Journal of Computational Mathematics, Vol.24, No.5, 2006, 561–578.

A NONCONFORMING ANISOTROPIC FINITE ELEMENT APPROXIMATION WITH MOVING GRIDS FOR STOKES PROBLEM *1)

Dong-yang Shi
(*Department of Mathematics, Zhengzhou University, Zhengzhou 450052, China*)

Yi-ran Zhang
(*School of Mathematical Science, Peking University, Beijing 100871, China*)

Abstract

This paper is devoted to the five parameters nonconforming finite element schemes with moving grids for velocity-pressure mixed formulations of the nonstationary Stokes problem in 2-D. We show that this element has anisotropic behavior and derive anisotropic error estimations in some certain norms of the velocity and the pressure based on some novel techniques. Especially through careful analysis we get an interesting result on consistency error estimation, which has never been seen for mixed finite element methods in the previously literatures.

Mathematics subject classification: 65N30.
Key words: Stokes problem, Nonconforming finite element, Anisotropy, Moving grids, Error estimate.

1. Introduction

We usually apply the finite element methods to the spatial domain, but choose difference methods with respect to the time variable for solving partial differential equations depending on time. At the same time, different meshes of domain are used at different time level. As we all know, the solutions may have weak regularity at the beginning, therefore, lower order interpolation functions and the smaller meshes should be used. As the time goes on, the regularity of the solution becomes better, the higher order interpolation functions and the larger meshes can be used. That is the main idea of the finite element with moving grids.

Local interpolation error estimations for the finite element methods with moving grids are developed in the literatures [2,3]. But these results are based on isotropic meshes at any time and on any domain. In fact, many examples show that the solutions sometimes have anisotropic behaviors [1,6] on boundary or interior layers. That means that the solution varies significiently only in certain directions. In such cases it is an obvious idea to reflect this anisotropy in the discretization by using anisotropic meshes with a small size in the direction of the rapid variation of the solution and a larger mesh size in the perpendicular direction. That is, anisotropic meshes are necessarily used.

Recently, there have appeared some studies on anisotropic meshes [1,8,14,15,16,18,19], but most of all only considered the interpolation error estimation and conforming elements for elliptic boundary problems. The nonconforming elements and Stokes problem on anisotropic meshes are hardly treated, [9] studied the anisotropic error of Crouzeix-Raviart type elements and applied them to possion problem, [6] studied the quasi-wilson element under a new framework. However, as to anisotropic meshes with moving grids there have been no articles published on

* Received February 17, 2004; final revised December 7, 2005.
1) This research is supported by the National Science Foundation of China(No.10371113).

this respect. Because of the restrictions of the BB conditions, nonconforming finite elements are particularly interested in mixed methods for problems like the velocity-pressure type of Stokes equations, which are advanced in simply structure, economic computing and matching of error orders. The well-known examples include nonconforming Crouzeix-Raviart element [13], the rotated Q_1 element [4] and so on, but these elements can only be used to deal with Stokes equations with moving grids under the regular assumption [10]. [5] developed a kind of nonconforming rectangular element and gave the error estimation for stationary Stokes equations on isotropic meshes. In this paper, we will first show that the element in [5] has anisotropic behavior , and then we derive the error estimation of the stationary Stokes equations on anisotropic meshes. Furthermore, with the idea of moving grids, we study the nonstationary Stokes equations and address the anisotropic nonconforming error estimations based on some results of the stationary problem and some novel approaches. At the same time, by careful analysis we will prove a very interesting and more important result, that is , when the solution $(u,p) \in (H^2(\Omega))^2 \times H^1(\Omega)$ the consistency error is of order $O(h^2)$, one order higher than that of interpolation error which is similar to the reports of [11] for the triangular quasi-conforming and generalized conforming finite elements of fourth order plate bending problem, and the double set 12-parameter rectangular element obtained in [12], and that of [16] for the quasi-wilson element for the second order problems.

Throughout the paper we use the following concerning indices. For the sake of simplity, let $\Omega \subset R^2$ be a rectangular domain with boundary $\partial\Omega$ parallel to x-axis or y-axis. Let Γ_h be a family of rectangular subdivisions, i.e., $\overline{\Omega} = \bigcup\limits_{K\in\Gamma_h} K$. Denote by h_K the diameter of the finite element K, and by ρ_K the supremum of the diameters of all balls contained in K. Then the regularity assumption in the classical finite element theory is $\frac{h_K}{\rho_K} \leq C, \forall K \in \Gamma_h$ (The C is a positive constant independent of Γ_h and of the function under consideration). This assumption is no longer valid in the case of anisotropic meshes. Conversely, anisotropic element K is characterized by $\frac{h_K}{\rho_K} \longrightarrow \infty$ where the limit can be considered as $h \longrightarrow 0, h = \max\limits_K h_K$. In this paper the C will also denote the positive constant, not necessarily the same at different occurances which is independent of $\frac{h_K}{\rho_K}$ and h. For the general element K, we denote the lenghths of sides parallel to x-axis and y-axis by $2h_x$ and $2h_y$ respectively, and the central point of K by (x_K, y_K). Let $\widehat{K}$ be a reference element(see Fig.1.), and $\widehat{K} = [-1,1] \times [-1,1]$ with vertices $\widehat{d_1} = (-1,-1), \widehat{d_2} = (1,-1), \widehat{d_3} = (1,1), \widehat{d_4} = (-1,1)$. Let $\widehat{l_1} = \overline{\widehat{d_1}\widehat{d_2}}, \widehat{l_2} = \overline{\widehat{d_2}\widehat{d_3}}, \widehat{l_3} = \overline{\widehat{d_3}\widehat{d_4}}, \widehat{l_4} = \overline{\widehat{d_4}\widehat{d_1}}$ be the four sides of $\widehat{K}$. The transformation of $F_K : \widehat{K} \longrightarrow K$ is defined by

$$\begin{cases} x = x_K + h_x\xi, \\ y = y_K + h_y\eta. \end{cases} \tag{1.1}$$

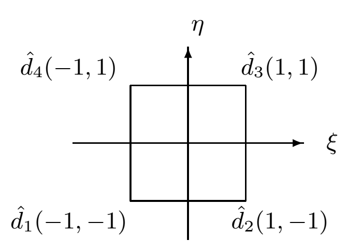

Fig.1. the reference element $\widehat{K}$

2. Construction of the Nonconforming Five-parameter Rectangular Element and Its Anisotropism

We define a finite element $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ as follows:

$$\widehat{P} = span\{1, \xi, \eta, \varphi(\xi), \varphi(\eta)\},$$

$$\widehat{\Sigma} = \{\widehat{v}_1, \widehat{v}_2, \widehat{v}_3, \widehat{v}_4, \widehat{v}_5\},$$

where

$$\varphi(t) = \tfrac{1}{2}(5t^4 - 3t^2) \quad , or \quad \varphi(t) = \tfrac{1}{2}(3t^2 - 1),$$

$$\widehat{v}_i = \tfrac{1}{|\widehat{l}_i|}\int_{\widehat{l}_i} \widehat{v} d\widehat{s}, \qquad i = 1, 2, 3, 4,$$

$$\widehat{v}_5 = \tfrac{1}{|\widehat{K}|}\int_{\widehat{K}} \widehat{v} d\xi d\eta, \qquad \forall \widehat{v} \in H^1(\widehat{K}),$$

$|\widehat{l}_i| = meas(\widehat{l}_i), |\widehat{K}| = meas(\widehat{K})$.
Then we have the following lemma.
Lemma 2.1[5]**.** *$\forall \widehat{v} \in H^1(\widehat{K})$, there exists a unique interpolation operator $\widehat{\Pi} : H^1(\widehat{K}) \longrightarrow \widehat{P}$ such that*

$$\tfrac{1}{|\widehat{l}_i|}\int_{\widehat{l}_i} \widehat{\Pi}\widehat{v} d\widehat{s} = \tfrac{1}{|\widehat{l}_i|}\int_{\widehat{l}_i} \widehat{v} d\widehat{s} = \widehat{v}_i, \qquad i = 1, 2, 3, 4,$$

$$\tfrac{1}{|\widehat{K}|}\int_{\widehat{K}} \widehat{\Pi}\widehat{v} d\widehat{s} = \tfrac{1}{|\widehat{K}|}\int_{\widehat{K}} \widehat{v} d\xi d\eta = \widehat{v}_5$$

and

$$\widehat{\Pi}\widehat{v} = \widehat{v}_5 + \frac{1}{2}(\widehat{v}_2 - \widehat{v}_4)\xi + \frac{1}{2}(\widehat{v}_3 - \widehat{v}_1)\eta + \frac{1}{2}(\widehat{v}_2 + \widehat{v}_4 - 2\widehat{v}_5)\varphi(\xi) + \frac{1}{2}(\widehat{v}_3 + \widehat{v}_1 - 2\widehat{v}_5)\varphi(\eta). \quad (2.1)$$

Lemma 2.2. *For any $\alpha = (a_1, a_2)$ and $|\alpha| = 1$, we have*

$$\|\widehat{D}^\alpha(\widehat{v} - \widehat{\Pi}\widehat{v})\|_{0,\widehat{K}} \le C|\widehat{D}^\alpha \widehat{v}|_{1,\widehat{K}}. \quad (2.2)$$

Proof. Let $\alpha = (1, 0)$, then

$$\widehat{D}^\alpha \widehat{\Pi}\widehat{v} = \frac{\partial \widehat{\Pi}\widehat{v}}{\partial \xi} = \frac{1}{2}(\widehat{v}_2 - \widehat{v}_4) + \frac{1}{2}(\widehat{v}_2 + \widehat{v}_4 - 2\widehat{v}_5)\varphi'(\xi).$$

Obviously, $\{1, \varphi'(\xi)\}$ is a group of basis of $\widehat{D}^\alpha \widehat{P}$, i.e.,$\widehat{D}^\alpha \widehat{P} = span\{1, \varphi'(\xi)\}$. Therefore,

$$\widehat{D}^\alpha \widehat{\Pi}\widehat{v} = \beta_1 + \beta_2 \varphi'(\xi),$$

where

$$\beta_1 = \frac{1}{2}(\widehat{v}_2 - \widehat{v}_4), \qquad \beta_2 = \frac{1}{2}(\widehat{v}_2 + \widehat{v}_4 - 2\widehat{v}_5)$$

Rewrite β_1, β_2 as

$$\beta_1 = \tfrac{1}{4}[\int_{\widehat{l}_2} \widehat{v}(1, \eta) d\eta - \int_{\widehat{l}_4} \widehat{v}(-1, \eta) d\eta] = \tfrac{1}{|\widehat{K}|}\int_{\widehat{K}} \tfrac{\partial \widehat{v}}{\partial \xi} d\xi d\eta,$$

$$\beta_2 = \tfrac{1}{4}[\int_{\widehat{l}_2} \widehat{v}(1, \eta) d\eta + \int_{\widehat{l}_4} \widehat{v}(-1, \eta) d\eta - \int_{\widehat{K}} \widehat{v}(\xi, \eta) d\xi d\eta] = \tfrac{1}{|\widehat{K}|}\int_{\widehat{K}} \xi \tfrac{\partial \widehat{v}}{\partial \xi} d\xi d\eta.$$

Then we define the operators $F_j, j = 1, 2$ as

$$\begin{aligned} F_1(\widehat{w}) &= \tfrac{1}{|\widehat{K}|}\int_{\widehat{K}} \widehat{w} d\xi d\eta, \\ F_2(\widehat{w}) &= \tfrac{1}{|\widehat{K}|}\int_{\widehat{K}} \xi \widehat{w} d\xi d\eta, \qquad \forall \widehat{w} \in H^1(\widehat{K}). \end{aligned} \quad (2.3)$$

It is easy to see that $F_j \in (H^1(\widehat{K}))', j = 1, 2$.

Similarly, we can get the same result for the case $\alpha = (0, 1)$. The proof is completed by applying the fundamental anisotropism theory[6].

Remark 1. One can also check the anisotropy of the element by the method provided in [1].

3. The Anisotropic Error Estimation of the Stationary Stokes Equations

We consider the following problem:

$$\begin{cases} -\gamma \triangle u + \nabla p = f, & \text{in } \Omega, \\ divu = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $u = (u_1, u_2)$ is the velocity vector, p is the pressure, $\gamma > 0$ is the coeffient of kinematic viscosity, $f = (f_1, f_2)$ is a given vector function. Then (3.1)is equivalent to the following variational problem: Find $(u, p) \in V \times M$ such that

$$\begin{cases} a(u, v) + b(v, p) = f(v), & \forall v \in V, \\ b(u, q) = 0, & \forall q \in M, \end{cases} \tag{3.2}$$

where

$$V = (H_0^1(\Omega))^2, \quad M = \{q \in L^2(\Omega); \textstyle\int_\Omega q dx dy = 0\},$$

$$a(u, v) = \gamma \textstyle\int_\Omega \nabla u \nabla v dx dy,$$

$$b(v, q) = -\textstyle\int_\Omega q divv dx dy,$$

$$f(v) = \textstyle\int_\Omega f v dx dy, \quad \forall u, v \in V, \forall q \in M.$$

The spaces V and M satisfy the BB condition [7], i.e., there exists a constant $\beta > 0$ such that

$$\sup_{v \in V} \frac{b(v, q)}{|v|_V} \geq \beta \|q\|_{0,\Omega}. \tag{3.3}$$

Let us define

$$E_h = \{v_h; v_h|_K = \widehat{v} \circ F_K^{-1}, \widehat{v} \in \widehat{P}, \forall K \in \Gamma_h\},$$

$$V_h = E_h \times E_h, \quad M_h = \{q_h; q_h|_K \text{ is a constant}, \forall K \in \Gamma_h\},$$

and

$$|v_h|_h = (\sum_{K \in \Gamma_h} |v_h|_{1,K}^2)^{\frac{1}{2}}. \tag{3.4}$$

We have

Lemma 3.1. $|\cdot|_h$ *is a norm of the finite element spaces* V_h.

Proof. We only show that $v_h = 0$ when $|v_h|_h = 0, \forall v_h \in V_h$.

Since $v_h|_K = \widehat{v} \circ F_K^{-1}$ can be expressed as

$$v_h|_K = v_{h5} + \frac{1}{2}(v_{h2} - v_{h4})\xi + \frac{1}{2}(v_{h3} - v_{h1})\eta + \frac{1}{2}(v_{h2} + v_{h4} - 2v_{h5})\varphi(\xi) + \frac{1}{2}(v_{h3} + v_{h1} - 2v_{h5})\varphi(\eta). \tag{3.5}$$

Consequently,

$$\frac{\partial v_h|_K}{\partial x} = \frac{v_{h2} - v_{h4}}{2h_x} + \frac{v_{h2} + v_{h4} - v_{h5}}{2h_x}\varphi'(\xi), \tag{3.6}$$

$$\frac{\partial v_h|_K}{\partial y} = \frac{v_{h3} - v_{h1}}{2h_y} + \frac{v_{h3} + v_{h1} - v_{h5}}{2h_y}\varphi'(\eta). \tag{3.7}$$

If $|v_h|_h = 0$, then $|v_h|_{1,K} = 0, \forall K \in \Gamma_h$, i.e., $\frac{\partial v_h|_K}{\partial x} = \frac{\partial v_h|_K}{\partial y} = 0, \forall K \in \Gamma_h$. When K is an element on upper-left corner of Ω, we get $v_{h3} = v_{h4} = 0$. Combining (3.5),(3.6),(3.7) with $\varphi'(\xi)$ and $\varphi'(\eta)$ being nonconstants, we obtain $v_{h1} = v_{h2} = v_{h5} = 0$, therefore $|v_h|_K = 0$. Under the influence of the function values on sides between neighbour elements, we derive $v_h = 0$. The proof is completed.

The approximation problem corresponding to (3.2) reads as: Find $(u_h, p_h) \in V_h \times M_h$ such that

$$\begin{cases} a_h(u_h, v_h) + b_h(v_h, p_h) = (f, v_h), & \forall v_h \in V_h, \\ b(u_h, q_h) = 0, & \forall q_h \in M_h, \end{cases} \tag{3.8}$$

where

$$a_h(u_h, v_h) = \gamma \sum_{K\in\Gamma_h} \int_K \nabla u_h \nabla v_h dxdy,$$
$$b_h(v_h, q_h) = -\sum_{K\in\Gamma_h} \int_K q_h div v_h dxdy, \qquad \forall v_h \in V_h, \forall q_h \in M_h.$$

Obviously, $a_h(\cdot,\cdot)$ is coercive on $V_h \times V_h$, i.e.,

$$a_h(v_h, v_h) \geq \gamma |v_h|_h^2, \qquad \forall v_h \in V_h. \tag{3.9}$$

Let $\Pi_h : V \longrightarrow V_h$ is an interpolation operator such that, $\forall K \in \Gamma_h$,

$$\begin{aligned} &\tfrac{1}{|l_i|}\int_{l_i} \Pi_h v = \tfrac{1}{|l_i|}\int_{l_i} v, \qquad i = 1,2,3,4, \\ &\tfrac{1}{|K|}\int_K \Pi_h v = \tfrac{1}{|K|}\int_K v, \end{aligned} \tag{3.10}$$

where $l_i, i = 1,2,3,4$ are sides of K which can be obtained from $\widehat{l_i}$ by the affine mapping $F_K : \widehat{K} \longrightarrow K$, i.e., $F_K(\widehat{l_i}) = l_i, i = 1,2,3,4$.

Lemma 3.2. *The interpolation operator Π_h satisfies*

$$\begin{aligned} &|v - \Pi_h v|_h \leq C|v|_{1,\Omega}, \\ &|\Pi_h v|_h \leq C|v|_{1,\Omega}, \qquad \forall v \in H_0^1(\Omega). \end{aligned} \tag{3.11}$$

Proof. Define an operator

$$T : L^2(\widehat{K}) \longrightarrow span\{1, \varphi'(\xi)\},$$
$$Tw = F_1(w) + F_2(w)\varphi'(\xi),$$

where $F_1(w), F_2(w)$ are defined in (2.3).

It is easy to see that $Tw = w, \forall w \in \widehat{\mathcal{P}}_0(\widehat{K})$. Here and later we denote by $\widehat{\mathcal{P}}_t(\widehat{K})$ the set of all polynomials in two variables of degree $\leq t$ on element $\widehat{K}$. Hence, $\|w - Tw\|_{0,\widehat{K}} \leq C \parallel w \parallel_{0,\widehat{K}}$, $\forall w \in L^2(\widehat{K})$. With the inequality above, we obtain

$$\|\frac{\partial(\widehat{v} - \widehat{\Pi}\widehat{v})}{\partial \xi}\|_{0,\widehat{K}} = \|\frac{\partial \widehat{v}}{\partial \xi} - T\frac{\partial \widehat{v}}{\partial \xi}\|_{0,\widehat{K}} \leq C\|\frac{\partial \widehat{v}}{\partial \xi}\|_{0,\widehat{K}}.$$

Similarly,

$$\|\frac{\partial(\widehat{v} - \widehat{\Pi}\widehat{v})}{\partial \eta}\|_{0,\widehat{K}} \leq C\|\frac{\partial \widehat{v}}{\partial \eta}\|_{0,\widehat{K}}.$$

Using the affine mapping (1.1) and Lemma 2.2, we have

$$
\begin{aligned}
&|v-\Pi_h v|_h^2 = \sum_K (\| \tfrac{\partial (v-\Pi_h v)}{\partial x} \|_{0,K}^2 + \| \tfrac{\partial (v-\Pi_h v)}{\partial y} \|_{0,K}^2) \\
&= \sum_K (h_x^{-2} \| \tfrac{\partial (\widehat{v}-\widehat{\Pi v})}{\partial \xi} \|_{0,\widehat{K}}^2 + h_y^{-2} \| \tfrac{\partial (\widehat{v}-\widehat{\Pi v})}{\partial \eta} \|_{0,\widehat{K}}^2) h_x h_y \\
&\le C \sum_K (h_x^{-2} \| \tfrac{\partial \widehat{v}}{\partial \xi} \|_{0,\widehat{K}}^2 + h_y^{-2} \| \tfrac{\partial \widehat{v}}{\partial \eta} \|_{0,\widehat{K}}^2) h_x h_y \\
&\le C |v|_{1,\Omega}^2.
\end{aligned}
$$

Therefore, the first inequality of Lemma 3.2 is derived.

Using the triangle inequality and the inequality above, we obtain the second inequality of Lemma 3.2.

Remark 2. For vector functions $v=(v_1,v_2)^T \in (H_0^1(\Omega))^2$, Lemma 3.2 also holds.

Lemma 3.3. *The bilinear form $b_h(\cdot,\cdot)$ satisfies BB condition on $V_h \times M_h$, i.e., there exists a constant $\beta^* > 0$ such that*

$$
\sup_{v_h \in V_h} \frac{b_h(v_h,q_h)}{|v_h|_h} \ge \beta^* |q_h|_M, \qquad \forall q_h \in M_h. \tag{3.12}
$$

Proof. The operator Π_h satisfies

$$
b_h(\Pi_h v, q_h) = b_h(v,q_h), \qquad \forall v \in V. \tag{3.13}
$$

In fact,

$$
\begin{aligned}
&b_h(v-\Pi_h v, q_h) = -\sum_{K \in \Gamma_h} \int_K q_h div(v-\Pi_h v) dxdy \\
&= -\sum_{K \in \Gamma_h} q_h \int_K div(v-\Pi_h v) dxdy \\
&= -\sum_{K \in \Gamma_h} q_h \int_{\partial K} div(v-\Pi_h v) \cdot n ds \\
&= 0.
\end{aligned}
$$

Applying (3.13), (3.11) and (3.3), we have

$$
\sup_{v_h \in V_h} \tfrac{b_h(v_h,q_h)}{|v_h|_h} \ge \sup_{v \in V} \tfrac{b_h(\Pi_h v,q_h)}{|\Pi_h v|_h} = \sup_{v \in V} \tfrac{b_h(v,q_h)}{|\Pi_h v|_h} \ge \tfrac{1}{C} \sup_{v \in V} \tfrac{b_h(v,q_h)}{|v|_h} \ge \beta^* |q_h|_M,
$$

where $\beta^* = \frac{\beta}{C} > 0$. The proof of Lemma 3.3 is completed.

Due to (3.9) and (3.12), we know from the general theory of the mixed finite element methods that the approximation problem (3.8) exists a unique solution $(u_h,p_h) \in V_h \times M_h$ and there holds the following error estimation

$$
|u-u_h|_h + |p-p_h|_M \le C(\inf_{v_h \in V_h} |u-v_h|_h + \inf_{q_h \in M_h} |p-q_h|_M + \sup_{v_h \in V_h} \frac{|a_h(u,v_h)+b_h(v_h,p)-f(v_h)|}{|v_h|_h}). \tag{3.14}
$$

If $(u,p) \in (H^2(\Omega))^2 \times H^1(\Omega)$, then

$$
\begin{aligned}
&\inf_{v_h \in V_h} |u-v_h|_h \le |u-\Pi_h u|_h, \\
&\inf_{q_h \in M_h} |p-q_h|_M \le |p-P_0 p|_M,
\end{aligned} \tag{3.15}
$$

where $P_0 p|_K = \frac{1}{|K|} \int_K p dxdy, \forall K \in \Gamma_h$.

By the affine mapping $F_K:\widehat{K}\longrightarrow K$ and lemma 2.2, we have

$$\begin{aligned}
&|u-v_h|_h\le|u-\Pi_hu|_h^2=\sum_{K\in\Gamma_h}\int_K[(\tfrac{\partial(u-\Pi_hu)}{\partial x})^2+(\tfrac{\partial(u-\Pi_hu)}{\partial y})^2]dxdy\\
&=\sum_{K\in\Gamma_h}(h_x^{-1}h_y\|\tfrac{\partial(\widehat{u}-\widehat{\Pi}_h\widehat{u})}{\partial\xi}\|_{0,\widehat{K}}^2+h_xh_y^{-1}\|\tfrac{\partial(\widehat{u}-\widehat{\Pi}_h\widehat{u})}{\partial\eta}\|_{0,\widehat{K}}^2)\\
&\le C\sum_{K\in\Gamma_h}(h_x^{-1}h_y|\tfrac{\partial\widehat{u}}{\partial\xi}|_{1,\widehat{K}}^2+h_xh_y^{-1}|\tfrac{\partial\widehat{u}}{\partial\eta}|_{1,\widehat{K}}^2)\\
&\le C\sum_{K\in\Gamma_h}[h_x^{-1}h_y(\|\tfrac{\partial^2u}{\partial x^2}\|_{0,K}^2h_x^4+\|\tfrac{\partial^2u}{\partial x\partial y}\|_{0,K}^2h_x^2h_y^2)h_x^{-1}h_y^{-1}\\
&\qquad+h_xh_y^{-1}(\|\tfrac{\partial^2u}{\partial x\partial y}\|_{0,K}^2h_x^2h_y^2+\|\tfrac{\partial^2u}{\partial y^2}\|_{0,K}^2h_y^4)h_x^{-1}h_y^{-1}]\\
&\le Ch^2\sum_{K\in\Gamma_h}|u|_{2,K}^2\\
&=Ch^2|u|_{2,\Omega}^2.
\end{aligned}\tag{3.16}$$

Therefore,

$$\inf_{v_h\in V_h}|u-v_h|_h\le|u-\Pi_hu|_h\le Ch|u|_{2,\Omega}.\tag{3.17}$$

We now turn to estimate $\inf_{q_h\in M_h}|p-q_h|_M$. Let $\widehat{P}_0\widehat{p}=\frac{1}{|\widehat{K}|}\int_{\widehat{K}}\widehat{p}d\xi d\eta$. With the local interpolation theory on the reference element $\widehat{K}$, we have

$$\begin{aligned}
&|p-P_0p|_M^2=\sum_{K\in\Gamma_h}\int_K(p-P_0p)^2dxdy=\sum_{K\in\Gamma_h}h_xh_y\int_{\widehat{K}}(\widehat{p}-\widehat{P}_0\widehat{p})^2d\xi d\eta\\
&\le C\sum_{K\in\Gamma_h}h_xh_y|\widehat{p}|_{1,\widehat{K}}^2\\
&\le C\sum_{K\in\Gamma_h}h_xh_y\int_K[h_x^2(\tfrac{\partial p}{\partial x})^2+h_y^2(\tfrac{\partial p}{\partial y})^2](h_xh_y)^{-1}dxdy\\
&\le Ch^2|p|_{1,\Omega}^2.
\end{aligned}\tag{3.18}$$

Hence,

$$\inf_{q_h\in M_h}|p-q_h|_M\le|p-P_0p|_M\le Ch|p|_{1,\Omega}.\tag{3.19}$$

In the following, we will estimate the consistency error. By the Green's formula and the equation (3.1), the consistency error can be written as

$$\begin{aligned}
&a_h(u,v_h)+b_h(v_h,p)-f(v_h)\\
&=\sum_{K\in\Gamma_h}(\gamma\int_K\nabla u\nabla v_hdxdy-\int_Kp\,div v_hdxdy-\int_Kfv_hdxdy)\\
&=\sum_{K\in\Gamma_h}(\gamma\int_{\partial K}\tfrac{\partial u}{\partial n}v_hds-\int_{\partial K}pv_h\cdot nds)\\
&=\gamma T_1(u,v_h)-T_2(p,v_h),\qquad\forall v_h\in V_h,
\end{aligned}\tag{3.20}$$

where

$$\begin{aligned}
T_1(u,v_h)&=\sum_{K\in\Gamma_h}\int_{\partial K}\tfrac{\partial u}{\partial n}v_hds,\\
T_2(p,v_h)&=\sum_{K\in\Gamma_h}\int_{\partial K}pv_h\cdot nds.
\end{aligned}$$

Now, we set to deal with the first term of the right hand side of (3.20).

$$T_1(u,v_h)=\sum_{K\in\Gamma_h}(-\int_{l_1}v_h\frac{\partial u}{\partial y}dx+\int_{l_2}v_h\frac{\partial u}{\partial x}dy+\int_{l_3}v_h\frac{\partial u}{\partial y}dx-\int_{l_4}v_h\frac{\partial u}{\partial x}dy).\tag{3.21}$$

For any element $K, \forall v \in H^1(K)$, we define

$$P_{0i}v = \frac{1}{2h_x}\int_{l_i} vdx, i = 1, 3,$$

$$P_{0i}v = \frac{1}{2h_y}\int_{l_i} vdy, i = 2, 4,$$

$$P_0 v = \frac{1}{|K|}\int_K vdxdy.$$

For vector function $v = (v_1, v_2) \in (H^1(K))^2$, we can define

$$P_{0i}v = (P_{0i}v_1, P_{0i}v_2), \qquad i = 1, 2, 3, 4,$$

$$P_0 v = (P_0 v_1, P_0 v_2).$$

It is easy to see that all these operators mentioned above are affine equivalent, the corresponding operators on the reference element $\widehat{K}$ are read respectively $\widehat{P}_{0i}, \widehat{P}_0, i = 1, 2, 3, 4$. Using the definition of V_h, we have $\forall v_h \in V_h$,

$$\begin{aligned} T_1(u, v_h) = & \sum_{K\in\Gamma_h}[-\int_{l_1}(v_h - P_{01}v_h)(\tfrac{\partial u}{\partial y} - P_0\tfrac{\partial u}{\partial y})dx + \int_{l_2}(v_h - P_{02}v_h)(\tfrac{\partial u}{\partial x} - P_0\tfrac{\partial u}{\partial x})dy \\ & + \int_{l_3}(v_h - P_{03}v_h)(\tfrac{\partial u}{\partial y} - P_0\tfrac{\partial u}{\partial y})dx - \int_{l_4}(v_h - P_{04}v_h)(\tfrac{\partial u}{\partial x} - P_0\tfrac{\partial u}{\partial x})dy]. \end{aligned} \tag{3.22}$$

Let

$$\begin{aligned} Lv_h &= \tfrac{x-(x_K-h_x)}{2h_x}P_{02}v_h - \tfrac{x-(x_K+h_x)}{2h_x}P_{04}v_h \\ &= \tfrac{1}{2}(1+\xi)\widehat{P}_{02}\widehat{v}_h - \tfrac{1}{2}(1-\xi)\widehat{P}_{04}\widehat{v}_h \\ &= \widehat{L}\widehat{v}_h, \end{aligned} \tag{3.23}$$

$$\begin{aligned} Nv_h &= \tfrac{y-(y_K-h_y)}{2h_y}P_{03}v_h - \tfrac{y-(y_K+h_x)}{2h_y}P_{01}v_h \\ &= \tfrac{1}{2}(1+\eta)\widehat{P}_{02}\widehat{v}_h - \tfrac{1}{2}(1-\eta)\widehat{P}_{04}\widehat{v}_h \\ &= \widehat{N}\widehat{v}_h. \end{aligned} \tag{3.24}$$

Obviously, L and N are the linear interpolation operators,which are affine equivalent. $\widehat{L}$ and $\widehat{N}$ are the corresponding operators on the reference element $\widehat{K}$. We rewrite (3.22) as follows.

$$\begin{aligned} T_1(u, v_h) = & \sum_{K\in\Gamma_h}\{\int_K \tfrac{\partial}{\partial y}[(v_h - Nv_h)(\tfrac{\partial u}{\partial y} - P_0\tfrac{\partial u}{\partial y})]dxdy \\ & + \int_K \tfrac{\partial}{\partial x}[(v_h - Lv_h)(\tfrac{\partial u}{\partial x} - P_0\tfrac{\partial u}{\partial x})]dxdy\} \\ & = \sum_{K\in\Gamma_h}(A_K + B_K), \end{aligned} \tag{3.25}$$

where

$$A_K = \int_K \tfrac{\partial}{\partial y}[(v_h - Nv_h)(\tfrac{\partial u}{\partial y} - P_0\tfrac{\partial u}{\partial y})]dxdy,$$

$$B_K = \int_K \tfrac{\partial}{\partial x}[(v_h - Lv_h)(\tfrac{\partial u}{\partial x} - P_0\tfrac{\partial u}{\partial x})]dxdy.$$

At first, we estimate A_K,

$$\begin{aligned} A_K &= \int_K \tfrac{\partial}{\partial y}[(v_h - Nv_h)(\tfrac{\partial u}{\partial y} - P_0\tfrac{\partial u}{\partial y})]dxdy, \\ &= \int_K (v_h - Nv_h)\tfrac{\partial^2 u}{\partial y^2}dxdy + \int_K(\tfrac{\partial u}{\partial y} - P_0\tfrac{\partial u}{\partial y})(\tfrac{\partial v_h}{\partial y} - \tfrac{\partial Nv_h}{\partial y})dxdy \\ &= A_{K_1} + A_{K_2}, \end{aligned}$$

where

$$\begin{aligned} A_{K_1} &= \int_K (v_h - Nv_h)\tfrac{\partial^2 u}{\partial y^2}dxdy, \\ A_{K_2} &= \int_K(\tfrac{\partial u}{\partial y} - P_0\tfrac{\partial u}{\partial y})(\tfrac{\partial v_h}{\partial y} - \tfrac{\partial Nv_h}{\partial y})dxdy. \end{aligned}$$

It can be seen that

$$|A_{K_1}| \leq \| v_h - Nv_h \|_{0,K} \|\frac{\partial^2 u}{\partial y^2}\|_{0,K} \leq C(h_x h_y)^{\frac{1}{2}} \|\widehat{v}_h - \widehat{N}\widehat{v}_h\|_{0,\widehat{K}} |u|_{2,K}.$$

Note that $\widehat{N}\widehat{w} = \widehat{w}, \forall \widehat{w} \in \widehat{\mathcal{P}}_0(\widehat{K})$. Using the local interpolation theory on the reference element $\widehat{K}$, we have

$$|A_{K_1}| \leq C(h_x h_y)^{\frac{1}{2}} |\widehat{v}_h|_{1,\widehat{K}} |u|_{2,K} \leq C(\| \frac{\partial v_h}{\partial x} \|^2_{0.K} h_x^2 + \| \frac{\partial v_h}{\partial y} \|^2_{0.K} h_y^2)^{\frac{1}{2}} |u|_{2,K} \leq Ch|v_h|_{1,K}|u|_{2,K}. \tag{3.26}$$

Consequently, we consider A_{K_2}. Note that

$$\frac{\partial Nv}{\partial y} = \frac{1}{2h_y}(P_{03}v_h - P_{01}v_h) = \frac{1}{|K|}\int_K \frac{\partial v_h}{\partial y} dxdy = P_0 \frac{\partial v_h}{\partial y}, \tag{3.27}$$

we have

$$\|\frac{\partial N v_h}{\partial y}\|_{0,K} = \frac{1}{|K|}\int_K |\frac{\partial v_h}{\partial y} dxdy| \cdot |K|^{\frac{1}{2}} \leq [\int_K (\frac{\partial v_h}{\partial y})^2 dxdy]^{\frac{1}{2}} = \|\frac{\partial v_h}{\partial y}\|_{0,K}. \tag{3.28}$$

Applying (3.28) to A_{K_2} and letting $\frac{\partial u}{\partial y} = w$, we obtain

$$\begin{aligned} |A_{K_2}| &\leq \|w - P_0 w\|_{0,K} \|\tfrac{\partial v_h}{\partial y} - \tfrac{\partial N v_h}{\partial y}\|_{0,K} \\ &\leq 2 \| \tfrac{\partial v_h}{\partial y} \|_{0,K} \| w - P_0 w \|_{0,K} \\ &\leq 2|v_h|_{1,K} \|\widehat{w} - \widehat{P}_0 \widehat{w}\|_{0,\widehat{K}} (h_x h_y)^{\frac{1}{2}} \\ &\leq C|v_h|_{1,K} |\widehat{w}|_{1,\widehat{K}} (h_x h_y)^{\frac{1}{2}} \\ &\leq C|v_h|_{1,K} (\|\tfrac{\partial w}{\partial x}\|^2_{0,K} h_x^2 + \|\tfrac{\partial w}{\partial y}\|^2_{0,K} h_y^2)^{\frac{1}{2}} \\ &\leq Ch|v_h|_{1,K}|w|_{1,K} \\ &\leq Ch|v_h|_{1,K}|u|_{2,K}. \end{aligned} \tag{3.29}$$

Combining (3.29),(3.26) with A_K, we get

$$|A_K| \leq Ch|u|_{2,K}|v_h|_{1,K}. \tag{3.30}$$

By analogy with the estimation of A_K,

$$|B_K| \leq Ch|u|_{2,K}|v_h|_{1,K}. \tag{3.31}$$

Subsitituting (3.30) and (3.31) into (3.25), we have

$$|T_1(u, v_h)| \leq Ch|u|_{2,\Omega}|v_h|_h. \tag{3.32}$$

In the following, we will estimate the second term on the right hand side of (3.20).

$$\begin{aligned} |T_2(p, v_h)| &= \sum_{K \in \Gamma_h} (-\textstyle\int_{l_1} p v_{h2} dx + \int_{l_3} p v_{h2} dx + \int_{l_2} p v_{h1} dy - \int_{l_4} p v_{h1} dy) \\ &= \sum_{K \in \Gamma_h} [-\textstyle\int_{l_1} (p - P_0 p)(v_{h2} - P_{01} v_{h2}) dx + \int_{l_3} (p - P_0 p)(v_{h2} - P_{03} v_{h2}) dx \\ &\quad + \textstyle\int_{l_2} (p - P_0 p)(v_{h1} - P_{02} v_{h1}) dy - \int_{l_4} (p - P_0 p)(v_{h1} - P_{04} v_{h1})] dy \\ &= \sum_{K \in \Gamma_h} \{\textstyle\int_K \frac{\partial}{\partial y}[(p - P_0 p)(v_{h2} - N v_{h2})] dxdy + \int_K \frac{\partial}{\partial x}[(p - P_0 p)(v_{h1} - L v_{h1})] dxdy\} \\ &= \sum_{K \in \Gamma_h} (S_K + T_K), \end{aligned} \tag{3.33}$$

where

$$S_K = \int_K \frac{\partial}{\partial y}[(p - P_0 p)(v_{h2} - N v_{h2})]dxdy,$$
$$T_K = \int_K \frac{\partial}{\partial x}[(p - P_0 p)(v_{h1} - L v_{h1})]dxdy.$$

By analogy with the estimation in A_K and B_K of $T_1(u, v_h)$, we conclude

$$S_K \leq Ch|p|_{1,K}|v_{h2}|_{1,K}, \tag{3.34}$$

$$T_K \leq Ch|p|_{1,K}|v_{h1}|_{1,K}. \tag{3.35}$$

Thus

$$|T_2(p, v_h)| \leq Ch|p|_{1,\Omega}|v_h|_h. \tag{3.36}$$

Combining (3.32), (3.36) and (3.20) yields

$$|a_h(u, v_h) + b_h(v_h, p) - f(v_h)| \leq Ch(|u|_{2,\Omega} + |p|_{1,\Omega})|v_h|_h, \quad \forall v_h \in V_h. \tag{3.37}$$

therefore

$$\sup_{v_h \in V_h} \frac{|a_h(u, v_h) + b_h(v_h, p) - f(v_h)|}{|v_h|_h} \leq Ch(|u|_{2,\Omega} + |p|_{1,\Omega}) \tag{3.38}$$

Furthermore, combining (3.38),(3.17),(3.19) and (3.14) the following theorem is derived.

Theorem 3.4. *Assume that (u, p) and (u_h, p_h) are the solutions of Stokes problem (3.2) and the corresponding finite element problem (3.8) respectively, if $(u, p) \in (H^2(\Omega))^2 \times H^1(\Omega)$, then*

$$|u - u_h|_h + |p - p_h|_M \leq Ch(|u|_{2,\Omega} + |p|_{1,\Omega}). \tag{3.39}$$

From now on, we will prove an interesting result on consistency error estimation in the following.

Theorem 3.5. *If $(u, p) \in (H^3(\Omega))^2 \times H^2(\Omega)$, then*

$$|a_h(u, v_h) + b_h(v_h, p) - f(v_h)| \leq Ch^2(|u|_{3,\Omega} + |p|_{2,\Omega})|v_h|_h, \quad \forall v_h \in V_h, \tag{3.40}$$

Proof. From (3.21) we may rewrite $T_1(u, v_h)$ as follows

$$T_1(u, v_h) = \sum_{K \in \Gamma_h} (I_1 + I_2 + I_3 + I_4),$$

where

$$I_1 = -\int_{l_1} (v_h - P_{01} v_h)(\frac{\partial u}{\partial y} - P_{01} \frac{\partial u}{\partial y})dx,$$
$$I_2 = \int_{l_2} (v_h - P_{02} v_h)(\frac{\partial u}{\partial x} - P_{02} \frac{\partial u}{\partial x})dy,$$
$$I_3 = \int_{l_3} (v_h - P_{03} v_h)(\frac{\partial u}{\partial y} - P_{03} \frac{\partial u}{\partial y})dx,$$
$$I_4 = -\int_{l_4} (v_h - P_{04} v_h)(\frac{\partial u}{\partial x} - P_{04} \frac{\partial u}{\partial x})dy.$$

We consider firstly $I_1 + I_3$.

$$\begin{aligned} & I_1 + I_3 \\ = & -\int_{l_1} (v_h - P_{01} v_h)(\frac{\partial u}{\partial y} - P_{01} \frac{\partial u}{\partial y})dx + \int_{l_3} (v_h - P_{03} v_h)(\frac{\partial u}{\partial y} - P_{03} \frac{\partial u}{\partial y})dx. \end{aligned}$$

Note that $\frac{\partial v_h}{\partial x} \in \{1, \varphi'(x)\}$ and $\frac{\partial v_h}{\partial y} \in \{1, \varphi'(y)\}$, we have

$$\begin{aligned}
&(v_h - P_{01}v_h)|_{l_1} \\
&= v_h(x, y_K - h_y) - \tfrac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x} v_h(x, y_K - h_y)dx \\
&= \tfrac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}[v(x, y_K - h_y) - v_h(t, y_K - h_y)]dt \\
&= \tfrac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\int_t^x \tfrac{\partial v_h}{\partial z}(z, y_K - h_y)dzdt \\
&= \tfrac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\int_t^x \tfrac{\partial v_h}{\partial z}(z, y_K + h_y)dzdt \\
&= v_h(x, y_K + h_y) - \tfrac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x} v_h(x, y_K + h_y)dx \\
&= (v_h - P_{03}v_h)|_{l_3}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&(\tfrac{\partial u}{\partial y} - P_{01}\tfrac{\partial u}{\partial y})|_{l_1} \\
&= \tfrac{\partial u}{\partial y}(x, y_K - h_y) - \tfrac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\tfrac{\partial u}{\partial y}(t, y_K - h_y)dt \\
&= \tfrac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\int_t^x \tfrac{\partial^2 u}{\partial z\partial y}(z, y_K - h_y)dzdt.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&(\tfrac{\partial u}{\partial y} - P_{03}\tfrac{\partial u}{\partial y})|_{l_3} \\
&= \tfrac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\int_t^x \tfrac{\partial^2 u}{\partial z\partial y}(z, y_K + h_y)dzdt.
\end{aligned}$$

So

$$\begin{aligned}
&I_1 + I_3 \\
&= \int_{x_K-h_x}^{x_K+h_x}(v_h - P_{03}v_h)|_{l_3}\cdot[(\tfrac{\partial u}{\partial y} - P_{03}\tfrac{\partial u}{\partial y})|_{l_3} - (\tfrac{\partial u}{\partial y} - P_{01}\tfrac{\partial u}{\partial y})|_{l_1}]dx \\
&= \tfrac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}[\int_{x_K-h_x}^{x_K+h_x}\int_t^x\int_{y_K-h_y}^{y_K+h_y}\tfrac{\partial^3 u}{\partial z\partial y^2}(z, y)dydzdt] \\
&\quad\cdot[v_h(x, y_K + h_y) - \tfrac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x} v_h(x, y_K + h_y)dx]dx.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&|v_h(x, y_K + h_y) - \tfrac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x} v_h(x, y_K + h_y)dx|^2 \\
&= \tfrac{1}{4h_x^2}|\int_{x_K-h_x}^{x_K+h_x}\int_t^x \tfrac{\partial v_h}{\partial z}(z, y_K - h_y)dzdt|^2 \\
&\le \tfrac{1}{4h_x^2}\int_{x_K-h_x}^{x_K+h_x}\int_t^x |\tfrac{\partial v_h}{\partial z}(z, y_K - h_y)|^2dzdt\int_{x_K-h_x}^{x_K+h_x}|x - t|dt,
\end{aligned}$$

hence,

$$\begin{aligned}
&\int_{x_K-h_x}^{x_K+h_x}|v_h(x, y_K + h_y) - \tfrac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x} v_h(x, y_K + h_y)dx|^2dx \\
&\le \tfrac{1}{4h_x^2}\int_{x_K-h_x}^{x_K+h_x}(\int_{x_K-h_x}^{x_K+h_x}\int_t^x|\tfrac{\partial v_h}{\partial z}(z, y_K - h_y)|^2dzdt\int_{x_K-h_x}^{x_K+h_x}|x - t|dt)dx \\
&\le \tfrac{4h_x^2}{3}\int_{x_K-h_x}^{x_K+h_x}|\tfrac{\partial v_h}{\partial x}|^2dx \\
&= \tfrac{2h_x^2}{3h_y}\int_{x_K-h_x}^{x_K+h_x}\int_{y_K-h_y}^{y_K+h_y}|\tfrac{\partial v_h}{\partial x}|^2dxdy.
\end{aligned} \tag{3.41}$$

By (3.14) and Cauchy-Schwartz inequality we have

$$|I_1 + I_3| \le Ch_x^2\|\frac{\partial^3 u}{\partial x\partial y^2}\|_{0,K}|\frac{\partial v_h}{\partial x}|_{0,K}.$$

Similarly

$$|I_2+I_4|\leq Ch_y^2\|\frac{\partial^3 u}{\partial x^2\partial y}\|_{0,K}|\frac{\partial v_h}{\partial y}|_{0,K}.$$

Combining the two inequalities above, we have

$$T_1(u,v_h)=\sum_K\int_{\partial K}\frac{\partial u}{\partial n}v_h ds\leq Ch^2|u|_{3,\Omega}|v_h|_h.$$

By analogy with the estimate of $T_1(u,v_h)$, we have

$$T_2(p,v_h)=\sum_K\int_{\partial K}pv_h\cdot nds\leq Ch^2|p|_{2,\Omega}|v_h|_h.$$

Therefore,

$$|a_h(u,v_h)+b_h(v_h,p)-f(v_h)|\leq Ch^2(|u|_{3,\Omega}+|p|_{2,\Omega})|v_h|_h,\qquad \forall v_h\in V_h.$$

4. The Nonconforming Five-parameter Anisotropic Finite Element with Moving Grids of the Nonstationary Stokes Equations

We consider the following nonstationary Stokes equations

$$\begin{cases}\frac{\partial u}{\partial t}-\gamma\triangle u+\nabla p=f,\qquad \forall(x,t)\in\Omega\times(0,T],\\ divu=0,\qquad \forall(x,t)\in\Omega\times(0,T],\\ u=0,\qquad \forall(x,t)\in\partial\Omega\times(0,T],\\ u|_{t=0}=u_0,\qquad \forall x\in\Omega.\end{cases}\tag{4.1}$$

Then (4.1) is equivalent to the following variational problem. Find $(u,p)\in V\times M,\forall t\in(0,T)$ such that

$$\begin{cases}(\frac{\partial u}{\partial t},v)+a(u,v)-(p,divv)=(f,v),\qquad \forall v\in V,\\ (q,divu)=0,\qquad \forall q\in M,\\ u|_{t=0}=u_0,\qquad \forall x\in\Omega,\end{cases}\tag{4.2}$$

where $(\cdot,\cdot)$ is an inner-product in $L^2(\Omega)$ or $(L^2(\Omega))^2$.

Let $\{\Gamma_h\}_h$ be a family of rectangular subdivisions without regular assumption or quasiuniform assumption [10],V_h and M_h be the approximation spaces of V and M, respectively. The approximation problem corresponding to (4.2) reads as: Find $(u_h,p_h)\in V_h\times M_h$ such that

$$\begin{cases}(\frac{\partial u_h}{\partial t},v)+a_h(u_h,v)-(p_h,div_hv)=(f,v),\qquad \forall v\in V_h,\\ (q,div_hu_h)=0,\qquad \forall q\in M_h,\\ u_h|_{t=0}=u_0^h,\qquad \forall x\in\Omega.\end{cases}\tag{4.3}$$

Where u_0^h is an approximation problem of u_0 in V_h, $div_h|_K=divv,\quad \forall v\in V_h,\forall K\in\Gamma_h$. Now we apply the idea of moving grids to (4.3) and develop the scheme of the nonconforming five-parameter anisotropic finite element with moving grids. Let $0=t_0<t_1\cdots<t_N=T$ be a partition of the time interval [0,T]. For any time level t_n, let$\{\Gamma_h\}_h$ be a family of rectangular subdivisions for spatial domain at this time. We denote by V_n^h and M_n^h the finite element spaces on the spatial domain at the time level t_n, definitions of V_n^h and M_n^h are the same as V_h and

M_h. Due to Lemma 3.3, the spaces V_n^h and M_n^h satisfy the BB condition, i.e., there exists a constant $\beta^* > 0$ such that

$$\sup_{v \in V_n^h} \frac{|(q, div_h v)|}{|v|_h} \geq \beta^* \|q\|_{0,\Omega}, \qquad \forall q \in M_n^h. \tag{4.4}$$

At the time point $t_n (n = 0, 1, 2, \cdots, N.)$,we let

$$V_n = \{v(x, t_n); v \in V\}, \qquad M_n = \{q(x, t_n); q \in M\}. \tag{4.5}$$

We choose the approximations spaces $S \times Z$ of (u, p) in the following way: The approximation solution $u^h(x, t) \in S$ is the piecewise linear function with respect to the time subdivisions $0 = t_0 < t_1 \cdots < t_N = T$, its function value $u^h(x, t_n)$ at the time level t_n is the interpolation function of the solution u on the finite element space $V_n^h (n = 0, 1, 2, \cdots, N)$. The approximation function $p^h(x, t) \in Z$ can be obtained in the same way.

We introduce the five-parameter rectangular anisotropic finite element with moving grids (see[2,3]) and determine the function values $u_n^h = u^h(x, t_n)$ and $p_n^h = p^h(x, t_n)$ of the approximation solutions $u^h(x, t)$ and $p^h(x, t) (n = 1, 2, \cdots, N)$ as follows:

$$(\widehat{u}_n^h - u_n^h, v) = 0, \qquad \forall v \in V_{n+1}^h, \text{for } n > 0, \tag{4.6}$$

$$(\widehat{u}_0^h - u_0, v) = 0, \qquad \forall v \in V_1^h, \text{for } n = 0, \tag{4.7}$$

$$(u_{n+1}^h - \widehat{u}_n^h, v) + a_h(u_{n+1}^h, v)\triangle t_n - (p_{n+1}^h, div_h v)\triangle t_n = (f_{n+1}, v)\triangle t_n, \qquad \forall v \in V_{n+1}^h, \tag{4.8}$$

$$(q, div_h u_{n+1}^h) = 0, \qquad \forall q \in M_{n+1}^h, \tag{4.9}$$

where $f_n = f(x, t_n)$.

(4.6) implies that $\widehat{u}_n^h = u_n^h$ when $V_n^h = V_{n+1}^h$. Furthermore, (4.6) is a L^2-projection modification scheme for the former space when the two spaces $V_n^h$and V_{n+1}^h have different meshes or different interpolation functions. (4.8) is the general trapezoid difference scheme. We get $\widehat{u}_n^h$ from u_n^h in (4.6) and get u_{n+1}^h from $\widehat{u}_n^h$ in (4.8). u_n^h, p_n^h are uniquely from (4.6)–(4.9) by partial differential equation theory.

5. Error Estimate of the Nonconforming Five-parameter Anisotropic Approximation Scheme with Moving Grids

The main error between the solution $u(x, t)$ and the approximation solution $u^h(x, t)$ consists of three parts: the interpolation error with the finite element method, the difference error with respect to the time, and the error of moving grids.

Firstly, we will prove the following Lemma 5.1 which plays an important role in our error analysis.

Lemma 5.1. $\|v_h\|_0 \leq C|v_h|_h, \quad \forall v_h \in V_h$.

Proof. By using duality argument, we consider the following elliptic problem,

$$\begin{cases} -\triangle w = g, & in \Omega \\ w = 0, & on \partial\Omega. \end{cases} \tag{5.1}$$

Then by the regularity of the solution of the elliptic problem,(5.1) has a unique solution $w \in H^2(\Omega) \cap H_0^1(\Omega)$ satifying $\|w\|_2 \leq C\|g\|_0$. Let

$$E(w, v_h) = \sum_K \int_{\partial K} \frac{\partial w}{\partial n} v_h ds, \qquad \forall v_h \in V_h,$$

due to the estimate of $T_1(u, v_h)$ in (3.21) – (3.32), we know

$$|E(w, v_h)| \leq Ch|w|_{2,\Omega}|v_h|_h. \tag{5.2}$$

Furthermore, by Green's formula,

$$\begin{aligned}|\int_\Omega gv_h dxdy| &= |-\int_\Omega \triangle w v_h dxdy| = |\sum_K \int_K \nabla w \nabla v_h dxdy - E(w,v_h)| \\ &\le |w|_{1,\Omega}|v_h|_h + Ch|w|_{2,\Omega}|v_h|_h \le C\|w\|_{2,\Omega}|v_h|_h, \quad \forall v_h \in V_h.\end{aligned}$$

Therefore,

$$|\int_\Omega gv_h dxdy| \le C\|g\|_0 |v_h|_h.$$

Let $g = v_h$, we obtain $\|v_h\|_0 \le C|v_h|_h, \quad \forall v_h \in V_h$.

Remark 3. The method used in this lemma is different from the conventional ones. Because the meshes are anisotropic, the estimate of (5.2) (see section 3 of this paper) becomes more difficult and complicated than that of the isotropic case . This is typical character of the anisotropic elements.

Then we have the following lemma about the difference error.

Lemma 5.2. *Let $u_n = u(x,t_n), p_n = p(x,t_n)$, the following equation holds*

$$(u_{n+1} - u_n, v) + a_h(u_{n+1}, v)\triangle t_n - (p_{n+1}, div_h v)\triangle t_n = (f_{n+1}, v)\triangle t_n + E_n(v), \quad \forall v \in V_{n+1}^h, \tag{5.3}$$

where

$$|E_n(v)| \le C[(\int_{t_n}^{t_{n+1}} \|\frac{\partial f}{\partial t}\|_0^2 dt)^{\frac{1}{2}} + (\int_{t_n}^{t_{n+1}} \|\frac{\partial p}{\partial t}\|_0^2 dt)^{\frac{1}{2}} + (\int_{t_n}^{t_{n+1}} \|\frac{\partial u}{\partial t}\|_1^2 dt)^{\frac{1}{2}}](\triangle t_n)^{\frac{3}{2}}|v|_h + Ch\triangle t_n |v|_h. \tag{5.4}$$

Proof. We get from (4.1),

$$(\frac{\partial u}{\partial t}, v) + a_h(u,v) - (p, div_h v) = (f,v) + T_h(u,p,v), \quad \forall v \in V_{n+1}^h, \tag{5.5}$$

where

$$T_h(u,p,v) = \sum_K \int_{\partial K} (\gamma \frac{\partial u}{\partial n} - pn) v ds.$$

By use of (3.32) and (3.36), we obtain

$$T_h(u,p,v) \le Ch(|u|_{2,\Omega} + |p|_{1,\Omega})|v|_h, \quad \forall v \in V_{n+1}^h.$$

Using the regularity of solutions of the Stokes problem, i.e., $|u|_{2,\Omega} + |p|_{1,\Omega} \le C\|f\|_0, \quad \forall v \in V_{n+1}^h$, we get

$$|T_h(u,p,v)| \le Ch\|f\|_0 |v|_h. \tag{5.6}$$

Intergrating with $t_n \le t \le t_{n+1}$ in (5.5) we know

$$(u_{n+1} - u_n, v) + a_h(\int_{t_n}^{t_{n+1}} u dt, v) - (\int_{t_n}^{t_{n+1}} p dt, div_h v) = (\int_{t_n}^{t_{n+1}} f dt, v) + \int_{t_n}^{t_{n+1}} T_h(u,p,v) dt, \tag{5.7}$$

thus, combining (5.3) and (5.7) we have

$$\begin{aligned} E_n(v) &= (\int_{t_n}^{t_{n+1}} (f - f_{n+1}) dt, v) + (\int_{t_n}^{t_{n+1}} (p - p_{n+1}) dt, div_h v) \\ &\quad - a_h(\int_{t_n}^{t_{n+1}} (u - u_{n+1}) dt, v) + \int_{t_n}^{t_{n+1}} T_h(u,p,v) dt \end{aligned} \tag{5.8}$$

Using Cauchy-Schwartz inequality and Lemma (5.1) yields

$$|(\int_{t_n}^{t_{n+1}}(f-f_{n+1})dt,v)| \le C(\int_{t_n}^{t_{n+1}}\|\tfrac{\partial f}{\partial t}\|_0^2dt)^{\frac{1}{2}}(\triangle t_n)^{\frac{3}{2}}|v|_h,$$

$$|(\int_{t_n}^{t_{n+1}}(p-p_{n+1})dt,div_hv)| \le C(\int_{t_n}^{t_{n+1}}\|\tfrac{\partial p}{\partial t}\|_0^2dt)^{\frac{1}{2}}(\triangle t_n)^{\frac{3}{2}}|v|_h,$$

$$|a_h(\int_{t_n}^{t_{n+1}}(u-u_{n+1})dt,v)| \le C(\int_{t_n}^{t_{n+1}}\|\tfrac{\partial u}{\partial t}\|_1^2dt)^{\frac{1}{2}}(\triangle t_n)^{\frac{3}{2}}|v|_h,$$

$$|\int_{t_n}^{t_{n+1}}T_h(u,p,v)dt| \le Ch\triangle t_n|v|_h,$$

Lemma 5.2 follows from the combination of above inequalities.

Introduce the Stokes projection $(R_nu,R_np)\in V_n^h\times M_n^h(n=0,1,2,\cdots,N.)$ which is determined by

$$a_h(u-R_nu,v)-(p-R_np,div_hv)=T_h(u,p,v),\qquad \forall v\in V_n^h, \tag{5.9}$$

$$(q,div_h(u-R_nu))=0,\qquad \forall q\in M_n^h. \tag{5.10}$$

And let $f^*=-\gamma\triangle u+\nabla p$,then $(R_nu,R_np)\in V_n^h\times M_n^h$ satisfies

$$\begin{cases} a_h(R_nu,v)-(R_np,div_hv)=(f^*,v), & \forall v\in V_n^h,\\ (q,div_hR_nu)=0, & \forall q\in M_n^h. \end{cases}$$

Obviously $(R_nu,R_np)\in V_n^h\times M_n^h$ is the finite element approximation of $(u,p)\in V\times M$. Using the duality argument and the techniques used in (3.20)– (3.39), there holds the following interpolation estimates on anisotropic meshes

$$h^{-1}\|u-R_nu\|_0+|u-R_nu|_h\le Ch^{r-1}\|u\|_{r,\Omega},\qquad (r=1 \text{ or } 2), \tag{5.11}$$

$$\|p-R_np\|_0\le Ch\|p\|_{1,\Omega}. \tag{5.12}$$

Let

$$\begin{aligned} &e_0=0, && \rho_0=0, &&\\ &e_n=u_n^h-R_nu_n, && \rho_n=u_n-R_nu_n, && n=1,2,\cdots,N,\\ &\widehat{e}_n=\widehat{u}_n^h-R_{n+1}u_n, && \widehat{\rho}_n=u_n-R_{n+1}u_n, && n=0,1,2,\cdots,N,\\ &\varepsilon_n=p_n^h-R_np_n, && \widehat{\varepsilon}_n=p_n-R_np_n, && n=1,2,\cdots,N. \end{aligned}$$

We have

Lemma 5.3. $\forall 0<\alpha<1$, *there holds*

$$(1-\alpha)\|e_{n+1}\|_0^2-\|e_n\|_0^2+(1-\alpha)a_h(e_{n+1},e_{n+1})\triangle t_n\le\frac{1}{\alpha}\|\widehat{\rho}_n-\rho_n\|_0^2+\theta_n, \tag{5.13}$$

where

$$|\theta_n|\le C[\int_{t_n}^{t_{n+1}}(\|\frac{\partial f}{\partial t}\|_0^2+\|\frac{\partial p}{\partial t}\|_0^2+\|\frac{\partial u}{\partial t}\|_1^2)dt](\triangle t_n)^2+Ch^2\int_{t_n}^{t_{n+1}}\|\frac{\partial u}{\partial t}\|_1^2\,dt+Ch^2\triangle t_n. \tag{5.14}$$

Proof. In full analogy of Lemma 3 in the literature [2], Lemma 5.3 is completed.

Lemma 5.4. $\forall L,1\le L\le N$,*the following estimation holds*

$$\|e_L\|_0^2+\sum_{n=0}^{L-1}a_h(e_{n+1},e_{n+1})\triangle t_n\le C\{m(m+1)\max_n\|\widehat{\rho}_n-\rho_n\|_0^2+\sum_{n=0}^{L-1}\theta_n\}, \tag{5.15}$$

where m is the varying times of meshes.

Proof. Using Lemma 5.3 and by analogy with the proof of Lemma 3 in the literture [3], Lemma 5.4 holds.

Theorem 5.5. *If the solution (u,p) of (4.1) and the right hand term f satisfy that*

$$\frac{\partial u}{\partial t}\in L^2(0,T;(H^2(\Omega))^2),\ \frac{\partial p}{\partial t}\in L^2(0,T;H^1(\Omega))\ \text{and}\ \frac{\partial f}{\partial t}\in L^2(0,T;\Omega),$$

then there holds the following error estimation

$$\max_n\|u_n^h-u_n\|_0^2+\sum_{n=0}^{N-1}a_h(u_{n+1}^h-u_{n+1},u_{n+1}^h-u_{n+1})\triangle t_n\le C[(m^2h^2+1)h^2+\triangle t^2],\qquad(5.16)$$

where $\triangle t=\max\limits_n\triangle t_n$.

Proof. Since

$$\|u_n^h-u_n\|_0^2\le C(\|\ e_n\ \|_0^2+\|\ \rho_n\ \|_0^2)$$

and

$$a_h(u_{n+1}^h-u_{n+1},u_{n+1}^h-u_{n+1})\le C[a_h(e_{n+1},e_{n+1})+a_h(\rho_{n+1},\rho_{n+1})],$$

using (5.11),(5.12) and Lemma (5.4), we may deduce the estimation (5.16).

Theorem 5.6. *If the solution (u,p) of(4.1) and the right-hand term f satisfy that*

$$\frac{\partial u}{\partial t}\in L^2(0,T;(H^2(\Omega))^2),\ \frac{\partial p}{\partial t}\in L^2(0,T;H^1(\Omega))\ \text{and}\ \frac{\partial f}{\partial t}\in L^2(0,T;\Omega),$$

then the error estimation of the pressure reads as

$$\sum_{n=0}^{N-1}\|p_{n+1}^h-p_{n+1}\|_0^2\triangle t_n\le C(h^2+\triangle t^2),$$

where $\triangle t=\max\limits_n\triangle t_n, mh$ *and* $\max\limits_n\triangle t_n/\min\limits_n\triangle t_n$ *are given bounded and* m *is the varying times of meshes.*

Proof. Since $\frac{\partial u}{\partial t}\in L^2(0,T;(H^2(\Omega))^2)$, $\frac{\partial p}{\partial t}\in L^2(0,T;H^1(\Omega))$, $\frac{\partial f}{\partial t}\in L^2(0,T;\Omega)$, from the first equation of (4.1) we know that

$$u_{tt}=\gamma\triangle u_t-\nabla p_t+f_t\in L^2(0,T;\Omega),$$

that is

$$\int_0^T\|\frac{\partial^2u}{\partial t^2}\|_0^2dt\le C.$$

Now, we prove that

$$\int_{t_n}^{t_{n+1}}\|\frac{\partial(u-u^h)}{\partial t}\|_0^2dt\le C(\triangle t)^2.\qquad(5.17)$$

In fact, let $\varphi=\frac{\partial u}{\partial t}$, $t=(1-s)t_n+st_{n+1}$, $\varphi(x,t)=\widehat{\varphi}(x,s)$, then

$$\begin{aligned}&\textstyle\int_{t_n}^{t_{n+1}}\|\frac{\partial(u-u^h)}{\partial t}\|_0^2dt=\int_{t_n}^{t_{n+1}}\int_\Omega(\frac{\partial u}{\partial t}-\frac{u(x,t_{n+1})-u(x,t_n)}{\triangle t_n})^2dxdydt\\&\textstyle=\int_\Omega\int_{t_n}^{t_{n+1}}(\frac{\partial u}{\partial t}-\frac{1}{\triangle t_n}\int_{t_n}^{t_{n+1}}\frac{\partial u}{\partial t}dt)^2dtdxdy=\int_\Omega dxdy\int_0^1(\widehat{\varphi}-\int_0^1\widehat{\varphi}ds)^2ds(\Delta t_n)\\&\textstyle\le C\int_\Omega dxdy\int_0^1|\frac{\partial\widehat{\varphi}}{\partial s}|^2ds(\triangle t_n)\le C\int_\Omega dxdy\int_{t_n}^{t_{n+1}}(\frac{\partial\varphi}{\partial t})^2dt(\triangle t_n)^2\\&\textstyle\le C(\triangle t)^2\int_{t_n}^{t_{n+1}}\int_\Omega(\frac{\partial\varphi}{\partial t})^2dxdydt\\&\textstyle\le C(\triangle t)^2\int_{t_n}^{t_{n+1}}\|\frac{\partial^2u}{\partial t^2}\|_0^2dt\le C(\triangle t)^2\int_0^T\|\frac{\partial^2u}{\partial t^2}\|_0^2dt\le C(\triangle t)^2.\end{aligned}$$

Due to the BB condition of the discretization schemes, we deduce

$$\begin{aligned}&\|p_{n+1}^h-R_{n+1}p_{n+1}\|_0\triangle t_n\le C\sup_{v\in V_n^h}\tfrac{|(p_{n+1}^h-R_{n+1}p_{n+1},div_hv)|}{|v|_h}\triangle t_n\\&\le C\sup_{v\in V_n^h}\tfrac{|(p_{n+1}^h-p_{n+1},div_hv)|}{|v|_h}\triangle t_n+C\sup_{v\in V_n^h}\tfrac{|(p_{n+1}-R_{n+1}p_{n+1},div_hv)|}{|v|_h}\triangle t_n.\end{aligned}$$

By (5.11),(5.12) and (5.6), it follows

$$\begin{aligned}&\|p_{n+1}^h - R_{n+1}p_{n+1}\|_0\triangle t_n\\ &\leq C\sup_{v\in V_n^h}\frac{|(p_{n+1}^h - p_{n+1}, div_h v)|}{|v|_h}\triangle t_n + C\sup_{v\in V_n^h}\frac{|a_h(u_{n+1}-R_{n+1}u_{n+1},v)|}{|v|_h}\triangle t_n + Ch\triangle t_n.\end{aligned} \tag{5.18}$$

Consequently, by using (4.7) and (5.3) we have

$$\begin{aligned}&(p_{n+1}^h - p_{n+1}, div_h v)\triangle t_n\\ &= (u_{n+1}^h - \widehat{u}_n^h, v) - (u_{n+1}-u_n, v) + a_h(u_{n+1}^h - u_{n+1}, v)\triangle t_n + E_n(v), \qquad \forall v\in V_{n+1}^h.\end{aligned} \tag{5.19}$$

Furthermore,

$$|(u_{n+1}^h - \widehat{u}_n^h, v) - (u_{n+1}-u_n, v)| = |(\int_{t_n}^{t_{n+1}}\frac{\partial(u-u^h)}{\partial t}dt, v)| \leq C(\int_{t_n}^{t_{n+1}}\|\frac{\partial(u-u^h)}{\partial t}\|_0^2 dt)^{\frac12}(\triangle t_n)^{\frac12}|v|_h,$$

$$|a_h(u_{n+1}^h - u_{n+1}, v)\triangle t_n| \leq C[a_h(u_{n+1}^h - u_{n+1}, u_{n+1}^h - u_{n+1})]^{\frac12}\triangle t_n|v|_h,$$

$$|a_h(u_{n+1} - R_{n+1}u_{n+1}, v)|\triangle t_n \leq Ch\triangle t_n|v|_h.$$

Combining the inequalities above with (5.18), we obtain

$$\begin{aligned}&\|p_{n+1}^h - R_{n+1}p_{n+1}\|_0\triangle t_n\\ &\leq C\{(\int_{t_n}^{t_{n+1}}\|\frac{\partial(u-u^h)}{\partial t}\|_0^2dt)^{\frac12}(\triangle t_n)^{\frac12} + [a_h(u_{n+1}^h - u_{n+1}, u_{n+1}^h - u_{n+1})]^{\frac12}\triangle t_n + h\triangle t_n\}\\ &+C[(\int_{t_n}^{t_{n+1}}\|\frac{\partial f}{\partial t}\|_0^2dt)^{\frac12} + (\int_{t_n}^{t_{n+1}}\|\frac{\partial p}{\partial t}\|_0^2dt)^{\frac12} + (\int_{t_n}^{t_{n+1}}\|\frac{\partial u}{\partial t}\|_1^2dt)^{\frac12}](\triangle t_n)^{\frac32}.\end{aligned}$$

Hence,

$$\begin{aligned}&\|p_{n+1}^h - R_{n+1}p_{n+1}\|_0^2\triangle t_n\\ &\leq C\{\int_{t_n}^{t_{n+1}}\|\frac{\partial(u-u^h)}{\partial t}\|_0^2dt + a_h(u_{n+1}^h - u_{n+1}, u_{n+1}^h - u_{n+1})\triangle t_n + h^2\triangle t_n\}\\ &+C[\int_{t_n}^{t_{n+1}}(\|\frac{\partial f}{\partial t}\|_0^2 + \|\frac{\partial p}{\partial t}\|_0^2 + \|\frac{\partial u}{\partial t}\|_1^2)dt](\triangle t_n)^2.\end{aligned}$$

By using (5.16) and (5.17) and when mh is bounded, the estimation

$$\sum_{n=0}^{N-1}\|p_{n+1}^h - R_{n+1}p_{n+1}\|_0^2\triangle t_n \leq C(h^2 + \triangle t^2) \tag{5.20}$$

holds.

Since $\|p_{n+1}^h - p_{n+1}\|_0^2 \leq C(\|p_{n+1}^h - R_{n+1}p_{n+1}\|_0^2 + \|R_{n+1}p_{n+1} - p_{n+1}\|_0^2)$ and $\|R_{n+1}p_{n+1} - p_{n+1}\|_0^2 \leq Ch^2$, by use of (5.20), it follows

$$\sum_{n=0}^{N-1}\|p_{n+1}^h - p_{n+1}\|_0^2\triangle t_n \leq C(h^2 + \triangle t^2).$$

Which completes the proof.

Note that the energy norm of the velocity and error order of the pressure are optimal when mh is bounded.

Acknowledgement. The authors would like to thank the anonymous referees for their helpful suggestions.

References

[1] T. Apel, Anisotropic Finite Elements: Local Estimates and Applications, B.G. Teubner Stuttgart, Leipzig, 1999.

[2] Pei-liang Dai and Xue-jun Xu, A class of nonconforming finite element with moving grids of Stokes problem (in Chinese), *Numer. Math. J. Chinese Univ.,* **3**:1 (1995), 74-81.

[3] Guo-ping Liang, The finite element method with moving grids (in Chinese), *Mathematica Numerica Sinica,* **4** (1985), 377-384.

[4] R. Rannacher and S. Turek, A simple nonconforming quarilateral Stokes element, *Numer. Meth. Partial Differ. Equ.*, **8** (1992), 97-111.

[5] Hou-de Han, Nonconforming elements in the mixed finite element method, *J. Comput. Math.*, **2** (1984), 223-233.

[6] Shao-chun Chen, Dong-yang Shi and Yong-cheng Zhao, Anisotropic interpolation and quasi -Wilson element for narrow quadrilateral meshes, *IMA J. Numer. Anal.*, **24** (2004), 77-95.

[7] R. Teman, Navier-Stokes equations, North-Holland: Amsterdam, 1984.

[8] A. Zenisek and M. Vanmaele, The interpolation theorem for narrow quadrilateral isoparametric finite elements, *Numer. Math.,* **72** (1995), 123-141.

[9] T. Apel, Serge N., and Joachim S., Crouzeix-Raviart type finite elements on anisotropic meshes, *Numer. Math.,* **89** (2001), 193-223.

[10] P.G. Ciarlet, The Finite Element Method for Elliptic Problem, North-Holland: Amsterdam, 1978.

[11] Zhong-ci Shi, On the accuracy of the quasi-conforming and generalized conforming finite elements, *Chin. Ann. of Math.*, **11**:2 (1990), 148-155.

[12] Dong-yang Shi and Shao-chun Chen, Accuracy analysis of 12-parameter rectangular plate elements with geometric symmetry, *System Science and Mathematical Sciences,* **13**:2 (2000), 146-151.

[13] M. Crouzeix and P.A. Raviart, Conforming and non-conforming finite elements for solving the stationary Stokes equations, *R. A. I. R. O. Anal. Numer.*, **7** (1973), 33-76.

[14] T. Apel and M. Dobrowolski, Anisotropric interpolation with application to the finite element method, *Computing,* **42** (1992), 277-293.

[15] W. Rachowicz, An anisotropic h-type mesh refinement strategy, *Comput. Methods Appl. Mech. Engrg,* **109** (1993), 169-181.

[16] Shao-chun Chen and Dong-yang Shi, Accuracy analysis for quasi-wilson element, *Acta. Math. Scientia.*, **1** (2000), 44-48.

[17] R. Rannacher, On the finite element approximation of the nonstationary Navier-Stokes problems.

[18] Shao-chun Chen, Yong-cheng Zhao and Dong-yang Shi, Anisotropic interpolations with application to nonconforming elements, *Appl. Numer. Math.,* **49** (2004), 135-152.

[19] Shao-chun Chen, Dong-yang Shi and Guo-biao Ren, Three dimension quasi-wilsion element for flat hexahedron meshes, *J. Comput. Math.*, **2** (2004), 178-187.