Journal of Computational Mathematics, Vol.24, No.3, 2006, 444–450.

CALCULATION OF NAMBU MECHANICS *1)

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Dedicated to the 70th birthday of Professor Lin Qun

Abstract

In this paper, a fundamental fact that Nambu mechanics is source free was proved. Based on this property, and via the idea of prolongation, finite dimensional Nambu system was prolonged to difference jet bundle. Structure preserving numerical methods of Nambu equations were established. Numerical experiments were presented at last to demonstrate advantages of the structure preserving schemes.

Mathematics subject classification: 65L06.
Key words: Nambu equations, Differential forms, Difference schemes, Structure preserving methods.

1. Introduction

Nambu proposed an intriguing generalization of classical Hamiltonian mechanics [1], the idea was to extend the original Poisson bracket formulation of two function defined on R^2 to bracket of three functions h, f, g of $w = (x, y, z) \in R^3$, as

$$\{f, g, h\} = \frac{\partial(f, g, h)}{\partial(x, y, z)},$$

where $\frac{\partial(f,g,h)}{\partial(x,y,z)}$ is Jacobian. The equations of motion corresponding to f and g were written as

$$\dot{w} = \nabla f \times \nabla g. \tag{1}$$

Easy to see that for arbitrary function $h(x, y, z)$,

$$\frac{dh}{dt} = \{f, g, h\}. \tag{2}$$

Recent interest in this topic is due to Takhtajan [2] studied in particular the consistency requirement one should place on the generalization . Hietarinta [3] express the consistency conditions via Nambu tensor.

Briefly speaking, let $f_1, f_2, \cdots, f_n$ be n functions defined on

$$R^n = \{(x_1, x_2, \cdots, x_n) = w\},$$

Nambu bracket is defined as following

$$\{f_1, f_2, \cdots, f_n\} = \frac{\partial(f_1, f_2, \cdots, f_n)}{\partial(x_1, x_2, \cdots, x_n)}. \tag{3}$$

It is easy to see that the bracket satisfys the following condition [2]:

* Received March 1, 2006.
1) Supported by the National Basic Research under the Grant 2005cb32170x.

1 Skew symmetry

$$\{f_1, f_2, \cdots, f_n\} = (-1)^{\epsilon(\sigma)}\{f_{\sigma(1)}, \cdots, f_{\sigma(n)}\},$$

where σ is a permutation of $1, \cdots, n$ and $\varepsilon(\sigma)$ is its parity.

2 The Leibnitz rule

$$\{f_1 g, f_2, \cdots, f_n\} = g\{f_1, f_2, \cdots, f_n\} + f_1\{g, f_2, \cdots, f_n\}.$$

1. Generalised Jacobi identity

$$\begin{aligned} & \{\{g_1, \cdots, g_{n-1}, f_1\}, f_2, \cdots, f_n\} + \{f_1, \{g_1, \cdots, g_{n-1}, f_2\}, \cdots, f_n\} \\ & + \{f_1, f_2, \cdots, \{g_1, \cdots, g_{n-1}, f_n\}\} \\ = \; & \{g_1, \cdots, g_{n-1}, \{f_1, f_2, \cdots, f_n\}\}. \end{aligned}$$

For given functions $f_1, f_2, \cdots, f_{n-1}$, we use f to denote the vector valued function $(f_1, f_2, \cdots, f_{n-1})^T$, the corresponding equations of motion are then

$$\begin{cases} \dot{x}_1 = \{f_1, f_2, \cdots, f_{n-1}, x_1\} = N(f_1, f_2, \cdots, f_{n-1})_1 = N(f)_1 \\ \dot{x}_2 = \{f_1, f_2, \cdots, f_{n-1}, x_2\} = N(f_1, f_2, \cdots, f_{n-1})_2 = N(f)_2 \\ \vdots \\ \dot{x}_n = \{f_1, f_2, \cdots, f_{n-1}, x_n\} = N(f_1, f_2, \cdots, f_{n-1})_n = N(f)_n \end{cases} \tag{4}$$

here we use $N(f_1, f_2, \cdots, f_{n-1}) == N(f)$ to denote Nambu vector fields generated by $f_1, f_2, \cdots, f_{n-1}$.

It will be proved in next section that the phase flow of (4) preserving $\omega^n = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$. This make us construct structure preserving schemes: preserving ω^n, called volume preserving. Feng [4] proposed this idea for source free system, and established a systematic technique to design it via vector field splitting methods. Unfortunately this methods are unpractical, especially for large n, because it need first to split vector field to a sum of 2^n vector fields, the existence of such vector fields only be proved theoretically, then solve all the 2^n systems of differential equations numerically, the amount of calculation is as much as 2^n times of original one. There is until now no volume preserving schemes of source free systems have been found which is independent on the dimension of space and for arbitrary right term, only for special kind of systems, some first order methods have been designed out explicitly [4, 5]. By the technique of lift (4) to difference jet bundle, we established a systematical methods to design multi-step structure preserving schemes for Nambu mechanics. Numerical experiments have been made at last.

2. Phase Flow of Nambu System

In this section, we will mainly prove that phase flow of Nambu system preserves $n - form$

$$\omega^n = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

It is well known that the phase flow of source free system preserves ω^n, so we only need to prove (4) is source free, i.e.,

$$\frac{\partial}{\partial x_1} N(f)_1 + \frac{\partial}{\partial x_2} N(f)_2 + \cdots + \frac{\partial}{\partial x_n} N(f)_n = 0.$$

Theorem 1. *Nambu system* (4) *is a source free system* *proof.* From definition of $N(f)_i$ we know

$$N(f)_i = (-1)^{i-1} \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{i-1}} & \frac{\partial f_1}{\partial x_{i+1}} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_{n-1}}{\partial x_1} & \cdots & \frac{\partial f_{n-1}}{\partial x_{i-1}} & \frac{\partial f_{n-1}}{\partial x_{i+1}} & \cdots & \frac{\partial f_{n-1}}{\partial x_n} \end{vmatrix},$$

and

$$\begin{aligned} \frac{\partial}{\partial x_i} N(f)_i &= (-1)^{i-1} \begin{vmatrix} \frac{\partial}{\partial x_i}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{i-1}} & \frac{\partial f_1}{\partial x_{i+1}} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial}{\partial x_i}\frac{\partial f_{n-1}}{\partial x_1} & \cdots & \frac{\partial f_{n-1}}{\partial x_{i-1}} & \frac{\partial f_{n-1}}{\partial x_{i+1}} & \cdots & \frac{\partial f_{n-1}}{\partial x_n} \end{vmatrix} + \cdots \\ &+ (-1)^{i-1} \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{i-1}} & \frac{\partial f_1}{\partial x_{i+1}} & \cdots & \frac{\partial}{\partial x_i}\frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_{n-1}}{\partial x_1} & \cdots & \frac{\partial f_{n-1}}{\partial x_{i-1}} & \frac{\partial f_{n-1}}{\partial x_{i+1}} & \cdots & \frac{\partial}{\partial x_i}\frac{\partial f_{n-1}}{\partial x_n} \end{vmatrix} \\ &= (-1)^{i-1} M_i^1 + (-1)^{i-1} M_i^2 + \cdots + (-1)^{i-1} M_i^n. \end{aligned}$$

$$M_i^j = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial}{\partial x_i}\frac{\partial f_1}{\partial x_j} & \cdots & \frac{\partial f_1}{\partial x_{i-1}} & \frac{\partial f_1}{\partial x_{i+1}} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_{n-1}}{\partial x_1} & \cdots & \frac{\partial}{\partial x_i}\frac{\partial f_{n-1}}{\partial x_j} & \cdots & \frac{\partial f_{n-1}}{\partial x_{i-1}} & \frac{\partial f_{n-1}}{\partial x_{i+1}} & \cdots & \frac{\partial f_{n-1}}{\partial x_n} \end{vmatrix}$$

By the expression of M_i^j, we have $M_i^j = (-1)^{i-j-1} M_j^i$. Therefor

$$\begin{aligned} & \frac{\partial}{\partial x_1} N(f)_1 + \frac{\partial}{\partial x_2} N(f)_2 + \cdots + \frac{\partial}{\partial x_n} N(f)_n \\ = & \sum_{i,j=1, i\neq j}^{n} (-1)^{i-1} M_i^j = \sum_{i \lneqq j}^{n} (-1)^{i-1} M_i^j + \sum_{i \gneqq j}^{n} (-1)^{i-1} M_i^j \\ = & \sum_{i \lneqq j}^{n} (-1)^{i-1} M_i^j + \sum_{i \gneqq j}^{n} (-1)^{i-1+i-j-1} M_j^i = 0. \end{aligned}$$

This end the proof.

3. Jet Nambu Mechanics

In [6], the author introduced difference jet manifold and their prolongation. Consider simple fibre bundle $(R \otimes R^n, \pi, R)$, π is projection, $\pi : R \otimes R^n \mapsto R$, $\pi(t, x_1, \cdots, x_n) = t$. E is translation transformation defined on R by $Et = t + h$, where h is some constant called step size. Naturally its inverse E^{-1} is defined as $E^{-1}t = t - h$. Denote Θ as all section of bundle $(R \otimes R^n, \pi, R)$, that is $\Theta = \{\Psi : R \mapsto R^n\}$. Two elements ψ_1, ψ_2 of Θ are said to be equivalent in respect to fixed t, and denoted by $\psi_1 \sim_t \psi_2$, if

$$\psi_1(t + ih) = \psi_2(t + ih) \quad i = 0, \pm 1, \pm 2, \cdots.$$

$J^\infty = \cup_{t \in R} \Theta / \sim_t$ is called infinite difference jet bundle and easy to see

$$J^\infty \simeq \otimes_{-\infty}^{+\infty} R^n \otimes R.$$

Use $(t, w = w^0, w^{\pm 1}, \cdots)$ to denote general coordinates of $\otimes_{-\infty}^{+\infty} R^n \otimes R$, projection from J^∞ to R, or to $\otimes_l^m R^n \otimes R$. is as usual.

As special case of prolongation of vector fields, we now prolong Nambu vector field to J^∞. Introduce t direction, and note that $w^{\pm i} = (x_1^{\pm i}, \cdots x_n^{\pm i})$, Nambu vector field corresponding to Nambu vector valued function (or briefly Nambu function) $f(w)$ is cast into

$$\partial/\partial t + \sum_{i=1}^{n} N(f)_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial t} + N(f)\frac{\partial}{\partial w}.$$

The prolonged vector field is

$$\partial/\partial t + \sum_{j=-\infty}^{\infty} \hat{N}(\hat{f})^j \frac{\partial}{\partial w^j},$$

where

$$\hat{f} = \sum_{j=-\infty}^{\infty} f(w^j)$$

is the prolongation of Nambu function f , and

$$\hat{N}(\hat{f})_i^j = N(f)_i(w^j).$$

The prolongation of ω^n is

$$\hat{\omega}^n = \sum_{j=-\infty}^{\infty} dx_1^j \wedge dx_2^j \wedge \cdots \wedge dx_n^j.$$

We can now see that $\hat{N}(\hat{f})$ is Nambu-Hamilton vector field of $\hat{f}$ corresponding to prolonged $n-$ form $\hat{\omega}^n$. In other words, the prolonged vector field is

$$\begin{cases} \vdots \\ \dot{w}^1 = \hat{N}(\hat{f})^1 \\ \dot{w}^0 = \hat{N}(\hat{f})^0 \\ \dot{w}^{-1} = \hat{N}(\hat{f})^{-1} \\ \vdots \end{cases} \tag{5}$$

Equations (5) is infinity dimensional Nambu-Hamiltonian system, called jet Nambu-Hamiltonian mechanics.

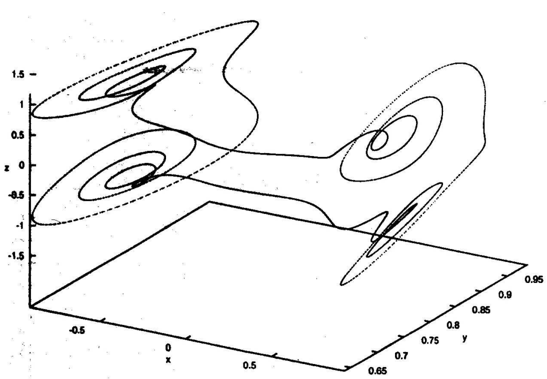

Figure 1: **E**xample1 calculated by Nambu scheme

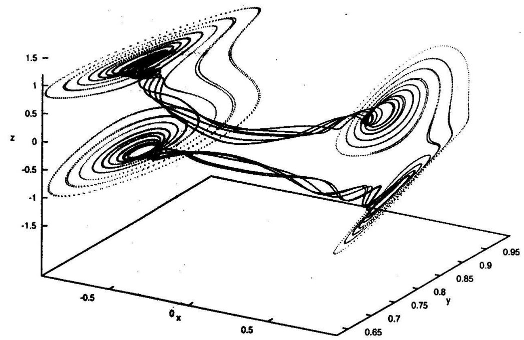

Figure 2: **E**xample1 calculated by non-Nambu scheme

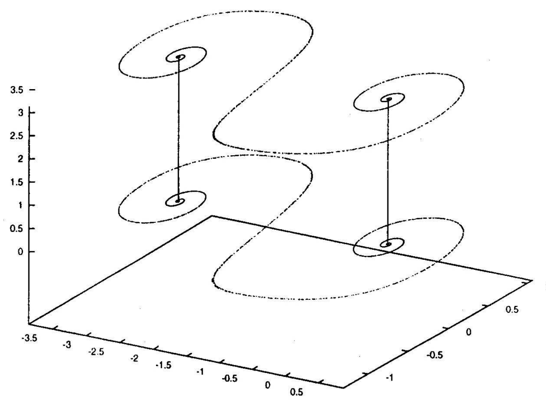

Figure 3: **E**xample2 calculated by Nambu scheme

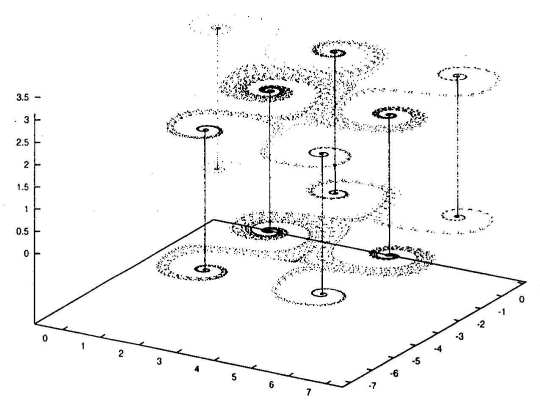

Figure 4: **E**xample2 calculated by non-Nambu scheme

4. Nambu Schemes

It is easy to prove that the phase flow of (5) preserves $\hat{\omega}^n$. As in finite dimensional case

[10, 11], We define difference scheme of (5) as jet Nambu-Hamiltonian scheme if it preserves $n-form\ \hat{\omega}^n$. Via the jet Nambu-Hamiltonian scheme, Nambu difference scheme can be defined as following:

Definition. *A difference scheme of* (3) *is called Nambu scheme if its prolongation is a jet Nambu-Hamilton scheme of* (5).

As examples, we show some Nambu scheme below. The proof of its structure-preserving is straightforward, only need to calculation carefully.

Here is a family of two step schemes

$$\begin{aligned} & w^{n+1} = 2w^n - w^{n-1} + \Delta t(\alpha N(f,g)(w^{n+1}) - \\ & (\beta+\alpha)N(f,g)(w^n) + \beta N(f,g)(w^{n-1})). \end{aligned} \tag{6}$$

In general, all the schemes in this family are first order. When $\alpha - \beta = 1$, they are second order, and the scheme for $\alpha = 0.5, \beta = -0.5$ is third order.

The following are three step methods:

$$\begin{aligned} & w^{n+2} = (1-a)w^{n+1} + (2a+1)w^n - (a+1)w^{n-1} \\ & +\Delta t(-\alpha N(f,g)(w^{n+2}) + (\alpha-\beta)N(f,g)(w^{n+1}) \\ & +(\beta-\gamma)N(f,g)(w^n) + \gamma N(f,g)(w^{n-1})). \end{aligned} \tag{7}$$

Second order condition is

$$\alpha + \beta + \gamma + a = -2.$$

When the remain parameters satisfy the additional condition:

$$3\alpha + \beta - \gamma = -2,$$

the family become third order. Similarly $\beta + 8\alpha = -4$ is fourth order condition and the only one fifth order scheme is for $\alpha = -1/3$.

On the other aspect, $a = -2, \gamma = -\alpha = -\beta$ are the reversible condition, we see that the schemes are at most of first order in this case.

Analogously we can construct four step schemes, five steps schemes, and so on. We have constructed a table of such structure preserving schemes, it will appear any where.

5. Numerical Experiment

In order to demonstrate advantages of the schemes which preserving the structure we give above, We used (6) with $\alpha = 0.5$ and $\beta = -0.5$ to calculate following Nambu systems.

Example1.

$$\left\{ \begin{array}{lllll} \frac{dx}{dt} & = & zsin(y) & - & ysin(z) \\ \frac{dy}{dt} & = & xsin(z) & - & zsin(x) \\ \frac{dz}{dt} & = & ysin(x) & - & xsin(y) \end{array} \right.$$

We calculated example1 by Nambu scheme(fig1)(20000 steps) and non-Nambu scheme (fig2) (10000000 steps) respectively, with initial point$(0.9, 0.7, 0.5)$ and stepsize 0.1.

Example2.

$$\left\{ \begin{array}{lllll} \frac{dx}{dt} & = & -cos(y)cos(z)sin(x) & - & 4cos(x)cos(z)sin(y) \\ \frac{dy}{dt} & = & -cos(x)cos(z)sin(y) & + & 4cos(y)cos(z)sin(x) \\ \frac{dz}{dt} & = & 2cos(y)cos(x)sin(z) & & \end{array} \right.$$

We calculated example2 by Nambu scheme(fig3) (10000 steps) and non-Nambu scheme(fig4) (10000000 steps) respectively, with initial point$(0.9, 0.7, 0.5)$ and stepsize 0.1.

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