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A NUMERICAL EMBEDDING METHOD FOR SOLVING THE NONLINEAR COMPLEMENTARITY PROBLEM(I)—THEORY*[1)]

Jian-jun Zhang De-ren Wang
(*Department of Mathematics, Shanghai University, Shanghai 200436, China*)

Abstract

In this paper, we extend the numerical embedding method for solving the smooth equations to the nonlinear complementarity problem. By using the nonsmooth theory, we prove the existence and the continuation of the following path for the corresponding homotopy equations. Therefore the basic theory of the numerical embedding method for solving the nonlinear complementarity problem is established. In part II of this paper, we will further study the implementation of the method and give some numerical exapmles.

Key words: B-differentiable equations, Nnlinear complementarity problem, Nmerical embedding method.

1. Introduction

The nonlinear complementarity problem is a very important mathematical programming problem. The development of theory and algorithms for this problem has a long history and there have numerous methods for solving this problem. See [1] for a comprehensive review of the literature.

Recently, based on the B-differentiable equations approach, many new methods for solving the nonlinear complementarity problem have been proposed. Harker and Xiao[2] established a damped-Newton method for solving the nonlinear complementarity problem and provided many numerical results. Pang and Garbriel[7] proposed an NE/SQP method for solving the nonlinear complementarity problem and proved its global and local quadratically convergence. These methods are important to the theories and algorithms for solving the nonlinear complementarity problem.

In this paper, based on the B-differentiable equations theory, we will study: how to extend the practical numerical embedding method to the nonlinear complementarity problem; How to prove the existence, the uniqueness and the continuation of the following path for the corresponding homotopy equations by using the B-differentiable theory; How to solve the nonlinear complementarity problem by numerical embedding method proposed in this paper. All this questions will be studied in this paper and the subsequent paper.

2. Preliminaries

We consider the following nonlinear complementarity problem:
Find $x \in R^n$ such that

$$x \geq 0, \quad f(x) \geq 0, \quad and \quad x^T f(x) = 0 \tag{2.1}$$

where $f : R^n \to R^n$ satisfies $|| \bigtriangledown f(x) - \bigtriangledown f(y)|| \leq L||x-y||_0$. This problem is denoted by NCP(f).

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Following the concept of a Minty-map[2][3], the NCP(f) can be converted into the B-differentiable equations:

$$F(x)=0 \tag{2.2}$$

where $F: R^n \to R^n$ is defined by

$$F(x)=f(x^+)+x^- \tag{2.3}$$

with $x_i^+ = \max(x_i,0)$, $x_i^- = \min(x_i,0)$, $x^+ = (x_1^+, x_2^+, \cdots, x_n^+)^T$ and $x^- = (x_1^-, x_2^-, \cdots, x_n^-)^T$. In other words, x solves (2.2) if and only if x^+ solves the nonlinear complementarity problem (2.1). Hence, by solving the systems (2.2), we can get the solution of the NCP(f) (2.1).

In order to present some properties of the mapping F defined by (2.3), let us review some notions in nonsmooth analysis.

The following definition is due to Robinson[13].

Definition 2.1.[6][13] *A function $F: D \subset R^n \to R^n$ is B-differentiable at a point $x \in D$ if there exists a positively homogeneous function $BF(x): R^n \to R^n$ (i.e., $BF(x)(\lambda v) = \lambda BF(x)v$ for all $\lambda \geq 0$ and $v \in R^n$), called the B-derivative of F at x, such that*

$$\lim_{v\to 0}[F(x+v)-F(x)-BF(x)v]/\|v\|=0.$$

If F is B-differentiable at all points $x \in D$, then F is called B-differentiable on D.

In a finite-dimensional Euclidean space R^n, Shapiro[14] showed that F is B-differentiable at x if and only if it is directionally differentiable at x. In this case, the B-derivative and the directional derivative are identical.

The basic properties of a B-differentiable function are summarized in the following theorem.

Theorem 2.2[6]. *Let $F: R^n \to R^n$ be locally Lipschitz continuous at a point x.*

(1) *If F is Fréchet differentiable at x, then it is B-differentiable at x and $BF(x) = \nabla F(x)$. Conversely, if F is B-differentiable at x and if the B-derivative $BF(x)v$ is linear in v, then F is Fréchet differentiable at x.*

(2) *If F is B-differentiable at x, then the B-derivative is unique. Moreover, $BF(x)$ is Lipschitz continuous with the same modulus as F.*

(3) *If F is B-differentiable at x, then F is directionally differentiable at x in any direction and $F'(x,d) = BF(x)d$.*

(4) *The addition, subtraction and chain rules hold for the B-derivative.*

Extending the notion of a strong F-derivative, Robinson[13] further introduced the following definition.

Definition 2.3.[13] *A function $F: D \subset R^n \to R^n$ is strong B-differentiable at a point $x \in D$ if F is B-differentiable and*

$$\lim_{\substack{y\to x\\ z\to x}}\{F(y)-F(z)-[BF(x)(y-x)-BF(x)(z-x)]\}/\|y-z\|=0.$$

If F is strong B-differentiable at all points $x \in R^n$, then F is called strong B-differentiable on D.

Using the above definitions, it is easy to prove that[2], the function F is B-differentiable everywhere, and its B-derivative is

$$(BF(x)v)_i = \sum_{j=1}^{n} BF_i^j(x)v_j, \tag{2.4}$$

where

$$BF_i^j(x)v_j = \begin{cases} f_{ij}(x^+)v_j & j \in \alpha(x) \\ f_{ij}(x^+)v_j^+ + I_{ij}v_j^- & j \in \beta(x) \\ I_{ij}v_j & j \in \gamma(x) \end{cases}$$

with $\alpha(x) = \{i|x_i(x) > 0\}$, $\beta(x) = \{i|x_i = 0\}$, $\gamma(x) = \{i|x_i < 0\}$.

$$f_{ij} = \partial f_i/\partial x_j, \quad I_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

For the convenience, let $\alpha = \alpha(x)$, $\beta = \beta(x)$ *and* $\gamma = \gamma(x)$. Following Harker and Xiao[2], we give the following definition.

Definition 2.4.[2] *We say that x is a regular vector for the function $F(x) = f(x^+) + x^-$, if*
(1) *$f_{\alpha\alpha}(x^+)$ is nonsingular,*
(2) *the Schur complement $f_{\beta\beta}(x^+) - f_{\beta\alpha}(x^+)f_{\alpha\alpha}^{-1}(x^+)f_{\alpha\beta}(x^+)$ is a P-matrix.*
The following theorems will be used in the subsequent discussion.
Theorem 2.5.[6] *Consider the mixed linear complementarity problem:*

$$\begin{cases} w = p + Mu + Nv \\ 0 = q + Xu + Yv \\ u \geq 0, \ w \geq 0, \ u^T w = 0, \end{cases} \tag{2.5}$$

where M and Y are square matrices.
(1) *A necessary and sufficient condition for a solution to exist and be unique for all vectors p and q is that*
(a) *Y is nonsingular;*
(b) *$M - NY^{-1}X$ is a P-matrix.*
(2) *If assumptions (a) and (b) of part (1) hold, then as a function in (p, q), the unique solution (u, v) is Lipschitzian.*
Theorem 2.6.[2] *Suppose the function $F(x) = f(x^+) + x^-$ is regular at x, then F is regular in a neighborhood of x. That is, if α', β' are some index sets satisfying $\alpha \subseteq \alpha' \subseteq \alpha \cup \beta, \beta' \subseteq \beta$ and $\alpha' \cap \beta' = \phi$, then*
(1) *$f_{\alpha'\alpha'}(x^+)$ is nonsingular,*
(2) *the Schur complement $f_{\beta'\beta'}(x^+) - f_{\beta'\alpha'}(x^+)f_{\alpha'\alpha'}^{-1}(x^+)f_{\alpha'\beta'}(x^+)$ is a P-matrix.*
Corollary 2.7. *Suppose F is regular at x. Then the matrix*

$$\begin{bmatrix} f_{\alpha\alpha} & f_{\alpha\beta^+} & 0 & 0 \\ f_{\beta^+\alpha} & f_{\beta^+\beta^+} & 0 & 0 \\ f_{\beta^-\alpha} & f_{\beta^-\beta^+} & I & 0 \\ f_{\gamma\alpha} & f_{\gamma\beta^+} & 0 & I \end{bmatrix}$$

is nonsingular, where the index sets β^+, β^- satisfy $\beta^+ \cup \beta^- = \beta$, $\beta^+ \cap \beta^- = \phi$.

3. The Existence and the Continuation of the Following Path

Consider the solution of the B-differentiable equations

$$F(x) = f(x^+) + x^- = 0 \tag{3.1}$$

we construct the embedding map $H : R^n \times [0,1] \to R^n$ as follows:

$$H(x,t) = F(x) + (t-1)F(x^0), \quad t \in [0,1] \tag{3.2}$$

where x^0 is some point in R^n. It is easy to see that the equation $H(x,1) = 0$ is equivalent to (3.1).

We now consider the solution of the equation

$$H(x,t) = 0. \tag{3.3}$$

For this purpose, we have to answer the following questions: (1). Does the equation (3.3) has a solution curve ? (2). If it does, how many solution curve?

The following theorems will answer the above questions.

Theorem 3.1. *Let $(x^*, t^*) \in R^n \times (0,1)$ satisfy (3.3) and F is regular at x^*. Then there exist neighborhoods $U \subset S(x^*, r) = \{x| \, \|x - x^*\| < r\}$ of x^* and $V \subset (0,1)$ of t^*, and a function $x : \overline{V} \to \overline{U}$, having the following properties*

$$\begin{cases} x(t^*) = x^* \\ H(x(t),t) = F(x(t)) + (t-1)F(x^0) = 0, \quad \forall \;\; t \in \overline{V} \end{cases}$$

Proof. Let the function $N(x,t) : R^n \times [0,1] \to R^n$ be defined according to the following equations:

$$BF(x^*)(N(x,t) - x^*) = BF(x^*)(x - x^*) - H(x,t). \tag{3.4}$$

We now prove that the function $N(x,t)$ defined by (3.4) exists and is unique, and is continuous with respect to (x,t).

Constitute the (2.4) of the B-derivative of F in (3.4), then (3.4) can be rewritten as

$$\left.\begin{aligned} & f_{\alpha\alpha}(N(x,t) - x^*)_\alpha + f_{\alpha\beta}(N(x,t) - x^*)^+_\beta \\ & = f_{\alpha\alpha}(x - x^*)_\alpha + f_{\alpha\beta}(x - x^*)^+_\beta - H_\alpha(x,t) \\ & f_{\beta\alpha}(N(x,t) - x^*)_\alpha + f_{\beta\beta}(N(x,t) - x^*)^+_\beta + (N(x,t) - x^*)^-_\beta \\ & = f_{\beta\alpha}(x - x^*)_\alpha + f_{\beta\beta}(x - x^*)^+_\beta + (x - x^*)^-_\beta - H_\beta(x,t) \\ & f_{\gamma\alpha}(N(x,t) - x^*)_\alpha + f_{\gamma\beta}(N(x,t) - x^*)^+_\beta + (N(x,t) - x^*)_\gamma \\ & = f_{\gamma\alpha}(x - x^*)_\alpha + f_{\gamma\beta}(x - x^*)^+_\beta + (x - x^*)_\gamma - H_\gamma(x,t) \end{aligned}\right\} \tag{3.5}$$

where

$\alpha = \alpha(x^*)$, $\beta = \beta(x^*)$, $\gamma = \gamma(x^*)$,
$f_{\gamma\alpha} = f_{\gamma\alpha}(x^{*+})$, $f_{\gamma\beta} = f_{\gamma\beta}(x^{*+})$, $f_{\alpha\alpha} = f_{\alpha\alpha}(x^{*+})$,
$f_{\alpha\beta} = f_{\alpha\beta}(x^{*+})$, $f_{\beta\alpha} = f_{\beta\alpha}(x^{*+})$, $f_{\beta\beta} = f_{\beta\beta}(x^{*+})$,
with $\quad x^{*+} = (x^*)^+$.

For the convenience, let us define

$$\begin{aligned} & q_\alpha(x,t) = -(f_{\alpha\alpha}(x - x^*)_\alpha + f_{\alpha\beta}(x - x^*)^+_\beta - H_\alpha(x,t)) \\ & q_\beta(x,t) = -(f_{\beta\alpha}(x - x^*)_\alpha + f_{\beta\beta}(x - x^*)^+_\beta + (x - x^*)^-_\beta - H_\beta(x,t)) \\ & q_\gamma(x,t) = -(f_{\gamma\alpha}(x - x^*)_\alpha + f_{\gamma\beta}(x - x^*)^+_\beta + (x - x^*)_\gamma - H_\gamma(x,t)) \\ & w_\beta = q_\beta(x,t) + f_{\beta\alpha}(N(x,t) - x^*)_\alpha + f_{\beta\beta}(N(x,t) - x^*)^+_\beta \end{aligned}$$

Notice that $w_\beta = -(N(x,t) - x^*)^-_\beta$, the system (3.5) is equivalent to the following mixed linear complementarity problem:

$$\begin{cases} q_\alpha(x,t) + f_{\alpha\alpha}(N(x,t) - x^*)_\alpha + f_{\alpha\beta}(N(x,t) - x^*)^+_\beta = 0 \\ w_\beta \geq 0, \quad (N(x,t) - x^*)^+_\beta \geq 0, \quad w^T_\beta (N(x,t) - x^*)^+_\beta = 0 \\ (N(x,t) - x^*)_\gamma = -q_\gamma - f_{\gamma\alpha}(N(x,t) - x^*)_\alpha - f_{\gamma\beta}(N(x,t) - x^*)^+_\beta \end{cases} \tag{3.6}$$

Therefore, by Theorem 2.5 the function $N(x,t)$ defined by (3.4) is uniquely defined. Since $q_\alpha(x,t), q_\beta(x,t), q_\gamma(x,t)$ is continuous with respect to (x,t), by (3.6) and Theorem 2.5, the function $N(x,t)$ is continuous with respect to (x,t).

Define $\beta^+ = \{i|\ i \in \beta(x^*), (N(x,t) - x^*)_i > 0\}$, $\beta^- = \beta \backslash \beta^+$. Then (3.6) can be rewritten as

$$\begin{aligned}
&f_{\alpha\alpha}(N(x,t) - x^*)_\alpha + f_{\alpha\beta^+}(N(x,t) - x^*)^+_{\beta^+} + q_\alpha(x,t) = 0\\
&f_{\beta^+\alpha}(N(x,t) - x^*)_\alpha + f_{\beta^+\beta^+}(N(x,t) - x^*)^+_{\beta^+} + q_{\beta^+}(x,t) = 0\\
&f_{\beta^-\alpha}(N(x,t) - x^*)_\alpha + f_{\beta^-\beta^+}(N(x,t) - x^*)^+_{\beta^+} + (N(x,t) - x^*)^-_{\beta^-} + q_{\beta^-}(x,t) = 0\\
&f_{\gamma\alpha}(N(x,t) - x^*)_\alpha + f_{\gamma\beta^+}(N(x,t) - x^*)^+_{\beta^+} + (N(x,t) - x^*)_\gamma + q_\gamma(x,t) = 0
\end{aligned}$$

That is

$$\begin{bmatrix} f_{\alpha\alpha} & f_{\alpha\beta^+} & 0 & 0 \\ f_{\beta^+\alpha} & f_{\beta^+\beta^+} & 0 & 0 \\ f_{\beta^-\alpha} & f_{\beta^-\beta^+} & I & 0 \\ f_{\gamma\alpha} & f_{\gamma\beta^+} & 0 & I \end{bmatrix} \begin{bmatrix} (N(x,t) - x^*)_\alpha \\ (N(x,t) - x^*)^+_{\beta^+} \\ (N(x,t) - x^*)^-_{\beta^-} \\ (N(x,t) - x^*)_\gamma \end{bmatrix} + \begin{bmatrix} q_\alpha(x,t) \\ q_{\beta^+}(x,t) \\ q_{\beta^-}(x,t) \\ q_\gamma(x,t) \end{bmatrix} = 0 \tag{3.7}$$

Since the function F is regular at x^*, by Corollary 2.7, the matrix

$$A = \begin{bmatrix} f_{\alpha\alpha} & f_{\alpha\beta^+} & 0 & 0 \\ f_{\beta^+\alpha} & f_{\beta^+\beta^+} & 0 & 0 \\ f_{\beta^-\alpha} & f_{\beta^-\beta^+} & I & 0 \\ f_{\gamma\alpha} & f_{\gamma\beta^+} & 0 & I \end{bmatrix}$$

is nonsingular. Assume that $\|A^{-1}\| \leq C$.

On the other hand, since $f_{\beta\beta} - f_{\beta\alpha}f_{\alpha\alpha}^{-1}f_{\alpha\beta}$ is a P-matrix, there exists some $\eta > 0$ such that for all vector v we have

$$\max_i v_i[f_{\beta\beta} - f_{\beta\alpha}f_{\alpha\alpha}^{-1}f_{\alpha\beta}v]_i \geq \frac{1}{\eta}\|v\|^2 \tag{3.8}$$

By the limit property, there exists $r \in (0, \min(\frac{1}{CL}, \frac{1}{2L\rho}))$ such that for all $x \in S(x^*, r)$, there hold

$$\begin{cases} \alpha(x) = \alpha(x^*) \cup (\alpha(x) \cap \beta(x^*)) \\ \gamma(x) = \gamma(x^*) \cup (\gamma(x) \cap \beta(x^*)), \end{cases} \tag{3.9}$$

where

$$\rho = 2 + (1 + \|f_{\gamma\alpha}\| + \|f_{\beta\alpha}\|)A_\alpha + (1 + \|f_{\gamma\beta}\| + \|f_{\beta\beta}\|)A_\beta$$

with

$$\begin{aligned}
A_\alpha &= \|f_{\alpha\alpha}^{-1}\|(1 + \eta\|f_{\alpha\beta}\| + \eta\|f_{\alpha\beta}\|\ \|f_{\beta\alpha}f_{\alpha\alpha}^{-1}\|)\\
A_\beta &= \eta(1 + \|f_{\beta\alpha}f_{\alpha\alpha}^{-1}\|).
\end{aligned}$$

Select $r_1 \in (0, \frac{r}{2C\|F(x^0)\|})$, then for all $x \in \overline{S}(x^*, r),\ t \in \overline{B}(t^*, r_1) = \{t|\ |t - t^*| \le r_1\}$, we have

$$\|N(x,t) - x^*\| = \left\| \begin{bmatrix} (N(x,t)-x^*)_\alpha \\ (N(x,t)-x^*)^+_{\beta^+} \\ (N(x,t)-x^*)^-_{\beta^-} \\ (N(x,t)-x^*)_\gamma \end{bmatrix} \right\| \le C \left\| \begin{bmatrix} q_\alpha(x,t) \\ q_{\beta^+}(x,t) \\ q_{\beta^-}(x,t) \\ q_\gamma(x,t) \end{bmatrix} \right\|$$
$$\le C \left\| \begin{bmatrix} f_\alpha(x^+) - f_\alpha(x^{*+}) + x^-_\alpha - x^{*-}_\alpha - f_{\alpha\alpha}(x-x^*)_\alpha - f_{\alpha\beta}(x-x^*)^+_\beta \\ f_\beta(x^+) - f_\beta(x^{*+}) + x^-_\beta - x^{*-}_\beta - f_{\beta\alpha}(x-x^*)_\alpha - f_{\beta\beta}(x-x^*)^+_\beta - (x-x^*)^-_\beta \\ f_\gamma(x^+) - f_\gamma(x^{*+}) + x^-_\gamma - x^{*-}_\gamma - f_{\gamma\alpha}(x-x^*)_\alpha - f_{\gamma\beta}(x-x^*)^+_\beta - (x-x^*)_\gamma \end{bmatrix} \right\|$$
$$+C|t-t^*| \cdot \|F(x^0)\|$$
$$= C \left\| \begin{bmatrix} f_\alpha(x^+) - f_\alpha(x^{*+}) - f_{\alpha\alpha}(x-x^*)_\alpha - f_{\alpha\beta}(x-x^*)^+_\beta \\ f_\beta(x^+) - f_\beta(x^{*+}) - f_{\beta\alpha}(x-x^*)_\alpha - f_{\beta\beta}(x-x^*)^+_\beta \\ f_\gamma(x^+) - f_\gamma(x^{*+}) - f_{\gamma\alpha}(x-x^*)_\alpha - f_{\gamma\beta}(x-x^*)^+_\beta \end{bmatrix} \right\| + C|t-t^*| \cdot \|F(x^0)\|$$
$$\le C(\|f(x^+) - f(x^{*+}) - \nabla f(x^{*+})(x^+ - x^{*+})\| + |t-t^*| \cdot \|F(x^0)\|)$$
$$\le C(\tfrac{L}{2}\|x-x^*\|^2 + |t-t^*| \cdot \|F(x^0)\|)$$
$$\le r.$$

Therefore, for any $t \in \overline{B}(t^*, r_1)$, $N(\cdot, t)$ is a continuous function mapping the $\overline{S}(x^*, r)$ into itself. By the Brouwer fixed-point theorem, for any $t \in \overline{B}(t^*, r_1)$, there exists a function $x : \overline{B}(t^*, r_1) \to \overline{S}(x^*, r)$ satisfying

$$BF(x^*)(x(t) - x^*) = BF(x^*)(x(t) - x^*) - H(x(t), t).$$

That is

$$H(x(t), t) = 0.$$

Theorem 3.2. *Suppose F is regular on $x(t)(t \in \overline{V})$ defined in the Theorem 3.1. Then the function x is continuous and single-valued with respect to the parameter t.*

Proof. Using the same notations as in the Theorem 3.1, we now prove the theorem.

Since for any $t_0, t_1 \in \overline{B}(t^*, r_1)$, there exist $x(t_0), x(t_1) \in \overline{S}(x^*, r)$ such that

$$H(x(t_0), t_0) = 0$$
$$H(x(t_1), t_1) = 0.$$

Therefore we also have

$$\begin{cases} BF(x^*)(x(t_0) - x^*) = BF(x^*)(x(t_0) - x^*) - H(x(t_0), t_0) \\ BF(x^*)(x(t_1) - x^*) = BF(x^*)(x(t_1) - x^*) - H(x(t_1), t_1) \end{cases} \tag{3.10}$$

Constitute the (2.4) in (3.10), then (3.10) can be rewritten as the following mixed linear complementarity problems:

$$\begin{cases} q_\alpha(x(t_0), t_0) + f_{\alpha\alpha}(x(t_0) - x^*)_\alpha + f_{\alpha\beta}(x(t_0) - x^*)^+_\beta = 0 \\ w_\beta(x(t_0), t_0) = q_\beta(x(t_0), t_0) + f_{\beta\alpha}(x(t_0) - x^*)_\alpha + f_{\beta\beta}(x(t_0) - x^*)^+_\beta \ge 0 \\ (x(t_0) - x^*)^+_\beta \ge 0, \quad w_\beta(x(t_0), t_0)^T (x(t_0) - x^*)^+_\beta = 0 \\ (x(t_0) - x^*)_\gamma = -q_\gamma(x(t_0), t_0) - f_{\gamma\alpha}(x(t_0) - x^*)_\alpha - f_{\gamma\beta}(x(t_0) - x^*)^+_\beta \end{cases} \tag{3.11}$$

$$\begin{cases} q_\alpha(x(t_1), t_1) + f_{\alpha\alpha}(x(t_1) - x^*)_\alpha + f_{\alpha\beta}(x(t_1) - x^*)^+_\beta = 0 \\ w_\beta(x(t_1), t_1) = q_\beta(x(t_1), t_1) + f_{\beta\alpha}(x(t_1) - x^*)_\alpha + f_{\beta\beta}(x(t_1) - x^*)^+_\beta \ge 0 \\ (x(t_1) - x^*)^+_\beta \ge 0, \quad w_\beta(x(t_1), t_1)^T (x(t_1) - x^*)^+_\beta = 0 \\ (x(t_1) - x^*)_\gamma = -q_\gamma(x(t_1), t_1) - f_{\gamma\alpha}(x(t_1) - x^*)_\alpha - f_{\gamma\beta}(x(t_1) - x^*)^+_\beta \end{cases} \tag{3.12}$$

Notice for any vectors u, v, it holds that

$$(u^+ - v^+)^T(u^- - v^-) = {u^+}^T u^- + {v^+}^T v^- - {u^+}^T v^- - {v^+}^T u^- \geq 0$$

$$w_\beta(x(t_1), t_1) = -(x(t_1) - x^*)_\beta^-, \quad w_\beta(x(t_0), t_0) = -(x(t_0) - x^*)_\beta^-.$$

It is easy to see that for any i, we have

$$\begin{aligned}&[(x(t_0)_\beta^+)_i - (x(t_1)_\beta^+)_i][(q_\beta(x(t_0), t_0) - q_\beta(x(t_1), t_1))\\ &-f_{\beta\alpha} f_{\alpha\alpha}^{-1}(q_\alpha(x(t_0), t_0) - q_\alpha(x(t_1), t_1))(f_{\beta\beta} - f_{\beta\alpha} f_{\alpha\alpha}^{-1} f_{\alpha\beta})(x(t_0)_\beta^+ - x(t_1)_\beta^+)]_i \leq 0\end{aligned}$$

Therefore by (3.8) and the triangular inequality, we obtain that

$$\begin{aligned}&\tfrac{1}{\eta}\|x(t_0)_\beta^+ - x(t_1)_\beta^+\|^2\\ &\leq \max_i((x(t_0)_\beta^+)_i - (x(t_1)_\beta^+)_i)[(f_{\beta\beta} - f_{\beta\alpha} f_{\alpha\alpha}^{-1} f_{\alpha\beta})(x(t_0)_\beta^+ - x(t_1)_\beta^+)]_i\\ &\leq \|x(t_0)_\beta^+ - x(t_1)_\beta^+\| \cdot \|q_\beta(x(t_0), t_0) - q_\beta(x(t_1), t_1)\|\\ &+\|x(t_0)_\beta^+ - x(t_1)_\beta^+\| \cdot \|f_{\beta\alpha} f_{\alpha\alpha}^{-1}\| \ \|q_\alpha(x(t_0), t_0) - q_\alpha(x(t_1), t_1)\|\end{aligned}$$

So

$$\begin{aligned}&\|x(t_0)_\beta^+ - x(t_1)_\beta^+\|\\ &\leq \eta(\|q_\beta(x(t_0), t_0) - q_\beta(x(t_1), t_1)\| + \|f_{\beta\alpha} f_{\alpha\alpha}^{-1}\| \ \|q_\alpha(x(t_0), t_0) - q_\alpha(x(t_1), t_1)\|)\\ &\leq \eta[\tfrac{L}{2}(\|x(t_0) - x^*\| + \|x(t_1) - x^*\|)\|x(t_0) - x(t_1)\| + |t_0 - t_1| \cdot \|F_\beta(x^0)\|\\ &+\tfrac{L}{2}\|f_{\beta\alpha} f_{\alpha\alpha}^{-1}\|(\|x(t_0) - x^*\| + \|x(t_1) - x^*\|)\|x(t_0) - x(t_1)\|\\ &+|t_0 - t_1| \ \|F_\alpha(x^0)\| \ \|f_{\beta\alpha} f_{\alpha\alpha}^{-1}\|]\\ &\leq LrA_\beta\|x(t_0) - x(t_1)\| + \eta|t_0 - t_1|[\|F_\beta(x^0)\| + \|F_\alpha(x^0)\| \cdot \|f_{\beta\alpha} f_{\alpha\alpha}^{-1}\|]\eta.\end{aligned}$$

We also have

$$\begin{aligned}&\|x(t_0)_\alpha - x(t_1)_\alpha\|\\ &= \|f_{\alpha\alpha}^{-1}[q_\alpha(x(t_0), t_0) - q_\alpha(x(t_1), t_1) + f_{\alpha\beta}(x(t_0)_\beta^+ - x(t_1)_\beta^+)]\|\\ &\leq \|f_{\alpha\alpha}^{-1}\|(\|q_\alpha(x(t_0), t_0) - q_\alpha(x(t_1), t_1)\| + \|f_{\alpha\beta}\| \ \|x(t_0)_\beta^+ - x(t_1)_\beta^+\|)\\ &\leq \|f_{\alpha\alpha}^{-1}\|(\tfrac{L}{2}(\|x(t_0) - x^*\| + \|x(t_1) - x^*\|)\|x(t_0) - x(t_1)\|\\ &+|t_0 - t_1| \ \|F_\alpha(x^0)\| + \|f_{\alpha\beta}\| \ \|x(t_0)_\beta^+ - x(t_1)_\beta^+\|)\\ &\leq \|f_{\alpha\alpha}^{-1}\|(Lr\|x(t_0) - x(t_1)\| + \|f_{\alpha\beta}\| \cdot A_\beta Lr\|x(t_0) - x(t_1)\| + |t_0 - t_1| \ \|F_\alpha(x^0)\|\\ &+\|f_{\alpha\beta}\| \cdot [\|F_\beta(x^0)\| + \|F_\alpha(x^0)\| \cdot \|f_{\beta\alpha} f_{\alpha\alpha}^{-1}\|\eta] \cdot |t_0 - t_1|).\end{aligned}$$

By (3.11) and (3.12), we can obtain the following inequalities

$$\begin{aligned}&\|x(t_0)_\gamma - x(t_1)_\gamma\|\\ &\leq \|q_\gamma(x(t_1), t_1) - q_\gamma(x(t_0), t_0)\| + \|f_{\gamma\alpha}\| \cdot \|x(t_1)_\alpha - x(t_0)_\alpha\| + \|f_{\gamma\beta}\| \ \|x(t_1)_\beta^+ - x(t_0)_\beta^+\|\\ &\leq \tfrac{L}{2}(\|x(t_1) - x^*\| + \|x(t_0) - x^*\|)\|x(t_0) - x(t_1)\|\\ &+\|f_{\gamma\alpha}\| \ \|x(t_0)_\alpha - x(t_1)_\alpha\| + \|f_{\gamma\beta}\| \ \|x(t_0)_\beta^+ - x(t_1)_\beta^+\|.\end{aligned}$$

$$\begin{aligned}&\|x(t_0)_\beta^- - x(t_1)_\beta^-\|\\ &\leq \|q_\beta(x(t_1), t_1) - q_\beta(x(t_0), t_0)\| + \|f_{\beta\alpha}\| \cdot \|x(t_1)_\alpha - x(t_0)_\alpha\| + \|f_{\beta\beta}\| \ \|x(t_1)_\beta^+ - x(t_0)_\beta^+\|\\ &\leq \tfrac{L}{2}(\|x(t_1) - x^*\| + \|x(t_0) - x^*\|)\|x(t_0) - x(t_1)\|\\ &+\|f_{\beta\alpha}\| \ \|x(t_0)_\alpha - x(t_1)_\alpha\| + \|f_{\beta\beta}\| \ \|x(t_0)_\beta^+ - x(t_1)_\beta^+\|.\end{aligned}$$

Therefore by the above inequalities, we obtain

$$\begin{aligned}
&\|x(t_0) - x(t_1)\| \\
&\le \|x(t_0)_\alpha - x(t_1)_\alpha\| + \|x(t_0)_\gamma - x(t_1)_\gamma\| + \|x(t_0)^+_\beta - x(t_1)^+_\beta\| + \|x(t_0)^-_\beta - x(t_1)^-_\beta\| \\
&\le \tfrac{1}{2}\|x(t_0) - x(t_1)\| + |t_0 - t_1| \cdot \{(\|F_\beta(x^0)\| + \|F_\alpha(x^0)\| \, \|f_{\beta\alpha} f_{\alpha\alpha}^{-1}\|)\eta \\
&+\|f_{\alpha\alpha}^{-1}\| \cdot \|f_{\alpha\beta}\|(\|F_\beta(x^0)\| + \|F_\alpha(x^0)\| \cdot \|f_{\beta\alpha} f_{\alpha\alpha}^{-1}\|)\eta + \|f_{\alpha\alpha}^{-1}\| \cdot \|F_\alpha(x^0)\| \\
&+(\|f_{\gamma\alpha}\| + \|f_{\beta\alpha}\|) \cdot (\|f_{\alpha\alpha}^{-1}\| \cdot [\|F_\alpha(x^0)\| + \|f_{\alpha\beta}\|(\|F_\beta(x^0)\| + \|F_\alpha(x^0)\| \cdot \|f_{\beta\alpha} f_{\alpha\alpha}^{-1}\|)\eta] \\
&+(\|f_{\gamma\beta}\| + \|f_{\beta\beta}\|)(\|F_\beta(x^0)\| + \|F_\alpha(x^0)\| \cdot \|f_{\beta\alpha} f_{\alpha\alpha}^{-1}\|)\eta\}
\end{aligned}$$

The above inequalities show that $x(t)$ is continuous with respect to t.

We now prove that the function $x(t)$ is single-valued with respect to t. Actually, if this is not true, then there exist some $t \in \overline{B}(t^*, r_1)$ and $x_1, x_2 \in S(x^*, r), x_1 \neq x_2$ satisfying

$$H(x_1, t) = 0$$

$$H(x_2, t) = 0$$

By the same procedure we can prove that

$$\|x_1 - x_2\| \le \frac{1}{2}\|x_1 - x_2\|.$$

This is impossible. Therefore $x(t)$ is single-valued with respect to t. So the function $x(t)$ satisfying the equation (3.3) is uniquely defined.

Let $\overline{U} = \overline{S}(x^*, r), \overline{V} = \overline{B}(t^*, r_1)$, then by the above discussion, there exists a continuous function $x : \overline{V} \to \overline{U}$ satisfying

$$H(x(t), t) = 0, \ \forall \ t \in \overline{V}$$

and it is uniquely defined.

Using the Theorem 3.1 and 3.2, we can prove the following theorem which shows that, the homotopy equation (3.3) has a continuous solution curve on $R^n \times [0, 1]$, and the solution curve is single-valued with respect to the parameter t.

Theorem 3.3. *Suppose the function $F(x) = f(x^+) + x^-$ is regular at x for any $x \in \{F(x) + (t-1)F(x^0) = 0, \ (x, t) \in R^n \times [0, 1]\}$, and there exists a constant $C > 0$ such that $\|A^{-1}(x)\| \le C$. Where*

$$A(x) = \begin{bmatrix} f_{\alpha\alpha}(x^+) & f_{\alpha\beta^+}(x^+) & 0 & 0 \\ f_{\beta^+\alpha}(x^+) & f_{\beta^+\beta^+}(x^+) & 0 & 0 \\ f_{\beta^-\alpha}(x^+) & f_{\beta^-\beta^+}(x^+) & I & 0 \\ f_{\gamma\alpha}(x^+) & f_{\gamma\beta^+}(x^+) & 0 & I \end{bmatrix}$$

with the index sets $\beta^+, \ \beta^-$ satisfy $\beta^+ \cup \beta^- = \beta, \ \beta^+ \cap \beta^- = \phi$, and $\alpha = \alpha(x), \ \beta = \beta(x), \ \gamma = \gamma(x)$. Then there exists a continuous solution curve $x : [0, 1] \to R^n$ which is unique, satisfying

$$\begin{cases} x(0) = x^0, \quad x(1) = x^* \\ H(x(t), t) = F(x(t)) + (t-1)F(x^0) = 0, \quad \forall \ \ t \in [0, 1] \end{cases}$$

where x^* is the solution of $F(x) = 0$.

Proof. The point $(x^0, 0)$ satisfies $H(x^0, 0) = 0$. Repeat the process of the Theorem 3.1 and 3.2 for $(x^0, 0)$, then it is easy to see that there exist $\epsilon > 0$ and a unique function $x : [0, \epsilon] \to R^n$ satisfying

$$\begin{cases} x(0) = x^0, \\ H(x(t), t) = 0, \quad \forall \ \ t \in [0, \epsilon]. \end{cases}$$

and $x(t)$ is continuous with respect to t. Repeat the above process for $(x(\epsilon),\epsilon)$ and so on. Since $x(t)$ is continuous and unique, $x(t)$ can be extended to a larger interval. Suppose $[0,b)$ where $b\leq 1$ is the largest interval that x can be extended to, then $x(t)$ is continuous on $[0,b)$ due to the Theorem 3.1 and 3.2. Thus for any $t_1,t_2\in[0,b)$ and $t_1<t_2, x(t)$ is uniformly continuous on $[t_1,t_2]$. Since $\{x(t)|\ t\in[t_1,t_2]\}$ is compact, we can choose

$$t_1=s_0<s_1<\cdots<s_N=t_2$$

so that

$$\begin{aligned}
&\alpha(x(s_i))=\alpha(x(s_{i+1}))\cup(\alpha(x(s_i))\cap\beta(x(s_{i+1})))\\
&\gamma(x(s_i))=\gamma(x(s_{i+1}))\cup(\gamma(x(s_i))\cap\beta(x(s_{i+1})))\\
&\|x(s_i)-x(s_{i+1})\|\leq\tfrac{1}{CL}
\end{aligned}$$

for each $i\in\{0,\cdots,N-1\}$. Therefore by the proof of the Theorem 3.2, there exists a constant $\overline{C}$ such that

$$\|x(s_i)-x(s_{i+1})\|\leq 2\overline{C}\|F(x^0)\|(s_{i+1}-s_i)$$

for $i=0,1,\cdots,N-1$. So

$$\begin{aligned}
\|x(t_2)-x(t_1)\|&\leq\sum_{i=0}^{N-1}\|x(s_{i+1})-x(s_i)\|\\
&\leq 2\overline{C}\|F(x^0)\|(t_2-t_1)
\end{aligned}\tag{3.13}$$

Let $\{t_k\}\subset[0,b)$ be any sequence satisfying $\lim\limits_{k\to\infty}t_k=b$. By (3.13) $\{x(t_k)\}$ is a Cauchy sequence, and therefore it converges to some point $x_b\in R^n$. Since

$$H(x(t),t)=0,\quad t\in[0,b),$$

by taking the limit we have x_b satisfies

$$H(x_b,b)=0.$$

Thus x can be extended to $[0,b]$. This is a contradiction.

Therefore there exists a unique continuous function $x:[0,1]\to D$, satisfying:

$$\begin{cases}
x(t^*)=x^*\\
H(x(t),t)=F(x(t))+(t-1)F(x^0)=0,\quad\forall\ \ t\in[0,1].
\end{cases}$$

By the Theorem 3.1, 3.2 and the above proof, we know $x(t)$ is single-valued with respect to t. Therefore the solution curve $x(t)$ has no turning point.

Since $H(x,1)=F(x)$ and function x is unique , we have $x(1)=x^*$.

Definition 3.4.[19] *f is a uniform P-function on the nonnegative orthant R^n_+ of R^n, if there exists a positive number α such that*

$$\max_i\{f_i(u)-f_i(v)\}(u_i-v_i)\geq\alpha\|u-v\|^2$$

for every $u,v\in R^n_+$.

Clearly, if the function f is a uniform P-function, then the function $F(x)=f(x^+)+x^-$ is regular at all $x\in R^n$[2]. So when f is a uniformly P-function, we have the following theorem.

Theorem 3.5. *Suppose the function f is a uniformly P-function. Then there exists a continuous solution curve $x:[0,1]\to R^n$ which is unique, satisfying*

$$\begin{cases}
x(0)=x^0,\quad x(1)=x^*\\
H(x(t),t)=F(x(t))+(t-1)F(x^0)=0,\quad\forall\ \ t\in[0,1]
\end{cases}$$

where x^* *is the solution of* $F(x) = 0$.

In part II of this paper, we will study the embedding algorithm and its convergence. Meanwhile, we will study the implementation of the algorithm and present a lot of numerical experiments.

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