Journal of Computational Mathematics, Vol.23, No.1, 2005, 49–66.

WAVEFORM RELAXATION METHODS OF NONLINEAR INTEGRAL-DIFFERENTIAL-ALGEBRAIC EQUATIONS [*1)]

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Abstract

In this paper we consider continuous-time and discrete-time waveform relaxation methods for general nonlinear integral-differential-algebraic equations. For the continuous-time case we derive the convergence condition of the iterative methods by invoking the spectral theory on the resulting iterative operators. By use of the implicit difference forms, namely the backward-differentiation formulae, we also yield the convergence condition of the discrete waveforms. Numerical experiments are provided to illustrate the theoretical work reported here.

Mathematics subject classification: 37M05, 45J05, 65L80, 65Y05.
Key words: Nonlinear integral-differential-algebraic equations, Waveform relaxation, Parallel solutions, Convergence of iterative methods, Engineering applications.

1. Introduction

We consider a system which is described by nonlinear integral-differential-algebraic equations (IDAEs) as follows

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & \tilde{f}_1(\dot{x}(t), x(t), y(t), \int_0^t \tilde{h}_1(x(s), y(s), s, t)ds, t), \quad x(0) = x_0, \\ y(t) & = & \tilde{f}_2(\dot{x}(t), x(t), y(t), \int_0^t \tilde{h}_2(x(s), y(s), s, t)ds, t), \quad t \in [0, T], \end{array}\right. \tag{1}$$

where t is the time variable, $x_0 \in \mathbf{R}^n$ is an initial value, $[0, T]$ is a given finite time interval, $x(t) \in \mathbf{R}^n$ and $y(t) \in \mathbf{R}^m$ are to be computed. We assume that the initial condition of (1) is consistent, that is, for a given $x_0 (= x(0))$ we can solve out $\dot{x}(0)$ and $y(0)$ from the following initial condition system:

$$\left\{\begin{array}{rcl} \dot{x}(0) & = & \tilde{f}_1(\dot{x}(0), x_0, y(0), 0, 0), \\ y(0) & = & \tilde{f}_2(\dot{x}(0), x_0, y(0), 0, 0). \end{array}\right. \tag{2}$$

We will also denote $y(0)$ as y_0 in this paper.

For a large and complex system like (1) waveform relaxation (WR) or dynamic iteration is a novel parallel algorithm of treating its numerical solutions [1 - 6]. Numerical algorithms with WR are suitable to be processed in parallel [7]. The general form of the WR algorithm for (1) is

$$\left\{\begin{array}{rcl} \dot{x}^{(k+1)}(t) & = & f_1(\dot{x}^{(k+1)}(t), \dot{x}^{(k)}(t), x^{(k+1)}(t), x^{(k)}(t), y^{(k+1)}(t), y^{(k)}(t), \\ & & \quad \int_0^t h_1(x^{(k+1)}(s), x^{(k)}(s), y^{(k+1)}(s), y^{(k)}(s), s, t)ds, t), \\ y^{(k+1)}(t) & = & f_2(\dot{x}^{(k+1)}(t), \dot{x}^{(k)}(t), x^{(k+1)}(t), x^{(k)}(t), y^{(k+1)}(t), y^{(k)}(t), \\ & & \quad \int_0^t h_2(x^{(k+1)}(s), x^{(k)}(s), y^{(k+1)}(s), y^{(k)}(s), s, t)ds, t), \\ & & x^{(k+1)}(0) = x_0, \quad t \in [0, T], \quad k = 0, 1, \cdots, \end{array}\right. \tag{3}$$

* Received December 30, 2002; final revised August 23, 2004.
1) This research work was supported by the Natural Science Foundation of China NSFC 10171080, the National High Technology Research and Development Program of China 2001AA111042 (863 Program), and the Excellent Young Teachers Program of MOE 1887, China.

where $[x^{(0)}(\cdot), y^{(0)}(\cdot)]^t$ is an initial guess, the nonlinear splitting functions $f_1 : (\mathbf{R}^n)^4 \times (\mathbf{R}^m)^2 \times \mathbf{R}^{l_1} \times [0,T] \to \mathbf{R}^n$, $f_2 : (\mathbf{R}^n)^4 \times (\mathbf{R}^m)^2 \times \mathbf{R}^{l_2} \times [0,T] \to \mathbf{R}^m$, $h_1 : (\mathbf{R}^n)^2 \times (\mathbf{R}^m)^2 \times [0,T]^2 \to \mathbf{R}^{l_1}$, and $h_2 : (\mathbf{R}^n)^2 \times (\mathbf{R}^m)^2 \times [0,T]^2 \to \mathbf{R}^{l_2}$ satisfy

$$\left\{ \begin{array}{l} f_1(w,w,x,x,y,y,z_1,t) = \tilde{f}_1(w,x,y,z_1,t), \\ f_2(w,w,x,x,y,y,z_2,t) = \tilde{f}_2(w,x,y,z_2,t), \end{array} \right. \tag{4}$$

and

$$\left\{ \begin{array}{l} h_1(x,x,y,y,s,t) = \tilde{h}_1(x,y,s,t), \\ h_2(x,x,y,y,s,t) = \tilde{h}_2(x,y,s,t), \end{array} \right. \tag{5}$$

where $w, x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, $z_1 \in \mathbf{R}^{l_1}$, $z_2 \in \mathbf{R}^{l_2}$, and $s, t \in [0,T]$. The above splitting functions are often adopted as Jacobi or Gauss-Seidel.

We also study in this paper the well-known Picard iteration for (1). This kind of iterations is a special form of WR in (3), namely

$$\left\{ \begin{array}{rcl} \dot{x}^{(k+1)}(t) & = & \tilde{f}_1(\dot{x}^{(k)}(t), x^{(k)}(t), y^{(k)}(t), \int_0^t \tilde{h}_1(x^{(k)}(s), y^{(k)}(s), s, t)ds, t), \\ y^{(k+1)}(t) & = & \tilde{f}_2(\dot{x}^{(k)}(t), x^{(k)}(t), y^{(k)}(t), \int_0^t \tilde{h}_2(x^{(k)}(s), y^{(k)}(s), s, t)ds, t), \\ & & x^{(k+1)}(0) = x_0, \quad t \in [0,T], \quad k = 0, 1, \cdots, \end{array} \right. \tag{6}$$

where $[x^{(0)}(\cdot), y^{(0)}(\cdot)]^t$ is an initial guess as before.

It is known that a circuit system with lumped elements may have the form of (1). For example, if all the elements of a circuit are linear we can then describe the circuit by a system of linear IDAEs [7]. For a high-speed integrated circuit, its equation form may be written as nonlinear differential-algebraic equations (DAEs) with multiple delays if the transmission lines are lossless [8]. As long as the distributed elements (R, L, C, and G) exist in a large circuit we will meet nonlinear IDAEs with multiple delays in the time-domain simulation, see [9]. In other words, some complex differential and integral equations are often arising in the modern circuit simulation field. Here, we will not concretely concern the modelling problems which are really beyond the scope of the paper.

As a simple case, namely without the term $y(\cdot)$ and the second part of (1), a discrete-time WR version is considered in [10]. Moreover, WR solutions of Volterra integral equations are studied in [11]. To take advantage of Lipschitz constants of the nonlinear functions in a system of DAEs, WR is successfully applied to compute their numerical solutions [12, 13]. By a different approach from the known ones we have presented a general convergence condition about continuous-time WR solutions of nonlinear DAEs in [14]. The interesting approach can easily treat complex systems with WR decoupling. It is the first time that the spectral approach is used in WR solutions of nonlinear IDAEs.

In this paper we mainly study the convergence conditions of the continuous-time WR algorithm (3) and the Picard iteration (6) based on operations of linear operators and spectral analysis in function space. We also discuss discrete-time WR solutions by a backward-differentiation formula (BDF). Some typical systems with WR are further included into the paper. Numerical experiments on a test example are provided to illustrate these new convergence conditions.

2. Continuous-time Waveform Relaxation

First we assume that the splitting function pairs (f_1, f_2) and (h_1, h_2) respectively satisfy the following Lipschitz conditions.

Condition (L_f). For four vector norms $\|\cdot\|_n$ in $\mathbf{R}^n$, $\|\cdot\|_m$ in $\mathbf{R}^m$, and $\|\cdot\|_{l_i}$ in $\mathbf{R}^{l_i}$ $(i = 1, 2)$, we assume that there are constants $a_i, b_i (i = 1, 2, \cdots, 6), \alpha$, and β such that

$$\begin{array}{rl} & \|f_1(u_1, u_2, \cdots, u_6, z_1, t) - f_1(v_1, v_2, \cdots, v_6, w_1, t)\|_n \\ \leq & \sum_{i=1}^4 a_i \|u_i - v_i\|_n + \sum_{i=5}^6 a_i \|u_i - v_i\|_m + \alpha \|z_1 - w_1\|_{l_1}, \end{array} \tag{7}$$

and

$$\begin{array}{rl} & \|f_2(u_1,u_2,\cdots,u_6,z_2,t)-f_2(v_1,v_2,\cdots,v_6,w_2,t)\|_m \\ \leq & \sum_{i=1}^4 b_i\|u_i-v_i\|_n+\sum_{i=5}^6 b_i\|u_i-v_i\|_m+\beta\|z_2-w_2\|_{l_2}, \end{array} \tag{8}$$

where $t\in[0,T]$, $u_i,v_i\in\mathbf{R}^n(i=1,2,3,4)$, $u_i,v_i\in\mathbf{R}^m(i=5,6)$, $z_i,w_i\in\mathbf{R}^{l_i}$ $(i=1,2)$.

Condition (L_h). For four vector norms $\|\cdot\|_n$ in $\mathbf{R}^n$, $\|\cdot\|_m$ in $\mathbf{R}^m$, and $\|\cdot\|_{l_i}$ in $\mathbf{R}^{l_i}$ $(i=1,2)$, we assume that there are constants c_i and $d_i(i=1,2,3,4)$ such that

$$\|h_1(u_1,u_2,u_3,u_4,s,t)-h_1(v_1,v_2,v_3,v_4,s,t)\|_{l_1}\leq\sum_{i=1}^2 c_i\|u_i-v_i\|_n+\sum_{i=3}^4 c_i\|u_i-v_i\|_m \tag{9}$$

and

$$\|h_2(u_1,u_2,u_3,u_4,s,t)-h_2(v_1,v_2,v_3,v_4,s,t)\|_{l_2}\leq\sum_{i=1}^2 d_i\|u_i-v_i\|_n+\sum_{i=3}^4 d_i\|u_i-v_i\|_m, \tag{10}$$

where $s,t\in[0,T]$, $u_i,v_i\in\mathbf{R}^n(i=1,2)$ and $u_i,v_i\in\mathbf{R}^m(i=3,4)$.

Let us define $a_7=\alpha c_1, a_8=\alpha c_2, a_9=\alpha c_3, a_{10}=\alpha c_4$, $b_7=\beta d_1$, $b_8=\beta d_2$, $b_9=\beta d_3$, and $b_{10}=\beta d_4$. We also need a simple 2×2 matrix as

$$H=\begin{bmatrix} a_1+a_2 & a_5+a_6 \\ b_1+b_2 & b_5+b_6 \end{bmatrix}, \tag{11}$$

where a_1,a_2,a_5,a_6, b_1,b_2,b_5,b_6 are Lipschitz constants appearing in (7) and (8).

Let $z(t)=\dot{x}(t)$ for $t\in[0,T]$ where $x(0)=x_0$. Then $x(\cdot)=(Jz)(\cdot)$, in which

$$(Jz)(t)=x_0+\int_0^t z(s)ds, \quad t\in[0,T].$$

By (4) and (5) on $[0,T]$ we can rewrite (1) as

$$\left\{\begin{array}{rcl} z(t) & = & f_1(z(t),z(t),Jz(t),Jz(t),y(t),y(t),\int_0^t h_1(Jz(s),Jz(s),y(s),y(s),s,t)ds,t), \\ y(t) & = & f_2(z(t),z(t),Jz(t),Jz(t),y(t),y(t),\int_0^t h_2(Jz(s),Jz(s),y(s),y(s),s,t)ds,t). \end{array}\right. \tag{12}$$

We also let $w(\cdot)=[z(\cdot),y(\cdot)]^t$, $w_i(\cdot)=[z_i(\cdot),y_i(\cdot)]^t(i=1,2)$, and $F=[F_1,F_2]^t$, in which

$$\begin{array}{rcl} F_1(w_1(t),w_2(t)) & = & f_1(z_1(t),z_2(t),Jz_1(t),Jz_2(t),y_1(t),y_2(t), \\ & & \int_0^t h_1(Jz_1(s),Jz_2(s),y_1(s),y_2(s),s,t)ds,t) \end{array}$$

and

$$\begin{array}{rcl} F_2(w_1(t),w_2(t)) & = & f_2(z_1(t),z_2(t),Jz_1(t),Jz_2(t),y_1(t),y_2(t), \\ & & \int_0^t h_2(Jz_1(s),Jz_2(s),y_1(s),y_2(s),s,t)ds,t), \end{array}$$

where $t\in[0,T]$. According to these technical terms the solution of (12), which is equivalent to that of (1), can be regarded as the solution of a fixed-point equation in $C([0,T];\mathbf{R}^{n+m})$ as

$$w=F(w,w). \tag{13}$$

To analyze the convergence of iteration methods (3) and (6) we need to present an elementary and important conclusion about the spectral result for the compact perturbation of a linear Volterra type operator in the continuous function space [15, 16]. Let $A\in\mathbf{R}^{l\times l}$ and $u\in C([0,T];\mathbf{R}^l)$, we define

$$\mathcal{A}u=Au+\mathcal{A}_c u, \tag{14}$$

where $\mathcal{A}_c u(t)=\int_0^t k_c(t-s)u(s)ds$ for $t\in[0,T]$ in which $k_c(\cdot)$ is a continuous matrix function.

Two different proofs of the following lemma on the spectral radius of $\mathcal{A}$ were given in [16]. Let $\rho(A)$ be the spectral radius for $A \in \mathbf{R}^{l\times l}$. Now we restate it without proof.

Lemma 1. *For $\mathcal{A}$ in (14), we have $\rho(\mathcal{A}) = \rho(A)$ in $C([0,T];\mathbf{R}^l)$.*

2.1. Convergence of the Picard iteration

We present a sufficient condition in this subsection that (1) has a unique solution and its Picard iteration converges to the solution by use of the fixed-point form (12) or (13). For this purpose we may rewrite the Picard iteration (6).

Let $z^{(i)}(t) = \dot{x}^{(i)}(t)$ $(i = k, k+1)$ for $t \in [0,T]$ where $x^{(i)}(0) = x_0$ $(i = k, k+1)$. That is, $x^{(i)}(\cdot) = (Jz^{(i)})(\cdot)$ $(i = k, k+1)$ where J is the integral operation defined before. The Picard iteration of (12) is

$$\left\{\begin{array}{lcl} z^{(k+1)}(t) & = & f_1(z^{(k)}(t), z^{(k)}(t), Jz^{(k)}(t), Jz^{(k)}(t), y^{(k)}(t), y^{(k)}(t), \\ & & \displaystyle\int_0^t h_1(Jz^{(k)}(s), Jz^{(k)}(s), y^{(k)}(s), y^{(k)}(s), s, t)ds, t), \\ y^{(k+1)}(t) & = & f_2(z^{(k)}(t), z^{(k)}(t), Jz^{(k)}(t), Jz^{(k)}(t), y^{(k)}(t), y^{(k)}(t), \\ & & \displaystyle\int_0^t h_2(Jz^{(k)}(s), Jz^{(k)}(s), y^{(k)}(s), y^{(k)}(s), s, t)ds, t), \quad k = 0, 1, \ldots. \end{array}\right. \tag{15}$$

A brief form of (15) is

$$w^{(k+1)} = F(w^{(k)}, w^{(k)}), \quad k = 0, 1, \ldots \tag{16}$$

if we invoke the fixed-point form (13) in $C([0,T];\mathbf{R}^{n+m})$.

We recall that the partial ordering relationship "$\geq$" means that $x \geq y \Longleftrightarrow x_i \geq y_i (i = 1, 2, \cdots, n)$ where $x, y \in \mathbf{R}^n$ and $A \geq B \Longleftrightarrow a_{ij} \geq b_{ij} (i, j = 1, 2, \cdots, n)$ where $A, B \in \mathbf{R}^{n\times n}$.

Theorem 1. *Let Conditions (L_f) and (L_h) be satisfied and $\rho(H) < 1$, then (12) has a unique solution $[z^{(*)}(\cdot), y^{(*)}(\cdot)]^t$ and the sequence $\{[x^{(k)}(\cdot), y^{(k)}(\cdot)]^t\}$ produced by the Picard iteration (6) converges to $[x^{(*)}(\cdot), y^{(*)}(\cdot)]$, where $x^{(*)}(\cdot) = Jz^{(*)}(\cdot)$, which is the unique solution of (1).*

Proof. For $z_i(\cdot) \in C([0,T];\mathbf{R}^n)(i = 1, 2)$, since $\|Jz_1(t) - Jz_2(t)\|_n \leq \displaystyle\int_0^t \|z_1(s) - z_2(s)\|_n ds$, one has

$$\begin{array}{cl} & \|F_1(w_1(t), w_1(t)) - F_1(w_2(t), w_2(t))\|_n \\ \leq & (a_1 + a_2)\|z_1(t) - z_2(t)\|_n + (a_3 + a_4)\|Jz_1(t) - Jz_2(t)\|_n + (a_5 + a_6)\|y_1(t) - y_2(t)\|_m \\ & \displaystyle +\alpha\int_0^t (\sum_{i=1}^{2} c_i\|Jz_1(s) - Jz_2(s)\|_n + \sum_{i=3}^{4} c_i\|y_1(s) - y_2(s)\|_m)ds \\ \leq & \displaystyle (a_1 + a_2)\|z_1(t) - z_2(t)\|_n + (a_3 + a_4)\int_0^t \|z_1(s) - z_2(s)\|_n ds + (a_5 + a_6)\|y_1(t) - y_2(t)\|_m \\ & \displaystyle +(a_7 + a_8)t\int_0^t \|z_1(s) - z_2(s)\|_n ds + (a_9 + a_{10})\int_0^t \|y_1(s) - y_2(s)\|_m ds \\ = & (a_1 + a_2)\|z_1(t) - z_2(t)\|_n + (a_5 + a_6)\|y_1(t) - y_2(t)\|_m + [(a_3 + a_4) + (a_7 + a_8)t] \\ & \displaystyle \times\int_0^t \|z_1(s) - z_2(s)\|_n ds + (a_9 + a_{10})\int_0^t \|y_1(s) - y_2(s)\|_m ds, \end{array} \tag{17}$$

where $w_i(\cdot) = [z_i(\cdot), y_i(\cdot)]^t (i = 1, 2)$ with $y_i \in C([0,T];\mathbf{R}^m)$. Similarly, one has also

$$\begin{array}{cl} & \|F_2(w_1(t), w_1(t)) - F_2(w_2(t), w_2(t))\|_m \\ \leq & (b_1 + b_2)\|z_1(t) - z_2(t)\|_n + (b_5 + b_6)\|y_1(t) - y_2(t)\|_m + [(b_3 + b_4) + (b_7 + b_8)t] \\ & \displaystyle \times\int_0^t \|z_1(s) - z_2(s)\|_n ds + (b_9 + b_{10})\int_0^t \|y_1(s) - y_2(s)\|_m ds. \end{array} \tag{18}$$

Now we compactly write the inequalities (17) and (18) together as

$$\begin{aligned}&\begin{bmatrix} \|F_1(w_1(t),w_1(t))-F_1(w_2(t),w_2(t))\|_n \\ \|F_2(w_1(t),w_1(t))-F_2(w_2(t),w_2(t))\|_m \end{bmatrix}\\ \leq\ &\left(\begin{bmatrix} a_1+a_2 & a_5+a_6 \\ b_1+b_2 & b_5+b_6 \end{bmatrix}+\begin{bmatrix} (a_3+a_4)\mathcal{V}_c+(a_7+a_8)t\mathcal{V}_c & (a_9+a_{10})\mathcal{V}_c \\ (b_3+b_4)\mathcal{V}_c+(b_7+b_8)t\mathcal{V}_c & (b_9+b_{10})\mathcal{V}_c \end{bmatrix}\right)\\ &\times\begin{bmatrix} \|z_1(t)-z_2(t)\|_n \\ \|y_1(t)-y_2(t)\|_m \end{bmatrix},\end{aligned} \tag{19}$$

where $\mathcal{V}_c : C([0,T];\mathbf{R}) \to C([0,T];\mathbf{R})$ is a linear Volterra integral operator, namely $(\mathcal{V}_c u)(t) = \int_0^t u(s)ds$ for $u \in C([0,T];\mathbf{R})$.

Let

$$\mathcal{L}_c = \begin{bmatrix} (a_3+a_4)\mathcal{V}_c+(a_7+a_8)t\mathcal{V}_c & (a_9+a_{10})\mathcal{V}_c \\ (b_3+b_4)\mathcal{V}_c+(b_7+b_8)t\mathcal{V}_c & (b_9+b_{10})\mathcal{V}_c \end{bmatrix}$$

and $\mathcal{L} = H + \mathcal{L}_c$. In $C([0,T];\mathbf{R}^2)$ the linear Volterra type integral operator $\mathcal{L}_c$ is compact and nonnegative [17]. Recall that a linear operator $\mathcal{H}$ in $C([0,T];\mathbf{R}^2)$ is called nonnegative if $(\mathcal{H}v)(t) \geq 0$ for $t \in [0,T]$ where $v = [v_1, v_2]^t \in C([0,T];\mathbf{R}^2)$ with $v_i(t) \geq 0 (i = 1,2)$ for $t \in [0,T]$. For the spectral radius of $\mathcal{L}$, it is known that $\rho(\mathcal{L}) = \rho(H)$ from Lemma 1. Thus $\rho(\mathcal{L}) < 1$ and $(\mathcal{I}-\mathcal{L})^{-1} = \sum_{i=0}^{\infty}\mathcal{L}^i$, namely $(\mathcal{I}-\mathcal{L})^{-1}$ is nonnegative.

Moreover, for any positive integer k' we have

$$\begin{aligned}\begin{bmatrix} \|z^{(k+k')}(t)-z^{(k)}(t)\|_n \\ \|y^{(k+k')}(t)-y^{(k)}(t)\|_m \end{bmatrix} &\leq \textstyle\sum_{i=0}^{k'-1}\begin{bmatrix} \|z^{(k+i+1)}(t)-z^{(k+i)}(t)\|_n \\ \|y^{(k+i+1)}(t)-y^{(k+i)}(t)\|_m \end{bmatrix}\\ &\leq \textstyle\sum_{i=0}^{k'-1}\mathcal{L}^i\begin{bmatrix} \|z^{(k+1)}(t)-z^{(k)}(t)\|_n \\ \|y^{(k+1)}(t)-y^{(k)}(t)\|_m \end{bmatrix}\\ &\leq \textstyle\sum_{i=0}^{k'-1}\mathcal{L}^{i+k}\begin{bmatrix} \|z^{(1)}(t)-z^{(0)}(t)\|_n \\ \|y^{(1)}(t)-y^{(0)}(t)\|_m \end{bmatrix}\\ &\leq (\mathcal{I}-\mathcal{L})^{-1}\mathcal{L}^k\begin{bmatrix} \|z^{(1)}(t)-z^{(0)}(t)\|_n \\ \|y^{(1)}(t)-y^{(0)}(t)\|_m \end{bmatrix},\end{aligned} \tag{20}$$

where $k = 0,1,\ldots$. It deduces that $\{[z^{(k)}(\cdot), y^{(k)}(\cdot)]^t\}$ is a Cauchy sequence and uniformly converges to the unique solution $w^{(*)}(\cdot) = [z^{(*)}(\cdot), y^{(*)}(\cdot)]^t$ of (15) in $C([0,T];\mathbf{R}^{n+m})$ since $\mathcal{L}^k \to 0 (k \to +\infty)$. It further follows that the sequence $\{[x^{(k)}(\cdot), y^{(k)}(\cdot)]^t\}$ produced by the Picard iteration (6) converges to the unique solution $[x^{(*)}(\cdot), y^{(*)}(\cdot)]^t$ of (1) where $x^{(*)}(\cdot) = Jz^{(*)}(\cdot) \in C^1([0,T];\mathbf{R}^n)$. This completes the proof of Theorem 1.

2.2 Convergence of the WR algorithm

First we rewrite the WR algorithm (3) as

$$\left\{\begin{aligned} z^{(k+1)}(t) &= f_1(z^{(k+1)}(t), z^{(k)}(t), Jz^{(k+1)}(t), Jz^{(k)}(t), y^{(k+1)}(t), y^{(k)}(t),\\ &\qquad \int_0^t h_1(Jz^{(k+1)}(s), Jz^{(k)}(s), y^{(k+1)}(s), y^{(k)}(s), s, t)ds, t),\\ y^{(k+1)}(t) &= f_2(z^{(k+1)}(t), z^{(k)}(t), Jz^{(k+1)}(t), Jz^{(k)}(t), y^{(k+1)}(t), y^{(k)}(t),\\ &\qquad \int_0^t h_2(Jz^{(k+1)}(s), Jz^{(k)}(s), y^{(k+1)}(s), y^{(k)}(s), s, t)ds, t), \quad k = 0,1,\ldots. \end{aligned}\right. \tag{21}$$

As the same before, (21) has a simple form as follows

$$w^{(k+1)} = F(w^{(k+1)}, w^{(k)}), \quad k = 0,1,\cdots, \tag{22}$$

where $w^{(k)}(\cdot) = [z^{(k)}(\cdot), y^{(k)}(\cdot)]^t$ and $w^{(0)}(\cdot)$ is an initial guess.

Let $H = H_1 + H_2$ where

$$H_1 = \begin{bmatrix} a_1 & a_5 \\ b_1 & b_5 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} a_2 & a_6 \\ b_2 & b_6 \end{bmatrix}, \tag{23}$$

and $H_0 = (I - H_1)^{-1}H_2$. As in [14] we have the following lemma (for brevity we omit its proof).

Lemma 2. *If $(I - H_1)^{-1} \geq 0$ and $\rho(H_0) < 1$, then $\rho(H) < 1$.*

Now, we give a convergence condition for the general WR algorithm (3) or (21).

Theorem 2. *Let (i) Conditions (L_f) and (L_h) be satisfied, (ii) $(I - H_1)^{-1} \geq 0$, and (iii) $\rho(H_0) < 1$. Then, the function sequence $\{w^{(k)}(\cdot)\}$, where $w^{(k)}(\cdot) = [z^{(k)}(\cdot), y^{(k)}(\cdot)]^t$, produced by (21), and the function sequence $\{[x^{(k)}(\cdot), y^{(k)}(\cdot)]^t\}$, where $x^{(k)}(\cdot) = Jz^{(k)}(\cdot)$, produced by the WR algorithm (3), respectively converge to the solution $w^{(*)}(\cdot) = [z^{(*)}(\cdot), y^{(*)}(\cdot)]^t$ of (12) and to the solution $[x^{(*)}(\cdot), y^{(*)}(\cdot)]^t$ of (1) where $x^{(*)}(\cdot) = Jz^{(*)}(\cdot)$.*

Proof. By Lemma 2 and Theorem 1, we know that (12) has a unique solution $w^*(\cdot) = [z^{(*)}(\cdot), y^{(*)}(\cdot)]^t$. Let us prove that the sequence $\{w^{(k)}\}$ of (22) in $C([0,T]; \mathbf{R}^{n+m})$ converges to this solution. First we have

$$\begin{aligned}
&\|z^{(k+1)}(t) - z^{(*)}(t)\|_n = \|F_1(w^{(k+1)}(t), w^{(k)}(t)) - F_1(w^{(*)}(t), w^{(*)}(t))\|_n \\
\leq\ & a_1\|z^{(k+1)}(t) - z^{(*)}(t)\|_n + a_2\|z^{(k)}(t) - z^{(*)}(t)\|_n + a_3\int_0^t \|z^{(k+1)}(s) - z^{(*)}(s)\|_n ds \\
& +a_4\int_0^t \|z^{(k)}(s) - z^{(*)}(s)\|_n ds + a_5\|y^{(k+1)}(t) - y^{(*)}(t)\|_m + a_6\|y^{(k)}(t) - y^{(*)}(t)\|_m \\
& +\alpha\int_0^t (c_1\|Jz^{(k+1)}(s) - Jz^{(*)}(s)\|_n + c_2\|Jz^{(k)}(s) - Jz^{(*)}(s)\|_n + c_3\|y^{(k+1)}(s) - y^{(*)}(s)\|_m \\
& +c_4\|y^{(k)}(s) - y^{(*)}(s)\|_m)ds \\
\leq\ & a_1\|z^{(k+1)}(t) - z^{(*)}(t)\|_n + a_5\|y^{(k+1)}(t) - y^{(*)}(t)\|_m + a_2\|z^{(k)}(t) - z^{(*)}(t)\|_n \\
& +a_6\|y^{(k)}(t) - y^{(*)}(t)\|_m + (a_3 + a_7 t)\int_0^t \|z^{(k+1)}(s) - z^{(*)}(s)\|_n ds \\
& +a_9\int_0^t \|y^{(k+1)}(s) - y^{(*)}(s)\|_m ds + (a_4 + a_8 t)\int_0^t \|z^{(k)}(s) - z^{(*)}(s)\|_n ds \\
& +a_{10}\int_0^t \|y^{(k)}(s) - y^{(*)}(s)\|_m ds
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
&\|y^{(k+1)}(t) - y^{(*)}(t)\|_m = \|F_2(w^{(k+1)}(t), w^{(k)}(t)) - F_2(w^{(*)}(t), w^{(*)}(t))\|_m \\
\leq\ & b_1\|z^{(k+1)}(t) - z^{(*)}(t)\|_n + b_5\|y^{(k+1)}(t) - y^{(*)}(t)\|_m + b_2\|z^{(k)}(t) - z^{(*)}(t)\|_n \\
& +b_6\|y^{(k)}(t) - y^{(*)}(t)\|_m + (b_3 + b_7 t)\int_0^t \|z^{(k+1)}(s) - z^{(*)}(s)\|_n ds \\
& +b_9\int_0^t \|y^{(k+1)}(s) - y^{(*)}(s)\|_m ds + (b_4 + b_8 t)\int_0^t \|z^{(k)}(s) - z^{(*)}(s)\|_n ds \\
& +b_{10}\int_0^t \|y^{(k)}(s) - y^{(*)}(s)\|_m ds.
\end{aligned} \tag{25}$$

Combining the inequalities (24) and (25), we obtain

$$\begin{aligned}
&\begin{bmatrix} \|z^{(k+1)}(t) - z^{(*)}(t)\|_n \\ \|y^{(k+1)}(t) - y^{(*)}(t)\|_m \end{bmatrix} \\
\leq\ & \begin{bmatrix} a_1 & a_5 \\ b_1 & b_5 \end{bmatrix} \begin{bmatrix} \|z^{(k+1)}(t) - z^{(*)}(t)\|_n \\ \|y^{(k+1)}(t) - y^{(*)}(t)\|_m \end{bmatrix} + \begin{bmatrix} a_2 & a_6 \\ b_2 & b_6 \end{bmatrix} \begin{bmatrix} \|z^{(k)}(t) - z^{(*)}(t)\|_n \\ \|y^{(k)}(t) - y^{(*)}(t)\|_m \end{bmatrix} \\
& + \begin{bmatrix} (a_3 + a_7 t)\mathcal{V}_c & a_9\mathcal{V}_c \\ (b_3 + b_7 t)\mathcal{V}_c & b_9\mathcal{V}_c \end{bmatrix} \begin{bmatrix} \|z^{(k+1)}(t) - z^{(*)}(t)\|_n \\ \|y^{(k+1)}(t) - y^{(*)}(t)\|_m \end{bmatrix} + \begin{bmatrix} (a_4 + a_8 t)\mathcal{V}_c & a_{10}\mathcal{V}_c \\ (b_4 + b_8 t)\mathcal{V}_c & b_{10}\mathcal{V}_c \end{bmatrix} \\
& \times \begin{bmatrix} \|z^{(k)}(t) - z^{(*)}(t)\|_n \\ \|y^{(k)}(t) - y^{(*)}(t)\|_m \end{bmatrix}, \quad t \in [0,T],
\end{aligned} \tag{26}$$

where $(\mathcal{V}_c u)(t) = \displaystyle\int_0^t u(s)ds$ for $u \in C([0,T];\mathbf{R})$. Since $(I-H_1)^{-1} \geq 0$, we get

$$\begin{aligned}
&\left[\begin{array}{c} \|z^{(k+1)}(t)-z^{(*)}(t)\|_n \\ \|y^{(k+1)}(t)-y^{(*)}(t)\|_m \end{array}\right] \\
\leq\ & H_0 \left[\begin{array}{c} \|z^{(k)}(t)-z^{(*)}(t)\|_n \\ \|y^{(k)}(t)-y^{(*)}(t)\|_m \end{array}\right] + (I-H_1)^{-1}\left[\begin{array}{cc} (a_3+a_7t)\mathcal{V}_c & a_9\mathcal{V}_c \\ (b_3+b_7t)\mathcal{V}_c & b_9\mathcal{V}_c \end{array}\right] \\
& \left[\begin{array}{c} \|z^{(k+1)}(t)-z^{(*)}(t)\|_n \\ \|y^{(k+1)}(t)-y^{(*)}(t)\|_m \end{array}\right] + (I-H_1)^{-1}\left[\begin{array}{cc} (a_4+a_8t)\mathcal{V}_c & a_{10}\mathcal{V}_c \\ (b_4+b_8t)\mathcal{V}_c & b_{10}\mathcal{V}_c \end{array}\right] \\
& \left[\begin{array}{c} \|z^{(k)}(t)-z^{(*)}(t)\|_n \\ \|y^{(k)}(t)-y^{(*)}(t)\|_m \end{array}\right] \\
\leq\ & H_0 \left[\begin{array}{c} \|z^{(k)}(t)-z^{(*)}(t)\|_n \\ \|y^{(k)}(t)-y^{(*)}(t)\|_m \end{array}\right] + (I-H_1)^{-1}\left[\begin{array}{cc} (a_3+a_7T)\mathcal{V}_c & a_9\mathcal{V}_c \\ (b_3+b_7T)\mathcal{V}_c & b_9\mathcal{V}_c \end{array}\right] \\
& \left[\begin{array}{c} \|z^{(k+1)}(t)-z^{(*)}(t)\|_n \\ \|y^{(k+1)}(t)-y^{(*)}(t)\|_m \end{array}\right] + (I-H_1)^{-1}\left[\begin{array}{cc} (a_4+a_8T)\mathcal{V}_c & a_{10}\mathcal{V}_c \\ (b_4+b_8T)\mathcal{V}_c & b_{10}\mathcal{V}_c \end{array}\right] \\
& \left[\begin{array}{c} \|z^{(k)}(t)-z^{(*)}(t)\|_n \\ \|y^{(k)}(t)-y^{(*)}(t)\|_m \end{array}\right].
\end{aligned}$$

Moreover, there exist nonnegative constants $\gamma_i, \delta_i (i=1,2,3,4)$ such that one can further write the above formula as follows

$$\begin{aligned}
&\left[\begin{array}{c} \|z^{(k+1)}(t)-z^{(*)}(t)\|_n \\ \|y^{(k+1)}(t)-y^{(*)}(t)\|_m \end{array}\right] \\
\leq\ & \left[\begin{array}{cc} \gamma_1\mathcal{V}_c & \gamma_2\mathcal{V}_c \\ \gamma_3\mathcal{V}_c & \gamma_4\mathcal{V}_c \end{array}\right]\left[\begin{array}{c} \|z^{(k+1)}(t)-z^{(*)}(t)\|_n \\ \|y^{(k+1)}(t)-y^{(*)}(t)\|_m \end{array}\right] + H_0\left[\begin{array}{c} \|z^{(k)}(t)-z^{(*)}(t)\|_n \\ \|y^{(k)}(t)-y^{(*)}(t)\|_m \end{array}\right] \\
& + \left[\begin{array}{cc} \delta_1\mathcal{V}_c & \delta_2\mathcal{V}_c \\ \delta_3\mathcal{V}_c & \delta_4\mathcal{V}_c \end{array}\right]\left[\begin{array}{c} \|z^{(k)}(t)-z^{(*)}(t)\|_n \\ \|y^{(k)}(t)-y^{(*)}(t)\|_m \end{array}\right].
\end{aligned} \tag{27}$$

Let $\mathcal{R}_\gamma, \mathcal{R}_\delta : C([0,T];\mathbf{R}^2) \to C([0,T];\mathbf{R}^2)$ be two linear operators as follows

$$\mathcal{R}_\gamma = \left[\begin{array}{cc} \gamma_1\mathcal{V}_c & \gamma_2\mathcal{V}_c \\ \gamma_3\mathcal{V}_c & \gamma_4\mathcal{V}_c \end{array}\right], \quad \mathcal{R}_\delta = \left[\begin{array}{cc} \delta_1\mathcal{V}_c & \delta_2\mathcal{V}_c \\ \delta_3\mathcal{V}_c & \delta_4\mathcal{V}_c \end{array}\right].$$

Both of them are the Volterra type integral operators which are compact and nonnegative. That is, $\rho(\mathcal{R}_\gamma) = 0$ by Lemma 1 or [17]. Further, $(\mathcal{I}-\mathcal{R}_\gamma)^{-1} = \sum_{i=0}^{\infty}\mathcal{R}_\gamma{}^i$. The operator $(\mathcal{I}-\mathcal{R}_\gamma)^{-1}$ is linear and nonnegative, too. Thus, from (27) we have that

$$\begin{aligned}
\left[\begin{array}{c} \|z^{(k)}(t)-z^{(*)}(t)\|_n \\ \|y^{(k)}(t)-y^{(*)}(t)\|_m \end{array}\right] &\leq \mathcal{R}\left[\begin{array}{c} \|z^{(k-1)}(t)-z^{(*)}(t)\|_n \\ \|y^{(k-1)}(t)-y^{(*)}(t)\|_m \end{array}\right] \\
&\leq \mathcal{R}^k\left[\begin{array}{c} \|z^{(0)}(t)-z^{(*)}(t)\|_n \\ \|y^{(0)}(t)-y^{(*)}(t)\|_m \end{array}\right], \quad t \in [0,T], \quad k=1,2,\ldots,
\end{aligned} \tag{28}$$

where $\mathcal{R} = (\mathcal{I}-\mathcal{R}_\gamma)^{-1}(H_0+\mathcal{R}_\delta)$. We can write the nonnegative bounded linear operator $\mathcal{R}$ as

$$\mathcal{R} = H_0 + \mathcal{R}_c, \tag{29}$$

where $\mathcal{R}_c = (\mathcal{I}-\mathcal{R}_\gamma)^{-1}\mathcal{R}_\zeta$ in which $\mathcal{R}_\zeta = \mathcal{R}_\gamma H_0 + \mathcal{R}_\delta$. It is also known that there are constants $\zeta_i (i=1,2,3,4)$ such that

$$\mathcal{R}_\zeta = \left[\begin{array}{cc} \zeta_1\mathcal{V}_c & \zeta_2\mathcal{V}_c \\ \zeta_3\mathcal{V}_c & \zeta_4\mathcal{V}_c \end{array}\right].$$

Now we define two matrices as follows

$$A_\gamma = \left[\begin{array}{cc} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{array}\right], \quad A_\zeta = \left[\begin{array}{cc} \zeta_1 & \zeta_2 \\ \zeta_3 & \zeta_4 \end{array}\right].$$

By elementary calculations, for $v \in C([0,T];\mathbf{R}^2)$ we have

$$(\mathcal{I}-\mathcal{R}_\gamma)^{-1}v(t) = v(t) + \int_0^t e^{A_\gamma(t-s)}A_\gamma v(s)ds, \quad t\in[0,T].$$

Further,

$$(\mathcal{I}-\mathcal{R}_\gamma)^{-1}\mathcal{R}_\varsigma v(t) = \int_0^t e^{A_\gamma(t-s)}A_\varsigma v(s)ds, \quad t\in[0,T].$$

It shows that $\mathcal{R}_c$ in (29) is a linear compact Volterra integral operator. Thus, $\rho(\mathcal{R}) = \rho(H_0)$ by Lemma 1. It follows that $\rho(\mathcal{R}) < 1$. According to the proof of Theorem 1, we arrive at $\lim_{k\to+\infty}\|z^{(k)} - z^*\| = 0$ in $C([0,T];\mathbf{R}^n)$ and $\lim_{k\to+\infty}\|y^{(k)} - y^*\| = 0$ in $C([0,T];\mathbf{R}^m)$. Similarly, $\{[x^{(k)}(\cdot), y^{(k)}(\cdot)]^t\}$ uniformly converges to the solution $[x^{(*)}(\cdot), y^{(*)}(\cdot)]^t$ of (1). This completes the proof of Theorem 2.

3. Discrete-time Waveform Relaxation

Iterative methods are effective tools of numerical computations for large systems in practical applications [18, 19]. In this section we adopt a BDF to numerically compute discrete waveforms of the WR algorithm (3). A q-step BDF method, when $q = 1$ it becomes the known Euler method, replaces the derivatives in (3) by a numerical differentiation formula that consists of a linear combination of the approximate function values at $q+1$ times $t_p, t_{p-1}, \ldots, t_{p-q}$.

Now the unknown are N discrete values of waveforms at time-points t_p $(p = 1, 2, \ldots, N)$ with an equip time-step Δt where $t_N = T$. For (3) we have

$$\left\{\begin{array}{rcl}
\frac{1}{\Delta t}\sum\limits_{i=0}^{q}\eta_i x_{p-i}^{(k+1)} & = & f_1(\frac{1}{\Delta t}\sum\limits_{i=0}^{q}\eta_i x_{p-i}^{(k+1)}, \frac{1}{\Delta t}\sum\limits_{i=0}^{q}\eta_i x_{p-i}^{(k)}, x_p^{(k+1)}, x_p^{(k)}, y_p^{(k+1)}, y_p^{(k)}, u_p^{(k+1,k)}, t_p),\\
y_p^{(k+1)} & = & f_2(\frac{1}{\Delta t}\sum\limits_{i=0}^{q}\eta_i x_{p-i}^{(k+1)}, \frac{1}{\Delta t}\sum\limits_{i=0}^{q}\eta_i x_{p-i}^{(k)}, x_p^{(k+1)}, x_p^{(k)}, y_p^{(k+1)}, y_p^{(k)}, v_p^{(k+1,k)}, t_p),\\
u_p^{(k+1,k)} & = & \Delta t\sum\limits_{j=0}^{p}\omega_{p,j}^{(1)}h_1(x_{p-j}^{(k+1)}, x_{p-j}^{(k)}, y_{p-j}^{(k+1)}, y_{p-j}^{(k)}, t_{p-j}, t_p),\\
v_p^{(k+1,k)} & = & \Delta t\sum\limits_{j=0}^{p}\omega_{p,j}^{(2)}h_2(x_{p-j}^{(k+1)}, x_{p-j}^{(k)}, y_{p-j}^{(k+1)}, y_{p-j}^{(k)}, t_{p-j}, t_p),\\
 & & \Delta t = t_p - t_{p-1}, \quad p = 1, 2, \ldots, N, \quad k = 0, 1, \ldots,
\end{array}\right. \tag{30}$$

where $\omega_{p,j}^{(l)}(l = 1, 2)$ are numerical integral weights and $\eta_i(i = 0, 1, \ldots, q)$ are the BDF parameters in which $\eta_0 \neq 0$. To show the convergence of the above discrete iterative waveforms we also need the approximate form of (1) by the q-step BDF method. This yields

$$\left\{\begin{array}{rcl}
\frac{1}{\Delta t}\sum\limits_{i=0}^{q}\eta_i x_{p-i} & = & f_1(\frac{1}{\Delta t}\sum\limits_{i=0}^{q}\eta_i x_{p-i}, \frac{1}{\Delta t}\sum\limits_{i=0}^{q}\eta_i x_{p-i}, x_p, x_p, y_p, y_p, u_p, t_p),\\
y_p & = & f_2(\frac{1}{\Delta t}\sum\limits_{i=0}^{q}\eta_i x_{p-i}, \frac{1}{\Delta t}\sum\limits_{i=0}^{q}\eta_i x_{p-i}, x_p, x_p, y_p, y_p, v_p, t_p),\\
u_p & = & \Delta t\sum\limits_{j=0}^{p}\omega_{p,j}^{(1)}h_1(x_{p-j}, x_{p-j}, y_{p-j}, y_{p-j}, t_{p-j}, t_p),\\
v_p & = & \Delta t\sum\limits_{j=0}^{p}\omega_{p,j}^{(2)}h_2(x_{p-j}, x_{p-j}, y_{p-j}, y_{p-j}, t_{p-j}, t_p),\\
 & & \Delta t = t_p - t_{p-1}, \quad p = 1, 2, \ldots, N.
\end{array}\right. \tag{31}$$

By (30) and (31), for $\|x_p^{(k+1)}-x_p\|_n$ ($k\in\{0,1,\dots\}$) one has an estimation as follows

$$\begin{cases}\eta_0\|x_p^{(k+1)}-x_p\|_n & \le \displaystyle\sum_{i=1}^{q}\eta_i\|x_{p-i}^{(k+1)}-x_{p-i}\|_n+a_1\sum_{i=0}^{q}\eta_i\|x_{p-i}^{(k+1)}-x_{p-i}\|_n\\ & \quad+a_2\displaystyle\sum_{i=0}^{q}\eta_i\|x_{p-i}^{(k)}-x_{p-i}\|_n+a_3\Delta t\|x_p^{(k+1)}-x_p\|_n+a_4\Delta t\|x_p^{(k)}-x_p\|_n\\ & \quad+a_5\Delta t\|y_p^{(k+1)}-y_p\|_m+a_6\Delta t\|y_p^{(k)}-y_p\|_m+\alpha\Delta t\|u_p^{(k+1,k)}-u_p\|_{l_1},\\ \|u_p^{(k+1,k)}-u_p\|_{l_1} & \le \Delta t\displaystyle\sum_{j=0}^{p}\omega_{p,j}^{(1)}(c_1\|x_{p-j}^{(k+1)}-x_{p-j}\|_n+c_2\|x_{p-j}^{(k)}-x_{p-j}\|_n\\ & \quad+c_3\|y_{p-j}^{(k+1)}-y_{p-j}\|_m+c_4\|y_{p-j}^{(k)}-y_{p-j}\|_m).\end{cases}$$

It deduces

$$\begin{array}{c}\eta_0\|x_p^{(k+1)}-x_p\|_n-a_1\eta_0\|x_p^{(k+1)}-x_p\|_n-a_3\Delta t\|x_p^{(k+1)}-x_p\|_n-a_7\omega_{p,0}^{(1)}\Delta t^2\|x_p^{(k+1)}-x_p\|_n\\ -a_5\Delta t\|y_p^{(k+1)}-y_p\|_m-a_9\omega_{p,0}^{(1)}\Delta t^2\|y_p^{(k+1)}-y_p\|_m\\ \le a_2\eta_0\|x_p^{(k)}-x_p\|_n+a_4\Delta t\|x_p^{(k)}-x_p\|_n+a_8\omega_{p,0}^{(1)}\Delta t^2\|x_p^{(k)}-x_p\|_n+a_6\Delta t\|y_p^{(k)}-y_p\|_m\\ +a_{10}\omega_{p,0}^{(1)}\Delta t^2\|y_p^{(k)}-y_p\|_m+\varphi(\Delta t),\end{array}$$

where

$$\begin{aligned}\varphi(\Delta t) &= \sum_{i=1}^{q}\eta_i\|x_{p-i}^{(k+1)}-x_{p-i}\|_n+a_1\sum_{i=1}^{q}\eta_i\|x_{p-i}^{(k+1)}-x_{p-i}\|_n+a_2\sum_{i=1}^{q}\eta_i\|x_{p-i}^{(k)}-x_{p-i}\|_n\\ &\quad+\Delta t^2\sum_{j=1}^{p}\omega_{p,j}^{(1)}(a_7\|x_{p-j}^{(k+1)}-x_{p-j}\|_n+a_8\|x_{p-j}^{(k)}-x_{p-j}\|_n+a_9\|y_{p-j}^{(k+1)}-y_{p-j}\|_m\\ &\quad+a_{10}\|y_{p-j}^{(k)}-y_{p-j}\|_m).\end{aligned}$$

That is

$$\begin{array}{c}[(1-a_1)\eta_0-a_3\Delta t-a_7\omega_{p,0}^{(1)}\Delta t^2]\|x_p^{(k+1)}-x_p\|_n-(a_5+a_9\omega_{p,0}^{(1)}\Delta t)\Delta t\|y_p^{(k+1)}-y_p\|_m\\ \le(a_2\eta_0+a_4\Delta t+a_8\omega_{p,0}^{(1)}\Delta t^2)\|x_p^{(k)}-x_p\|_n+(a_6+a_{10}\omega_{p,0}^{(1)}\Delta t)\Delta t\|y_p^{(k)}-y_p\|_m+\varphi(\Delta t).\end{array}\tag{32}$$

Similarly, for $\|y_p^{(k+1)}-y_p\|_m$ ($k\in\{0,1,\dots\}$) one has also an estimation as follows

$$\begin{array}{c}-(b_1\eta_0+b_3\Delta t+b_7\omega_{p,0}^{(2)}\Delta t^2)\|x_p^{(k+1)}-x_p\|_n+[(1-b_5)-b_9\omega_{p,0}^{(2)}\Delta t]\Delta t\|y_p^{(k+1)}-y_p\|_m\\ \le(b_2\eta_0+b_4\Delta t+b_8\omega_{p,0}^{(2)}\Delta t^2)\|x_p^{(k)}-x_p\|_n+(b_6+b_{10}\omega_{p,0}^{(2)}\Delta t)\Delta t\|y_p^{(k)}-y_p\|_m+\psi(\Delta t),\end{array}\tag{33}$$

where

$$\begin{aligned}&\psi(\Delta t)\\ =\ & b_1\sum_{i=1}^{q}\eta_i\|x_{p-i}^{(k+1)}-x_{p-i}\|_n+b_2\sum_{i=1}^{q}\eta_i\|x_{p-i}^{(k)}-x_{p-i}\|_n+\Delta t^2\sum_{j=1}^{p}\omega_{p,j}^{(2)}(b_7\|x_{p-j}^{(k+1)}-x_{p-j}\|_n\\ &+b_8\|x_{p-j}^{(k)}-x_{p-j}\|_n+b_9\|y_{p-j}^{(k+1)}-y_{p-j}\|_m+b_{10}\|y_{p-j}^{(k)}-y_{p-j}\|_m).\end{aligned}$$

We now write (32) and (33) together for $k=0,1,\dots$

$$\begin{array}{l}\begin{bmatrix}(1-a_1)\eta_0-a_3\Delta t-a_7\omega_{p,0}^{(1)}\Delta t^2 & -(a_5+a_9\omega_{p,0}^{(1)}\Delta t)\\ -(b_1\eta_0+b_3\Delta t+b_7\omega_{p,0}^{(2)}\Delta t^2) & (1-b_5)-b_9\omega_{p,0}^{(2)}\Delta t\end{bmatrix}\begin{bmatrix}\|x_p^{(k+1)}-x_p\|_n\\ \Delta t\|y_p^{(k+1)}-y_p\|_m\end{bmatrix}\\ \le\begin{bmatrix}a_2\eta_0+a_4\Delta t+a_8\omega_{p,0}^{(1)}\Delta t^2 & a_6+a_{10}\omega_{p,0}^{(1)}\Delta t\\ b_2\eta_0+b_4\Delta t+b_8\omega_{p,0}^{(2)}\Delta t^2 & b_6+b_{10}\omega_{p,0}^{(2)}\Delta t\end{bmatrix}\begin{bmatrix}\|x_p^{(k)}-x_p\|_n\\ \Delta t\|y_p^{(k)}-y_p\|_m\end{bmatrix}+\begin{bmatrix}\varphi(\Delta t)\\ \psi(\Delta t)\end{bmatrix}.\end{array}\tag{34}$$

Let us define

$$E_1(\Delta t)=\begin{bmatrix}(a_3+a_7\omega_{p,0}^{(1)}\Delta t)/\eta_0 & a_9\omega_{p,0}^{(1)}\\ (b_3+b_7\omega_{p,0}^{(2)}\Delta t)/\eta_0 & b_9\omega_{p,0}^{(2)}\end{bmatrix}$$

and

$$E_2(\Delta t) = \left[\begin{array}{cc} (a_4 + a_8\omega_{p,0}^{(1)}\Delta t)/\eta_0 & a_{10}\omega_{p,0}^{(1)} \\ (b_4 + b_8\omega_{p,0}^{(2)}\Delta t)/\eta_0 & b_{10}\omega_{p,0}^{(2)} \end{array} \right].$$

We may rewrite (34) by a simple form as

$$\begin{aligned} &[(I - H_1) - \Delta t E_1(\Delta t)]H_{\eta_0} \left[\begin{array}{c} \|x_p^{(k+1)} - x_p\|_n \\ \Delta t\|y_p^{(k+1)} - y_p\|_m \end{array} \right] \\ &\leq [H_2 + \Delta t E_2(\Delta t)]H_{\eta_0} \left[\begin{array}{c} \|x_p^{(k)} - x_p\|_n \\ \Delta t\|y_p^{(k)} - y_p\|_m \end{array} \right] + \left[\begin{array}{c} \varphi(\Delta t) \\ \psi(\Delta t) \end{array} \right], \end{aligned} \tag{35}$$

where

$$H_{\eta_0} = \left[\begin{array}{cc} \eta_0 & 0 \\ 0 & 1 \end{array} \right].$$

It is obvious that $[(I - H_1) - \Delta t E_1(\Delta t)]^{-1} \geq 0$ if $(I - H_1)^{-1} \geq 0$ and Δt is small. Under this restriction we also let

$$D(\Delta t) = H_{\eta_0}^{-1}[(I - H_1)^{-1} - \Delta t E_1(\Delta t)]^{-1}[H_2 + \Delta t E_2(\Delta t)]H_{\eta_0}.$$

Next, without loss of generality we assume that the function values x_p, y_p, $x_p^{(i)}$, and $y_p^{(i)}$ $(p = 0, 1, \ldots, q-1; i = k, k+1)$ are known. Based on this assumption, we can give a convergence condition. The condition is similar to that of Theorem 2 in the continuous-time case for small Δt. Let us define the large unknown vectors $W^{(i)}(\Delta t) = [\|x_q^{(i)} - x_q\|_n, \Delta t\|y_q^{(i)} - y_q\|_m, \ldots, \|x_N^{(i)} - x_N\|_n, \Delta t\|y_N^{(i)} - y_N\|_m]^t$ $(i = k, k+1)$ with $2(N-q+1)$ dimensions. By use of (35) and combing all $N - q + 1$ inequalities for p $(p = q, q+1, \ldots, N)$, we can know the form of the iterative matrix $S(\Delta t)$ $(\in \mathbf{R}^{2(N-q+1)\times 2(N-q+1)})$ about the discrete system. Namely, $S(\Delta t)$ is a block lower-triangular matrix with the block diagonal matrices $D(\Delta t)$. We also assume that $\varphi(\Delta t)$ and $\psi(\Delta t)$ are very small at each iteration. Thus, we have proven the following theorem for the convergence of (30).

Theorem 3. *Let (i) Conditions (L_f) and (L_h) be satisfied, (ii) $(I - H_1)^{-1} \geq 0$, and (iii) $\rho(H_0) < 1$. Then, for small enough Δt the vector sequence $\{[x_p^{(k)}, y_p^{(k)}]^t\}$ of the discrete-time WR algorithm (30) converges to the solution $[x_p, y_p]^t$ of (31) for any fixed $p(\in \{1, 2, \ldots, N\})$.*

The discrete form of the Picard iteration (6) by a q-step BDF method is

$$\left\{ \begin{array}{rcl} \frac{1}{\Delta t}\sum_{i=0}^{q}\eta_i x_{p-i}^{(k+1)} & = & f_1(\frac{1}{\Delta t}\sum_{i=0}^{q}\eta_i x_{p-i}^{(k)}, \frac{1}{\Delta t}\sum_{i=0}^{q}\eta_i x_{p-i}^{(k)}, x_p^{(k)}, x_p^{(k)}, y_p^{(k)}, y_p^{(k)}, u_p^{(k)}, t_p), \\ y_p^{(k+1)} & = & f_2(\frac{1}{\Delta t}\sum_{i=0}^{q}\eta_i x_{p-i}^{(k)}, \frac{1}{\Delta t}\sum_{i=0}^{q}\eta_i x_{p-i}^{(k)}, x_p^{(k)}, x_p^{(k)}, y_p^{(k)}, y_p^{(k)}, v_p^{(k)}, t_p), \\ u_p^{(k)} & = & \Delta t\sum_{j=0}^{p}\omega_{p,j}^{(1)} h_1(x_{p-j}^{(k)}, x_{p-j}^{(k)}, y_{p-j}^{(k)}, y_{p-j}^{(k)}, t_{p-j}, t_p), \\ v_p^{(k)} & = & \Delta t\sum_{j=0}^{p}\omega_{p,j}^{(2)} h_2(x_{p-j}^{(k)}, x_{p-j}^{(k)}, y_{p-j}^{(k)}, y_{p-j}^{(k)}, t_{p-j}, t_p), \\ & & \Delta t = t_p - t_{p-1}, \quad p = 1, 2, \ldots, N, \quad k = 0, 1, \ldots. \end{array} \right. \tag{36}$$

Let

$$P_{\eta_0} = \left[\begin{array}{cc} a_1 + a_2 & \frac{1}{\eta_0}(a_5 + a_6) \\ (b_1 + b_2)\eta_0 & b_5 + b_6 \end{array} \right].$$

It is obvious that $P_{\eta_0} = H_{\eta_0}{}^{-1} H H_{\eta_0}$. Similarly, we have

Theorem 4. *Let (i) Conditions* (L_f) *and* (L_h) *be satisfied and* $\rho(H) < 1$, *then, for small enough* Δt *the vector sequence* $\{[x_p^{(k)}, y_p^{(k)}]^t\}$ *of the discrete-time Picard iteration (36) converges to the solution* $[x_p, y_p]^t$ *of (31) for any fixed* $p(\in \{1, 2, \ldots, N\})$.

To start up the recursion (30) or (36) for a given waveform iteration (we denote it as the k_0-th one), one needs initial values for the first $q-1$ approximate solutions $x_0^{(k_0)}(= x_0)$, $x_1^{(k_0)}, \ldots, x_{q-1}^{(k_0)}$. If $q > 1$, to compute $x_i^{(k_0)}(1 \le i \le q-1)$, we may use an i-step BDF method in practical application. This is because we already have values for $x_0^{(k_0)}(= x_0)$, $x_1^{(k_0)}, \ldots, x_{i-1}^{(k_0)}$.

On the other hand, if a BDF method is applied to compute the decoupled systems for a fixed WR splitting the discrete point function value at a time-point t_p is often a solution of a nonlinear algebraic system, for example, see (30). Let us denote this system as $G(w_p^{(k+1)}) = 0$ where $w_p^{(k+1)} \in \mathbf{R}^n$. We can use the standard Newton method to solve the resulting algebraic system. That is, for a fixed k, we can construct an iterative process as

$$\begin{cases} w_{p,k}^{(i+1)} = w_{p,k}^{(i)} - \omega\left(\dfrac{\partial G(w_{p,k}^{(i)})}{\partial w_p}\right)^{-1} G(w_{p,k}^{(i)}), \\ w_{p,k}^{(0)} = w_p^{(k)}, \quad i = 0, 1, \ldots, \end{cases} \tag{37}$$

where ω is an algorithm weight. Or its variants are used for the wanted solution, see also [18].

4. Two Typical Nonlinear Systems

Let us first discuss a subsystem of (1), which is a simplified form of its first part, as

$$\dot{x}(t) = \tilde{f}\left(x(t), \int_0^t \tilde{h}(x(s), s, t)ds, t\right), \quad x(0) = x_0, \quad t \in [0, T], \tag{38}$$

where $x_0 \in \mathbf{R}^n$ is an initial value, $x(t)$ is to be computed, and $\tilde{h} : \mathbf{R}^n \times [0, T]^2 \to \mathbf{R}^l$ is a nonlinear function. The above system is described by nonlinear integral-differential equations (IDEs).

The Picard iteration and the WR algorithm of (38) are, respectively

$$\dot{x}^{(k+1)}(t) = \tilde{f}\left(x^{(k)}(t), \int_0^t \tilde{h}(x^{(k)}(s), s, t)ds, t\right), \quad x^{(k+1)}(0) = x_0, \quad t \in [0, T] \tag{39}$$

and

$$\dot{x}^{(k+1)}(t) = f\left(x^{(k+1)}(t), x^{(k)}(t), \int_0^t h(x^{(k+1)}(s), x^{(k)}(s), s, t)ds, t\right), \quad x^{(k+1)}(0) = x_0, \quad t \in [0, T]. \tag{40}$$

In(40) the splitting functions $f : (\mathbf{R}^n)^2 \times \mathbf{R}^l \times [0, T] \to \mathbf{R}^n$ and $h : (\mathbf{R}^n)^2 \times [0, T]^2 \to \mathbf{R}^l$ satisfy

$$f(x, x, z, t) = \tilde{f}(x, z, t), \quad h(x, x, s, t) = \tilde{h}(x, s, t),$$

where $x \in \mathbf{R}^n$, $z \in \mathbf{R}^l$, and $s, t, \in [0, T]$. As before we assume that f is Lipschitz continuous with respect to its first three arguments, similarly for h with respect to its first two. The corresponding matrices H, H_1, and H_0 are now zero matrices. Based on Theorems 1 and 2 the Picard iteration (39) and the WR algorithm (40) are convergent for the nonlinear system (38). Their discrete-time WR algorithm and discrete-time Picard iteration are

$$\begin{cases} \dfrac{1}{\Delta t}\displaystyle\sum_{i=0}^{q} \eta_i x_{p-i}^{(k+1)} = f(x_p^{(k+1)}, x_p^{(k)}, u_p^{(k+1,k)}, t_p), \ u_p^{(k+1,k)} = \Delta t \displaystyle\sum_{j=0}^{p} \omega_{p,j} h(x_{p-j}^{(k+1)}, x_{p-j}^{(k)}, t_{p-j}, t_p), \\ \qquad \Delta t = t_p - t_{p-1}, \quad p = 1, 2, \ldots, N, \quad k = 0, 1, \ldots \end{cases} \tag{41}$$

and

$$\begin{cases} \dfrac{1}{\Delta t}\sum_{i=0}^{q}\eta_i x_{p-i}^{(k+1)} = \tilde{f}(x_p^{(k)}, u_p^{(k)}, t_p), \quad u_p^{(k)} = \Delta t\sum_{j=0}^{p}\omega_{p,j}\tilde{h}(x_{p-j}^{(k)}, t_{p-j}, t_p), \\ \Delta t = t_p - t_{p-1}, \quad p = 1, 2, \ldots, N, \quad k = 0, 1, \ldots. \end{cases} \tag{42}$$

By the same reasoning as before the convergence conditions of the discrete versions (41) and (42) can be obtained by Theorems 3 and 4 for a small time-step Δt. If the system of (38) includes delays, the error estimates of its continuous-time WR could be also obtained by use of integral inequalities. This work was done in [20], which is different from our brief approach given here. Moreover, the discrete-time WR version of (40) is considered in [10], too. However, no convergence results were presented therein for the continuous-time WR case.

In the following content we discuss another nonlinear system, which is the second kind Volterra equation. That is

$$y(t) = g(t) + \int_0^t \tilde{h}(y(s), s, t)ds, \quad t \in [0, T]. \tag{43}$$

It is a simplified form of the second part in (1). Its Picard iteration is the following well-known format:

$$y^{(k+1)}(t) = g(t) + \int_0^t \tilde{h}(y^{(k)}(s), s, t)ds, \quad t \in [0, T]. \tag{44}$$

The WR algorithm of (43) is

$$y^{(k+1)}(t) = g(t) + \int_0^t h(y^{(k+1)}(s), y^{(k)}(s), s, t)ds. \quad t \in [0, T]. \tag{45}$$

The nonlinear splitting function h in the above iterative process is supposed to be Lipschitz continuous with respect to its first two arguments. Let the Lipschitz constants be d_3 and d_4. Now, for (44) and (45) the resulting matrices H, H_1, and H_0 are also zero. Thus, both of the iterations are convergent by Theorems 1 and 2.

In [11] the convergence of (45) is proven only on a small time interval $[0, T_1]$, namely T_1 should be restricted to $T_1 < \dfrac{1}{d_3 + d_4}$. This restriction is in fact not necessary due to our preceding analysis. The discrete versions of (44) and (45) are

$$\begin{cases} y_p^{(k+1)} = g_p + \Delta t\sum_{j=0}^{p}\omega_{p,j}\tilde{h}(y_{p-j}^{(k)}, t_{p-j}, t_p), \\ \Delta t = t_p - t_{p-1}, \quad p = 1, 2, \ldots, N, \quad k = 0, 1, \ldots \end{cases} \tag{46}$$

and

$$\begin{cases} y_p^{(k+1)} = g_p + \Delta t\sum_{j=0}^{p}\omega_{p,j}h(y_{p-j}^{(k+1)}, y_{p-j}^{(k)}, t_{p-j}, t_p), \\ \Delta t = t_p - t_{p-1}, \quad p = 1, 2, \ldots, N, \quad k = 0, 1, \ldots, \end{cases} \tag{47}$$

where $g_p = g(t_p)$. The above iterative processes are also convergent for a small time-step Δt by Theorems 3 and 4.

5. Initial Iterations

For an iterative process in function space the selection of its initial iterations is often crucial to convergence behaviors for early iterations. Referring to the above discussion we now may give a simple choice on initial iterations. This way needs only to solve a two-dimensional system of IDAEs.

First, we suppose that two basic guesses $[x_\alpha^{(0)}(\cdot), y_\alpha^{(0)}(\cdot)]^t$ and $[x_\beta^{(0)}(\cdot), y_\beta^{(0)}(\cdot)]^t$ of the WR algorithm (3) are already obtained where $x_\alpha^{(0)}(\cdot)$ and $x_\beta^{(0)}(\cdot)$ are continuously differential functions. These two guesses should conform to the consistent initial condition (2).

Denote that $z_\alpha^{(0)}(t) = \dot{x}_\alpha^{(0)}(t)$ and $z_\beta^{(0)}(t) = \dot{x}_\beta^{(0)}(t)$ on $[0, T]$. By a similar reasoning as (26) we may let the two-dimensional function $[e_z(\cdot), e_y(\cdot)]^t$ be a solution of the following linear system:

$$\begin{aligned}(I - H_1)\begin{bmatrix} e_z(t) \\ e_y(t) \end{bmatrix} &= H_2 \begin{bmatrix} e_z^{(0)}(t) \\ e_y^{(0)}(t) \end{bmatrix} + \begin{bmatrix} (a_3 + a_7 t)\mathcal{V}_c & a_9\mathcal{V}_c \\ (b_3 + b_7 t)\mathcal{V}_c & b_9\mathcal{V}_c \end{bmatrix} \begin{bmatrix} e_z(t) \\ e_y(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} (a_4 + a_8 t)\mathcal{V}_c & a_{10}\mathcal{V}_c \\ (b_4 + b_8 t)\mathcal{V}_c & b_{10}\mathcal{V}_c \end{bmatrix} \begin{bmatrix} e_z^{(0)}(t) \\ e_y^{(0)}(t) \end{bmatrix}, \quad t \in [0, T],\end{aligned} \tag{48}$$

where $e_z^{(0)} = \|z_\alpha^{(0)}(t) - z_\beta^{(0)}(t)\|_n$ and $e_y^{(0)} = \|y_\alpha^{(0)}(t) - y_\beta^{(0)}(t)\|_m$. For $e_z(\cdot)$ in (48) we define $e_x(t) = \int_0^t e_z(s)ds$. The initial iteration of (3) can be now taken as

$$\begin{cases} x^{(0)}(t) = x_0 + e_x(t)u_x, \\ y^{(0)}(t) = y_0 + e_y(t)u_y, \quad t \in [0, T], \end{cases} \tag{49}$$

where $u_x = [1, \ldots, 1]^t \in \mathbf{R}^n$ and $u_y = [1, \ldots, 1]^t \in \mathbf{R}^m$.

For the Picard iteration (6), by use of (19) the two-dimensional function $[e_z(\cdot), e_y(\cdot)]^t$ is directly given by

$$\begin{bmatrix} e_z(t) \\ e_y(t) \end{bmatrix} = H \begin{bmatrix} e_z^{(0)}(t) \\ e_y^{(0)}(t) \end{bmatrix} + \begin{bmatrix} (a_3 + a_4)\mathcal{V}_c + (a_7 + a_8)t\mathcal{V}_c & (a_9 + a_{10})\mathcal{V}_c \\ (b_3 + b_4)\mathcal{V}_c + (b_7 + b_8)t\mathcal{V}_c & (b_9 + b_{10})\mathcal{V}_c \end{bmatrix} \begin{bmatrix} e_z^{(0)}(t) \\ e_y^{(0)}(t) \end{bmatrix}, \tag{50}$$

where $t \in [0, T]$.

Next, let us consider the choice of $[x_\alpha^{(0)}(\cdot), y_\alpha^{(0)}(\cdot)]^t$ and $[x_\beta^{(0)}(\cdot), y_\beta^{(0)}(\cdot)]^t$. In practical applications we may choose them by approximate solutions of (1) or its experimental values. As an alternative way we can let $x_\alpha^{(0)}(t) = x_0$ and $y_\alpha^{(0)}(t) = y_0$ for $t \in [0, T]$. For $[x_\beta^{(0)}(\cdot), y_\beta^{(0)}(\cdot)]^t$ we adopt a simplified version of the means presented in [21]. Let

$$x_\beta^{(0)}(t) = x_\beta(t)x_0, \quad y_\beta^{(0)}(t) = y_\beta(t)y_0, \quad t \in [0, T]. \tag{51}$$

The one-dimensional functions $x_\beta(t)$ and $y_\beta(t)$ can be found by minimizing

$$\|\dot{x}_\beta(t)x_0 - \tilde{f}_1(\dot{x}_\beta(t)x_0, x_\beta(t)x_0, y_\beta(t)y_0, \int_0^t \tilde{h}_1(x_\beta(s)x_0, y_\beta(s)y_0, s, t)ds, t)\|_n \tag{52}$$

and

$$\|y_\beta(t)y_0 - \tilde{f}_2(\dot{x}_\beta(t)x_0, x_\beta(t)x_0, y_\beta(t)y_0, \int_0^t \tilde{h}_2(x_\beta(s)x_0, y_\beta(s)y_0, s, t)ds, t)\|_m \tag{53}$$

with respect to $\dot{x}_\beta(t)$ and $y_\beta(t)$ on $[0, T]$.

If x_0, y_0 are not zero vectors and the above norms are the 2-norm, then (52) and (53) are equivalent to

$$\begin{cases} \dot{x}_\beta(t) = (x_0^t x_0)^{-1} x_0^t \tilde{f}_1(\dot{x}_\beta(t)x_0, x_\beta(t)x_0, y_\beta(t)y_0, \int_0^t \tilde{h}_1(x_\beta(s)x_0, y_\beta(s)y_0, s, t)ds, t), \\ y_\beta(t) = (y_0^t y_0)^{-1} y_0^t \tilde{f}_2(\dot{x}_\beta(t)x_0, x_\beta(t)x_0, y_\beta(t)y_0, \int_0^t \tilde{h}_2(x_\beta(s)x_0, y_\beta(s)y_0, s, t)ds, t), \\ x_\beta(0) = 1, \quad y_\beta(0) = 1, \quad t \in [0, T]. \end{cases} \tag{54}$$

If $x_0 = 0$ or $y_0 = 0$ we may let $x_\beta^{(0)}(t) = x_\beta(t)\tilde{x}_0$ where $x_\beta(0) = 0$ and $\tilde{x}_0 = [1, \ldots, 1]^t \in \mathbf{R}^n$ or $y_\beta^{(0)}(t) = y_\beta(t)\tilde{y}_0$ where $y_\beta(0) = 0$ and $\tilde{y}_0 = [1, \ldots, 1]^t \in \mathbf{R}^m$. For this situation, a similar relationship like (52) or (53) holds. Then, we also have a system like as (54). Here, we omit these expressions.

6. Numerical Experiments

Except the Picard splitting (6) we now also present some classic WR partitions of (1). They are the Jacobi and Gauss-Seidel splittings. By the splitting functions (4) and (5) we can write out their concrete expressions.

The Jacobi WR algorithm of (1) on $[0,T]$ for $k=0,1,\ldots$:

$$\begin{aligned}\dot{x}_i^{(k+1)}(t) &= (\tilde{f}_1)_i(\dot{x}_1^{(k)}(t),\ldots,\dot{x}_{i-1}^{(k)}(t),\dot{x}_i^{(k+1)}(t),\dot{x}_{i+1}^{(k)}(t),\ldots,\dot{x}_n^{(k)}(t),x_1^{(k)}(t),\ldots,x_{i-1}^{(k)}(t),\\ &\quad x_i^{(k+1)}(t),x_{i+1}^{(k)}(t),\ldots,x_n^{(k)}(t),y_1^{(k)}(t),\ldots,y_m^{(k)}(t),\int_0^t(\tilde{h}_1)_1^{(k+1,k)}(s,t)ds,\ldots,\\ &\quad \int_0^t(\tilde{h}_1)_{l_1}^{(k+1,k)}(s,t)ds,t),\quad i=1,2,\ldots,n,\end{aligned}$$

where $(\tilde{h}_1)_p^{(k+1,k)}(s,t)=(\tilde{h}_1)_p(x_1^{(k)}(s),\ldots,x_{i-1}^{(k)}(s),x_i^{(k+1)}(s),x_{i+1}^{(k)}(s),\ \ldots,\ x_n^{(k)}(s),\ y_1^{(k)}(s),\ \ldots,$ $y_m^{(k)}(s),s,t)$ for $p=1,2,\ldots,l_1$;

$$\begin{aligned}y_j^{(k+1)}(t) &= (\tilde{f}_2)_j(\dot{x}_1^{(k)}(t),\ldots,\dot{x}_n^{(k)}(t),x_1^{(k)}(t),\ldots,x_n^{(k)}(t),y_1^{(k)}(t),\ldots,y_{j-1}^{(k)}(t),y_j^{(k+1)}(t),\\ &\quad y_{j+1}^{(k)}(t),\ldots,y_m^{(k)}(t),\int_0^t(\tilde{h}_2)_1^{(k+1,k)}(s,t)ds,\ldots,\int_0^t(\tilde{h}_2)_{l_2}^{(k+1,k)}(s,t)ds,t),\\ &\quad j=1,2,\ldots,m,\end{aligned}$$

where $(\tilde{h}_2)_p^{(k+1,k)}(s,t)=(\tilde{h}_2)_p(x_1^{(k)}(s),\ldots,x_n^{(k)}(s),\ y_1^{(k)}(s),\ldots,\ y_{j-1}^{(k)}(s),\ y_j^{(k+1)}(s),\ y_{j+1}^{(k)}(s),\ \ldots,$ $y_m^{(k)}(s),s,t)$ for $p=1,2,\ldots,l_2$.

The Gauss-Seidel WR algorithm of (1) on $[0,T]$ for $k=0,1,\ldots$:

$$\begin{aligned}\dot{x}_i^{(k+1)}(t) &= (\tilde{f}_1)_i(\dot{x}_1^{(k+1)}(t),\ldots,\dot{x}_i^{(k+1)}(t),\dot{x}_{i+1}^{(k)}(t),\ldots,\dot{x}_n^{(k)}(t),x_1^{(k+1)}(t),\ldots,x_i^{(k+1)}(t),\\ &\quad x_{i+1}^{(k)}(t),\ldots,x_n^{(k)}(t),y_1^{(k)}(t),\ldots,y_m^{(k)}(t),\int_0^t(\tilde{h}_1)_1^{(k+1,k)}(s,t)ds,\ldots,\\ &\quad \int_0^t(\tilde{h}_1)_{l_1}^{(k+1,k)}(s,t)ds,t),\quad i=1,2,\ldots,n,\end{aligned}$$

where $(\tilde{h}_1)_p^{(k+1,k)}(s,t)=(\tilde{h}_1)_p(x_1^{(k+1)}(s),\ldots,x_i^{(k+1)}(s),x_{i+1}^{(k)}(s),\ \ldots,\ x_n^{(k)}(s),\ y_1^{(k)}(s),\ \ldots,\ y_m^{(k)}$ $(s),\ s,\ t)$ for $p=1,2,\ldots,l_1$;

$$\begin{aligned}y_j^{(k+1)}(t) &= (\tilde{f}_2)_j(\dot{x}_1^{(k+1)}(t),\ldots,\dot{x}_n^{(k+1)}(t),x_1^{(k+1)}(t),\ldots,x_n^{(k+1)}(t),y_1^{(k+1)}(t),\ldots,y_j^{(k+1)}(t),\\ &\quad y_{j+1}^{(k)}(t),\ldots,y_m^{(k)}(t),\int_0^t(\tilde{h}_2)_1^{(k+1,k)}(s,t)ds,\ldots,\int_0^t(\tilde{h}_2)_{l_2}^{(k+1,k)}(s,t)ds,t),\\ &\quad j=1,2,\ldots,m,\end{aligned}$$

where $(\tilde{h}_2)_p^{(k+1,k)}(s,t)=(\tilde{h}_2)_p(x_1^{(k+1)}(s),\ldots,x_n^{(k+1)}(s),\ y_1^{(k+1)}(s),\ldots,\ y_j^{(k+1)}(s),\ y_{j+1}^{(k)}(s),\ \ldots,$ $y_m^{(k)}(s),s,t)$ for $p=1,2,\ldots,l_2$.

Our numerical experiments are based on a test system which has the same form of (1). Its equations are

$$\left\{\begin{array}{lcl}
\frac{dx_1(t)}{dt} & = & \frac{1}{3}\frac{dx_2(t)}{dt} + tanh(y_1(t)-x_1(t)) + \int_0^t cos(x_1(s)-y_1(s))ds + sin(4\pi t),\\
\frac{dx_2(t)}{dt} & = & \frac{1}{2}\frac{dx_1(t)}{dt} + \frac{1}{3}\frac{dx_3(t)}{dt} + tanh(y_1(t)-x_2(t)) + tanh(y_2(t)-x_2(t))\\
 & & +2\int_0^t cos(x_2(s)-x_4(s))ds,\\
\frac{dx_3(t)}{dt} & = & \frac{1}{2}\frac{dx_2(t)}{dt} + \frac{1}{3}\frac{dx_4(t)}{dt} + tanh(y_1(t)-x_3(t)) + tanh(y_2(t)-x_3(t))\\
 & & +2\int_0^t cos(x_3(s)-x_1(s))ds,\\
\frac{dx_4(t)}{dt} & = & \frac{1}{3}\frac{dx_3(t)}{dt} + tanh(y_2(t)-x_4(t)) + \int_0^t cos(x_4(s)-y_2(s))ds,\\
y_1(t) & = & \frac{1}{5}tanh(y_1(t)) + \frac{1}{5}tanh(y_1(t)-x_1(t)) + \frac{1}{5}tanh(y_1(t)-x_2(t))\\
 & & +\frac{1}{5}tanh(y_1(t)-x_3(t)) + \int_0^t sin(y_1(s)-x_2(s))ds + t,\\
y_2(t) & = & \frac{1}{5}tanh(y_2(t)) + \frac{1}{5}tanh(y_2(t)-x_2(t)) + \frac{1}{5}tanh(y_2(t)-x_3(t))\\
 & & +\frac{1}{5}tanh(y_2(t)-x_4(t)) + \int_0^t sin(y_2(s)-x_3(s))ds,\\
 & & [x_1(0), x_2(0), x_3(0), x_4(0)]^t = [0,0,0,0]^t, \quad t\in[0,1],
\end{array}\right. \tag{55}$$

where $tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}$. Let us consider its numerical solutions by WR.

We respectively use the Picard iteration, the Jacobi and Gauss-Seidel splittings to numerically solve (55). Their discrete waveforms are computed by the 3-step BDF method where $\eta_0 = \frac{11}{6}, \eta_1 = -3, \eta_2 = \frac{3}{2}$, and $\eta_3 = -\frac{1}{3}$. The initial values of the BDF method are respectively given by the 2-step BDF method ($\eta_0 = \frac{3}{2}, \eta_1 = -2$, and $\eta_2 = \frac{1}{2}$) and the Euler method in our computations. Integrals are computed by the Trapezoid method, that is

$$\int_0^{t_p} \varrho(s)ds = \frac{\Delta t(\varrho(t_0) + \varrho(t_p))}{2} + \Delta t \sum_{i=1}^{p-1} \varrho(t_i)$$

for $p \geq 2$.

Let us now employ the 2-norm to give rise the Lipschitz constants of the splitting functions in (55). We need only to know the constants a_i and b_i $(i = 1, 2, \ldots, 6)$ for the checking convergence purpose. They may be easily taken as

$$\left\{\begin{array}{l}
a_1 = 0,\ a_2 = \frac{\sqrt{13}}{6},\ a_3 = 0,\ a_4 = \sqrt{2},\ a_5 = 0,\ a_6 = \sqrt{3},\\
b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ b_4 = \frac{\sqrt{2}}{5},\ b_5 = 0,\ b_6 = \frac{2}{5}
\end{array}\right.$$

for the Picard iteration;

$$\left\{\begin{array}{l}
a_1 = 0,\ a_2 = \frac{\sqrt{13}}{6},\ a_3 = \sqrt{2},\ a_4 = 0,\ a_5 = 0, a_6 = \sqrt{3},\\
b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ b_4 = \frac{\sqrt{2}}{5},\ b_5 = \frac{2}{5},\ b_6 = 0
\end{array}\right.$$

for the Jacobi splitting;

$$\begin{cases} a_1 = \dfrac{1}{2},\ a_2 = \dfrac{1}{3},\ a_3 = \sqrt{2},\ a_4 = 0,\ a_5 = 0,\ a_6 = \sqrt{3}, \\ b_1 = 0,\ b_2 = 0,\ b_3 = \dfrac{\sqrt{2}}{5},\ b_4 = 0,\ b_5 = \dfrac{2}{5},\ b_6 = 0 \end{cases}$$

for the Gauss-Seidel splitting. These Lipschitz constants obviously satisfy the previous convergence conditions of the continuous-time and discrete-time versions for the Picard iteration, the Jacobi splitting, and the Gauss-Seidel splitting.

Let the time-step Δt be 0.01. The number of the Newton iteration is taken as 5 in our numerical solver. The iterative error is defined as the sum of the squared difference of successive waveforms taken over all time-points, namely for two computed waveforms $w^{(i)}(t)(i = k-1, k)$ the iterative error is $Error(k) = \sqrt{\sum_{p=0}^{100} \|w^{(k)}(t_p) - w^{(k-1)}(t_p)\|^2}$ where $t_p = p\Delta t$ $(p = 0, 1, \ldots, 100)$. The computed results of the three WR algorithms, where the initial guesses are the zero function, are shown in Figure 1.

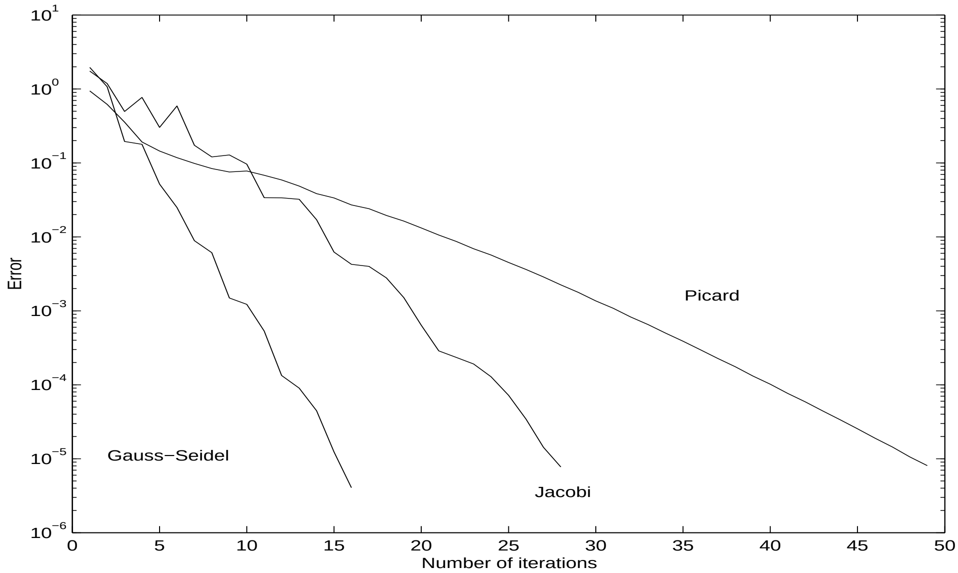

Figure 1: Convergence behaviors of the Picard iteration, the Jacobi WR algorithm, and the Gauss-Seidel WR algorithm for a nonlinear system.

We also use the constructed way shown in Section 5 to produce new initial guesses. During the process we need to form a simple system like (54). For our example, the system is

$$\begin{cases} \dot{x}_\beta(t) = \dfrac{18}{5} tanh(y_\beta(t) - x_\beta(t)) + \dfrac{6}{5}\displaystyle\int_0^t cos(x_\beta(s) - y_\beta(s))ds + \dfrac{3}{5} sin(4\pi t), \\ y_\beta(t) = \dfrac{1}{5} tanh(y_\beta(t)) + \dfrac{3}{5} tanh(y_\beta(t) - x_\beta(t)) + \displaystyle\int_0^t sin(y_\beta(s) - x_\beta(s))ds + \dfrac{1}{2}t, \\ x_\beta(0) = 0, \quad y_\beta(0) = 0, \quad t \in [0, 1]. \end{cases} \tag{56}$$

The numerical solution of the above system is simply solved by the Euler method.

Table 1. Numerical results of the Picard iteration with two different initial guesses: zero and nonzero

No. of it.	Error: zero/nonzero		
1 – 3	9.4158e-001/2.7235e+000,	6.1923e-001/1.5254e+000,	3.5675e-001/8.8115e-001
4 – 6	1.9240e-001/5.2744e-001,	1.4483e-001/3.2727e-001,	1.1792e-001/3.1063e-001
7 – 9	9.8432e-002/3.3499e-001,	8.4077e-002/2.7162e-001,	7.5591e-002/1.8185e-001
10 – 12	7.7739e-002/1.1693e-001,	6.8124e-002/7.4462e-002,	5.9003e-002/6.1977e-002
13 – 15	4.8666e-002/5.2232e-002,	3.8486e-002/4.2183e-002,	3.3674e-002/3.1806e-002
16 – 18	2.7093e-002/2.1586e-002,	2.4048e-002/1.4882e-002,	1.9506e-002/1.0969e-002
19 – 21	1.6346e-002/8.8143e-003,	1.3232e-002/7.7069e-003,	1.0608e-002/6.0277e-003

Table 2. Numerical results of the Jacobi splitting with two different initial guesses: zero and nonzero

No. of it.	Error: zero/nonzero		
1 – 3	1.7494e+000/4.7752e+000,	1.1830e+000/2.4166e+000,	4.9620e-001/1.4248e+000
4 – 6	7.6956e-001/7.7127e-001,	3.0282e-001/3.7690e-001,	5.8928e-001/1.1780e-001
7 – 9	1.7311e-001/5.8050e-002,	1.2075e-001/4.3542e-002,	1.2815e-001/4.3404e-002
10 – 12	9.6286e-002/1.9538e-002,	3.3875e-002/4.4143e-003,	3.3772e-002/3.2988e-003
13 – 15	3.2218e-002/2.0257e-003,	1.7085e-002/1.4448e-003,	6.2325e-003/1.2880e-003

Table 3. Numerical results of the Gauss-Seidel splitting with two different initial guesses: zero and nonzero

No. of it.	Error: zero/nonzero		
1 – 3	1.9599e+000/1.9262e+002,	1.0743e+000/2.6426e+001,	1.9491e-001/1.1807e+001
4 – 6	1.7853e-001/5.0226e+000,	5.1695e-002/1.8328e+000,	2.5007e-002/6.2396e-001
7 – 9	8.8815e-003/2.0024e-001,	6.1133e-003/4.3879e-002,	1.4943e-003/6.7010e-003

The first twenty-one, fifteen, and nine iterations for the Picard algorithm, the Jacobi WR algorithm, and the Gauss-Seidel WR algorithm, respectively, with the zero initial guess and the nonzero initial guess, are given in Tables 1, 2, and 3. The new initial guess can improve the convergence of the Picard algorithm and the Jacobi WR algorithm, especially for the former. The convergence effect based on the two different initial guesses is almost the same for the Gauss-Seidel WR algorithm. This is because the Gauss-Seidel WR algorithm has good convergence behavior and the first several iterations with the new initial guess has large iterative errors for this case.

7. Conclusions

For continuous-time and discrete-time waveform relaxation (WR) we have presented convergence conditions for a general system of nonlinear integral-differential-algebraic equations (IDAEs). In the continuous-time case the convergence condition, which tightly relates to Lipschitz constants of splitting functions in the decoupled system, is smoothly derived by a spectral property on linear operators. By referring to the continuous-time condition we also successfully show the convergence of discrete waveforms resulted from a backward-differentiation formula for small time-steps. The theoretical results reported here are new for nonlinear IDAEs in the WR literature. The convergence conditions are suitable to practical applications in parallel processing.

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