,$$
$$H^{[2]}w(s) := \int_{-L/2}^{L/2} k^{[2]}(s,\sigma)w(\sigma)\,d\sigma.$$

Then the original problem (1.5) can be written in the equivalent form

$$\left(H^{[1]}+H^{[2]}\right)w(s) = f(\nu(s)), \tag{2.16}$$

where $w(s) := u(\nu(s))$. It is known that the operator $H^{[1]}$ is invertible and $I + (H^{[1]})^{-1}H^{[2]}$ is a compact operator (see [2]). Hence, (1.5) and (2.16) are also equivalent to

$$(I + (H^{[1]})^{-1}H^{[2]})w(s) = (H^{[1]})^{-1}\, f(\nu(s)). \tag{2.17}$$

The key to the proof of this lemma is to view $\mathbf{I} + \left(\mathbf{D}^{[1]}\right)^{-1}\mathbf{D}^{[2]}$ as the approximation of $I + (H^{[1]})^{-1}H^{[2]}$. Denote $G := (H^{[1]})^{-1}H^{[2]}$. Corresponding to the operator G, the kernel $g(s,\sigma)$ is given by

$$g(s,\sigma) := (H^{[1]})^{-1}\, k^{[2]}(s,\sigma). \tag{2.18}$$

Let the matrix $\mathbf{T} := \left(t_{i,j}\right)_{i,j=1}^{n}$ be defined by

$$\begin{aligned} t_{i,j} &= \frac{1}{2}g(\bar{\alpha}_i,\bar{\alpha}_j)(\bar{\beta}_{k+1}-\bar{\beta}_{k-1}),\\ &\qquad j = \begin{cases} n-2[(k-1)-m/2]^2+(i-1), & k=1,\cdots,m/2,\\ (n+1)/2+2(k-m/2)^2+(i-1), & k=m/2+1,\cdots,m,\end{cases}\\ t_{i,j} &= 0, \quad \text{otherwise.}\end{aligned}$$

We now derive the upper bound for $\|\mathbf{T}-(\mathbf{D}^{[1]})^{-1}\mathbf{D}^{[2]}\|_{\mathcal{F}}$. Let $\mathbf{t}_\ell$ and $\mathbf{d}_\ell$ represent the ℓ-th row of the matrices $\mathbf{T}$ and $\mathbf{D}^{[2]}$, respectively. Define

$$\bar{g}(s,\bar{\alpha}_\ell) = \begin{cases} \frac{1}{2}g(s,\bar{\alpha}_\ell)(\bar{\beta}_{k+1}-\bar{\beta}_{k-1}),\\ \qquad \ell = \begin{cases} n-2[(k-1)-m/2]^2+(i-1), & k=1,\cdots,m/2,\\ (n+1)/2+2(k-m/2)^2+(i-1), & k=m/2+1,\cdots,m,\end{cases}\\ 0, \quad \text{otherwise,}\end{cases}$$

and

$$\bar{k}^{[2]}(s,\bar{\alpha}_\ell) = \begin{cases} \frac{1}{2}k^{[2]}(s,\bar{\alpha}_\ell)(\bar{\beta}_{k+1}-\bar{\beta}_{k-1}),\\ \qquad \ell = \begin{cases} n-2[(k-1)-m/2]^2+(i-1), & k=1,\cdots,m/2,\\ (n+1)/2+2(k-m/2)^2+(i-1), & k=m/2+1,\cdots,m,\end{cases}\\ 0, \quad \text{otherwise.}\end{cases}$$

Moreover, define a restriction operator $\mathbf{r}$ by

$$\mathbf{r}v(s) = [v(\bar{\alpha}_1),\cdots,v(\bar{\alpha}_n)]^\top, \qquad v\in C\left([-L/2,L/2]\right). \tag{2.19}$$

It is obvious that $\mathbf{t}_\ell = \mathbf{r}\bar{g}(s,\bar{\alpha}_\ell)$ and $\mathbf{d}_\ell = \mathbf{r}\bar{k}^{[2]}(s,\bar{\alpha}_\ell)$. Then using Lemmas 2.1 and 3.1 (see the next section) gives

$$\begin{aligned}&\left\|\mathbf{t}_\ell - (\mathbf{D}^{[1]})^{-1}\mathbf{d}_\ell\right\|_2 = \left\|(\mathbf{D}^{[1]})^{-1}\left(\mathbf{D}^{[1]}\mathbf{t}_\ell - \mathbf{d}_\ell\right)\right\|_2\\ =&\left\|(\mathbf{D}^{[1]})^{-1}\left(\mathbf{D}^{[1]}\mathbf{r}\bar{g}(s,\bar{\alpha}_\ell) - \mathbf{r}H^{[1]}\bar{g}(s,\bar{\alpha}_\ell)\right)\right\|_2 \le C\frac{\sqrt{n}\log n}{n-1},\end{aligned}$$

where $\|\cdot\|_2$ is Euclidean norm. Therefore,

$$\left\|\left(\mathbf{T} - (\mathbf{D}^{[1]})^{-1}\mathbf{D}^{[2]}\right)\mathbf{v}\right\|_2 \le \sqrt{\sum_{\ell=1}^{n}\left\|\mathbf{t}_\ell - (\mathbf{D}^{[1]})^{-1}\mathbf{d}_\ell\right\|_2^2}\,\|\mathbf{v}\|_2 \le C\frac{\sqrt{n}\log n}{n-1}. \tag{2.20}$$

Now we are ready to derive the lower bound for $\|\mathbf{I}+\mathbf{T}\|_{\mathcal{F}}$. The following inequality is known from [18]:

$$\|(I+G)v\|_{L^2} \ge C\|v\|_{L^2}, \qquad v \in L^2([-L/2,L/2]), \tag{2.21}$$

where the notation $\|\cdot\|_{L^2}$ stands for $\|\cdot\|_{L^2([-L/2,L/2])}$. The kernel $g(s,\sigma)$ in (2.18) is Lipschitz continuous with respect to the variables s and σ, respectively. Define a map

$$p_n:\ \mathbb{R}^n \longrightarrow L^2([-L/2,L/2])$$

by, for $\mathbf{v} = [v_1,\cdots,v_n]^\top$,

$$(p_n\mathbf{v})(s) = v_i, \quad \text{for } s\in(\alpha_i,\alpha_{i+1}),\ i = 1,\cdots,n-1,$$

i.e., $(p_n(\mathbf{v}))(s)$ is a piecewise constant function. It is easy to verify that

$$\|\mathbf{v}\|_2 = \|p_n\mathbf{v}\|_{L^2}. \tag{2.22}$$

Define a matrix $\widetilde{\mathbf{T}} := \left(\tilde{t}_{i,j}\right)_{i,j=1}^n$, where

$$\tilde{t}_{i,j} = \begin{cases}\displaystyle\int_{\bar{\beta}_{k-1}}^{\bar{\beta}_{k-1}} g(\bar{\alpha}_i,\sigma)\,d\sigma,\\ \qquad j = \begin{cases} n-2[(k-1)-m/2]^2+(i-1), & k=1,\cdots,m/2,\\ (n+1)/2+2(k-m/2)^2+(i-1), & k=m/2+1,\cdots,m,\end{cases}\\ 0, \quad \text{otherwise}.\end{cases}$$

Since $g(s,\sigma)$ is Lipschitz continuous with respect to s and σ, we can verify that

$$\left\|Gp_n\mathbf{v} - p_n(\widetilde{\mathbf{T}}\mathbf{v})\right\|_{L^2} \le C\frac{1}{n}\|\mathbf{v}\|_2, \tag{2.23}$$

$$\left\|(\widetilde{\mathbf{T}} - \mathbf{T})\mathbf{v}\right\|_2 \le \frac{1}{n}\|\mathbf{v}\|_2. \tag{2.24}$$

Applying (2.22) and the triangle inequality, together with (2.16) and (2.17), we derive

$$\begin{aligned}&\left\|(\mathbf{I}+\widetilde{\mathbf{T}})\mathbf{v}\right\|_2 = \left\|p_n\mathbf{v} + p_n\widetilde{\mathbf{T}}\mathbf{v}\right\|_{L^2}\\ \ge&\ \|(I+G)p_n\mathbf{v}\|_{L^2} - \left\|Gp_n\mathbf{v} - p_n\widetilde{\mathbf{T}}\mathbf{v}\right\|_{L^2} \ge C\left(1-\frac{1}{n}\right)\|\mathbf{v}\|_2.\end{aligned}$$

Then it follows again from the triangle inequality that

$$\|(\mathbf{I}+\mathbf{T})\mathbf{v}\|_2 \geq \left\|(\mathbf{I}+\widetilde{\mathbf{T}})\mathbf{v}\right\|_2 - \left\|(\widetilde{\mathbf{T}}-\mathbf{T})\mathbf{v}\right\|_2 \geq C\|\mathbf{v}\|_2. \tag{2.25}$$

Combining (2.20) and (2.25) leads to

$$\begin{aligned}\left\|\left(\mathbf{I}+(\mathbf{D}^{[1]})^{-1}\mathbf{D}^{[2]}\right)\right\|_2 &\geq \|(\mathbf{I}+\mathbf{T})\mathbf{v}\|_2 - \left\|(\mathbf{T}-(\mathbf{D}^{[1]})^{-1}\mathbf{D}^{[2]})\mathbf{v}\right\|_2 \\ &\geq C\left(1-\frac{\sqrt{n}\log n}{n-1}\right)\|\mathbf{v}\|_2. \end{aligned}\tag{2.26}$$

This completes the proof of Lemma 2.2. $\square$

The following stability result follows directly from Lemmas 2.1 and 2.2.

Theorem 2.1. *The numerical method using the graded mesh for the numerical integration, i.e.,* (1.11)*, is stable in the sense that the matrix* $\mathbf{D}$ *for the corresponding matrix equation* $\mathbf{DU}=\mathbf{F}$ *is non-singular. Furthermore, the sparse matrix* $\mathbf{D}$ *satisfies the estimate*

$$\left\|\boldsymbol{D}^{-1}\right\|_{\mathcal{F}} \leq C\sqrt{n-1}, \tag{2.27}$$

for sufficiently large n, where n is the total number of collocation points.

3. Convergence

The following two lemmas are important in establishing the convergence result for the scheme (1.11).

Lemma 3.1. *Let* $J(s,\sigma) := -\log|\nu(s)-\nu(s+\sigma)|$ *and* $W(s,\sigma) := u(\nu(s+\sigma))$*, where* $\nu(s)$ *is subject to the condition* (1.4) *and assume that* $u(\nu(s)) \in C^3([-L/2, L/2])$*. Then the error of the trapezoidal rule is given by*

$$\begin{aligned} Q(s) &:= \int_{-L/2}^{L/2} J(s,\sigma)W(s,\sigma)\,d\sigma - \frac{1}{2}\sum_{j=1}^{m} J(s,\bar\beta_j)W(s,\bar\beta_j)(\bar\beta_{j+1}-\bar\beta_{j-1}) \\ &= G(s)+E(s), \end{aligned}\tag{3.1}$$

where

$$G(s) := \frac{1}{12}\sum_{j=1}^{m}\left(J(s,\bar\beta_j)W(s,\bar\beta_j)\right)_{\sigma\sigma}(\bar\beta_{j+1}-\bar\beta_j)^3, \tag{3.2}$$

$$\begin{aligned} E(s) := \frac{1}{2}\sum_{j=1}^{m}\int_{\bar\beta_j}^{\bar\beta_{j+1}} &\left[\frac{\eta-\bar\beta_{j+1}}{\bar\beta_j-\bar\beta_{j+1}}\int_{\bar\beta_j}^{\eta}\left(J(s,\sigma)W(s,\sigma)\right)_{\sigma\sigma\sigma}(\sigma-\eta)^2\,d\sigma\right. \\ &\left.+\frac{\eta-\bar\beta_j}{\bar\beta_j-\bar\beta_{j+1}}\int_{\eta}^{\bar\beta_{j+1}}\left(J(s,\sigma)W(s,\sigma)\right)_{\sigma\sigma\sigma}(\sigma-\eta)^2\,d\sigma\right]d\eta. \end{aligned}\tag{3.3}$$

In (3.2) *and* (3.3)*,* $\bar\beta_j \in \bar B$ *with* $\bar B$ *given by* (1.9)*, and we also set* $\bar\beta_{-1} := \bar\beta_1$ *and* $\bar\beta_{m+1} := \bar\beta_m$*. Furthermore,* $G(s)$ *and* $E(s)$ *can be bounded by*

$$|G(s)| \leq C\frac{\log n}{m^2}, \qquad |E(s)| \leq C\frac{\log n}{m^3}. \tag{3.4}$$

Proof. Applying the Euler-Maclaurin theorem to the integrand with $\bar{\beta}_j$ and $\bar{\beta}_{j+1}$, respectively, gives

$$\begin{aligned} J(s,\sigma)W(s,\sigma) &= J(s,\bar{\beta}_j)W(s,\bar{\beta}_j) + \big(J(s,\bar{\beta}_j)W(s,\bar{\beta}_j)\big)_\sigma (\sigma-\bar{\beta}_j) \\ &+\frac{1}{2!}\big(J(s,\bar{\beta}_j)W(s,\bar{\beta}_j)\big)_{\sigma\sigma}(\sigma-\bar{\beta}_j)^2 + \frac{1}{2!}\int_{\bar{\beta}_j}^{\sigma}\left(J(s,\eta)W(s,\eta)\right)_{\eta\eta\eta}(\eta-\sigma)^2\,d\eta, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} J(s,\sigma)W(s,\sigma) &= J(s,\bar{\beta}_{j+1})W(s,\bar{\beta}_{j+1}) + \big(J(s,\bar{\beta}_{j+1})W(s,\bar{\beta}_{j+1})\big)_\sigma (\sigma-\bar{\beta}_{j+1}) \\ &+\frac{1}{2!}\big(J(s,\bar{\beta}_{j+1})W(s,\bar{\beta}_{j+1})\big)_{\sigma\sigma}(\sigma-\bar{\beta}_{j+1})^2 + \frac{1}{2!}\int_{\bar{\beta}_{j+1}}^{\sigma}\left(J(s,\eta)W(s,\eta)\right)_{\eta\eta\eta}(\eta-\sigma)^2\,d\eta. \end{aligned} \tag{3.6}$$

Multiplying (3.5) and (3.6) by $(\sigma-\bar{\beta}_{j+1})/(\bar{\beta}_j-\bar{\beta}_{j+1})$ and $(\sigma-\bar{\beta}_j)/(\bar{\beta}_{j+1}-\bar{\beta}_j)$ respectively, and adding the resulting quantities, yield (3.1). Moreover, similar to the proof of Lemma 3 in [6] we can obtain (3.4). The proof of Lemma 3.1 is thus complete. $\square$

Lemma 3.2. *Assume* $\psi \in C^4([0,2\pi])$ *and is* 2π*-periodic. Let vectors* $\{\mathbf{e}^j\}$ $(j=1,\cdots,n)$ *be given by*

$$[1, \exp(\mathbf{i}2\pi[j-1]/n), \exp(\mathbf{i}2\pi[2(j-1)]/n), \cdots, \exp(\mathbf{i}2\pi[(n-1)(j-1)]/n)]^\top.$$

Then

$$\left|<\mathbf{r}\psi''(s), \mathbf{e}^j>\right| \le \begin{cases} Cn^{-2}, & j=1, \\ C\Big(\dfrac{1}{j-1}+\dfrac{1}{n-(j-1)}\Big)^2, & j=2,\cdots,n, \end{cases}$$

where $\mathbf{r}$ *is the restriction operator defined in* (2.19) *and* $<\cdot,\cdot>$ *denotes the Euclidean inner product.*

Proof. It is known from [8] that the vectors $\mathbf{e}^j$ are the eigenvectors corresponding to the eigenvalues λ_k in (2.10). Then the remaining proof is exactly the same as that of Lemma 2.4 in [3]. $\square$

Due to the orthogonality property,

$$<\mathbf{e}^j, \mathbf{e}^k> = 2\pi\delta_{j,k} \quad (1 \le j,k \le n),$$

where $\delta_{j,k}$ is the Kronecker delta function, we may write the following expansion

$$\mathbf{r}\psi''(s) = \frac{1}{2\pi}\sum_{j=1}^{n} <\mathbf{r}\psi''(s), \mathbf{e}^j> \mathbf{e}^j.$$

It then follows from Lemma 3.2 that

$$\left\|(\mathbf{D}^{[1]})^{-1}\mathbf{r}\psi''\right\|_2^2 = \frac{1}{2\pi}\sum_{j=1}^{n}\lambda_j^{-2}\left|<\mathbf{r}\psi'', \mathbf{e}^j>\right|^2. \tag{3.7}$$

Applying Lemmas 3.2 and 2.1 gives

$$\left\|(\mathbf{D}^{[1]})^{-1}\mathbf{r}\psi''\right\|_2 \le C, \tag{3.8}$$

where the constant C is independent of n and m.

To provide error bounds of our numerical schemes, we use a discrete L^2 norm defined by (see, e.g., Cheng *et al.* [6])

$$\|u(s)\|_{\text{dis}} := \left[\frac{1}{n} < \mathbf{r}u(s), \mathbf{r}u(s) >\right]^{1/2} = \frac{1}{\sqrt{n}}\|\mathbf{r}u(s)\|_2. \tag{3.9}$$

Theorem 3.1. *Let $w(s) := u(\nu(s))$ and $w_n(s) := u_n(\nu(s))$ be the solutions of* (1.5) *and* (1.11)*, respectively, where $\nu(s)$ is subject to the condition* (1.4)*. Moreover, assume $w(s) \in C^4([-L/2, L/2])$. Then the a priori error estimate of the scheme* (1.11) *to the integral equation* (1.5) *is given by*

$$\|w(s) - w_n(s)\|_{dis} \le C\frac{\log n}{n},$$

where the discrete norm $\|\cdot\|_{\text{dis}}$ is given in (3.9)*.*

Proof. It is observed that

$$\|w(s) - w_n(s)\|_{\text{dis}} = \frac{1}{\sqrt{n}}\|\mathbf{r}w - \mathbf{r}w_n\|_2 = \frac{1}{\sqrt{n}}\|\mathbf{r}w - \mathbf{U}\|_2, \tag{3.10}$$

where $\mathbf{U}$ is given in (2.6). Therefore, it only needs to estimate $\|\mathbf{r}w - \mathbf{U}\|_2$. It follows from

$$\mathbf{D}(\mathbf{r}w - \mathbf{U}) = \mathbf{r}Q$$

that

$$\mathbf{r}w - \mathbf{U} = \mathbf{D}^{-1}\mathbf{r}Q = \mathbf{D}^{-1}\mathbf{r}G(s) + \mathbf{D}^{-1}\mathbf{r}E(s).$$

Using the inequality (3.8) with the change of the variable $s = \tau\frac{L}{2\pi} - \frac{L}{2}$, together with the stability results Theorem 2.1 and the quadrature error estimates in Lemma 3.1, yield

$$\|\mathbf{r}w - \mathbf{U}\|_2 \le C\frac{\sqrt{n}\log n}{n}.$$

Combining the above estimate and (3.10) completes the proof of this theorem. $\square$

Theorem 3.2. *Assume $w(s) \in C^4([-L/2, L/2])$, where $w(s) := u(\nu(s))$ is the solution of* (1.5)*. Assume the Simpson's rule is employed to approximate the integral involved in* (1.10) *and the graded mesh B is used such that $\beta_{j+1/2} := (\beta_j + \beta_{j+1})/2 \in B$ for all $\beta_j, \beta_{j+1} \in B$. Denote the resulting numerical solution by $w_n(s)$. Then the a priori error estimate of the numerical scheme is given by*

$$\|w(s) - w_n(s)\|_{\text{dis}} \le C\frac{\log n}{n^2}.$$

Proof. Since the proof of the above theorem is similar to that of Theorem 3.1, it will be omitted here. The detailed description of the numerical scheme using Simpson's rule can be found in [6]. $\square$

Remark 3.1. In our main theorems, Theorems 3.1 and 3.2, we require a quite strong regularity assumption, i.e., the solution of the underlying integral equations belongs to the space C^4 which implies that the boundary Γ as well as the function f should be sufficiently smooth. However, this requirement can be justified by noticing the equivalence between the solutions of (1.2) and (1.3) (see also [2] and [20]).

4. Concluding Remarks

This paper gives a rigorous convergence and stability proof for the collocation method proposed by Cheng et. al [6]. The underlying idea is to use graded meshes in the quadrature to approximate the singular integral involved. This approach allows fewer mesh points in the quadrature without decreasing the accuracy. As shown in the numerical tests in [6] and then confirmed by the rigorous proof in this paper, this method is more efficient than some traditional methods. For the benefit to the reader, a brief comparison with other methods, based on the first order accuracy, is listed in the following table.

Table 4.1: Comparison with other methods based on first order accuracy.

Discretization method	Preconditioning	Operations required
Trigonometric spectral method [11]	No	$\mathcal{O}(n^2)$
Trigonometric spectral method [13]	Two-grid	$\mathcal{O}(n\log n)$
Traditional collocation method	No	$\mathcal{O}(n^2)$
Method based on mesh grading	No	$\mathcal{O}(n^{3/2})$

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