Journal of Computational Mathematics, Vol.19, No.5, 2001, 449–458.

# ON THE $L_\infty$ CONVERGENCE AND THE EXTRAPOLATION METHOD OF A DIFFERENCE SCHEME FOR NONLOCAL PARABOLIC EQUATION WITH NATURAL BOUNDARY CONDITIONS*[1)]

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**Abstract**

In paper [4] (J. Comput. Appl. Math., 76 (1996), 137-146), a difference scheme for a class of nonlocal parabolic equations with natural boundary conditions was derived by the method of reduction of order and the unique solvability and second order convergence in $L_2$−norm are proved. In this paper, we prove that the scheme is second order convergent in $L_\infty$ norm and then obtain fourth order accuracy approximation in $L_\infty$ norm by extrapolation method. At last, one numerical example is presented.

*Key words*: Parabolic, Nonlocal, $L_\infty$ convergence, Extrapolation method.

## 1. Introduction

Nonlocal parabolic equations have many applications. For example, in considering fluid flow in a saturated porous medium, the equation governing the pore presure $p(r,t)$ in an annular cylindrical rock sample is given in [1] as

$$\frac{\partial p}{\partial t}+\frac{2(\nu_\mu-\nu)}{\eta(1-\nu_\mu)}\frac{dC(t)}{dt}=c(\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}),\quad 0<a<r<b,t>0 \tag{1.1}$$

together with

$$C(t)=(b^2-a^2)^{-1}\left[\frac{a^2}{2}p_0+\eta\int_a^b \rho p(\rho,t)d\rho\right] \tag{1.2}$$

where $c$ is a material constant with dimensions of velocity, the coefficient of consolidation, $\nu,\nu_\mu$ are shear coefficients and $\eta$ a material constant, $r$ denotes radial distance and $t$ time. If $t$ is replaced by $ct$ and $\frac{2(\nu_\mu-\nu)}{\eta(1-\nu_\mu)}$ by $q$ in (1.1), we get

$$\frac{\partial p}{\partial t}+q\frac{dC(t)}{dt}=\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r},\quad 0<a<r<b,t>0. \tag{1.3}$$

Various initial and boundary conditions can be considered. The existence and uniqueness of the solution of (1.3) with (1.2) under some initial and boundary conditions are proved in [2] provided

$$q\eta\neq -2. \tag{1.4}$$

* Received December 14, 1998.
[1)] The work is supported by Jiangsu Province's Natural Science Foundation (BK97004) and National Natural Science Foundation (19801007) of China.

It is also pointed out that, in practice, $q > 0$ and $\eta > 0$, so that (1.4) is normally met [2].

Substituting (1.2) into (1.3), we obtain an alternative form of (1.3)

$$\frac{\partial p}{\partial t} - \epsilon \int_a^b \rho \frac{\partial p}{\partial t}(\rho, t) d\rho = \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r},$$

where

$$\epsilon = -q\eta(b^2 - a^2)^{-1}.$$

(1.4) is equivalent to $\epsilon \neq 2(b^2 - a^2)^{-1}$ , or, $\frac{1}{2}(b^2 - a^2)\epsilon \neq 1$. When $q > 0$ and $\eta > 0$, we have $\epsilon < 0$. In the following, we suppose

$$\frac{1}{2}(b^2 - a^2)\epsilon < 1. \tag{1.5}$$

It is always valid when $\epsilon < 0$.

As usual, we write $r$ by $x$, $p$ by $u$. Lin, Tait [3] considered the finite difference solution to the nonlocal parabolic equation given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} + \epsilon \int_a^b \rho \frac{\partial u}{\partial t}(\rho, t) d\rho, \quad 0 < a < x < b, 0 < t \leq T, \tag{1.6}$$

subject to suitable initial and boundary conditions. If the initial-boundary conditions are of the form

$$\begin{array}{ll} u(x,0) = \phi(x), & a \leq x \leq b, \\ u(a,t) = f(t), & u(b,t) = g(t), \qquad 0 < t \leq T, \end{array} \tag{1.7}$$

a backward Euler scheme and a Crank-Nicolson scheme are presented, with the former giving rise to an error $O(\tau + h^2)$ and the latter to an error $O(\tau^2 + h^2)$. If the natural boundary conditions

$$\begin{array}{ll} u(x,0) = \phi(x), & a \leq x \leq b, \\ u(a,t) = f(t), & \frac{\partial u}{\partial x}(b,t) + u(b,t) = g(t), \qquad 0 < t \leq T \end{array} \tag{1.8}$$

are imposed, a difference scheme whose convergence order is only one in space and in time is presented. Sun [4] continually studied the finite difference solution to (1.6) with (1.8) and constructed a difference scheme by the method of reduction of order. He proved that the difference scheme is uniquely solvable and unconditionally convergent with the convergence order $O(\tau^2 + h^2)$ in energy norm. In this paper, we will prove that Sun's difference scheme is also second order convergent in $L_\infty-$norm and then obtain a fourth order accuracy approximation in $L_\infty-$norm by once extrapolation [5,6]. At last, we present a numerical example.

For generality, instead of (1.6), consider the inhomogeneous equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} + \epsilon \int_a^b \rho \frac{\partial u}{\partial t}(\rho, t) d\rho + \Phi(x,t), \quad 0 < a < x < b, 0 < t \leq T \tag{1.9}$$

together with (1.8). Let $M$ and $K$ be two positive integers and $h = \frac{b-a}{M}$, $\tau = \frac{T}{K}$. Denote

$$\Omega_h = \{x_i \mid x_i = a + ih, 0 \leq i \leq M\}, \quad \Omega_\tau = \{t_k \mid t_k = k\tau, 0 \leq k \leq K\},$$

$$x_{i-1/2} = (x_i + x_{i-1})/2, \qquad t_{k-1/2} = (t_k + t_{k-1})/2.$$

If $u = \{u_i \mid 0 \leq i \leq M\}$ and $v = \{v_i \mid 0 \leq i \leq M\}$ are two mesh functions on $\Omega_h$, take the notations

$$\begin{array}{l} u_{i-1/2} = (u_i + u_{i-1})/2, \qquad \delta_x u_{i-1/2} = (u_i - u_{i-1})/h, \\ \delta_x(x_i \delta_x u_i) = (x_{i+1/2}\delta_x u_{i+1/2} - x_{i-1/2}\delta_x u_{i-1/2})/h \\ (u,v) = h\sum_{i=1}^M x_{i-1/2} u_{i-1/2} v_{i-1/2}, \quad \|u\| = \sqrt{(u,u)}, \quad \|u\|_\infty = \max_{0\leq i\leq M} |u_i|, \\ \|\delta_x u\| = \sqrt{h\sum_{i=1}^M x_{i-1/2}(\delta_x u_{i-1/2})^2}, \quad \|\delta_x u\|_* = \sqrt{h\sum_{i=1}^M \frac{1}{x_{i-1/2}}(\delta_x u_{i-1/2})^2}. \end{array}$$

If $w = \{w^k \mid 0 \le k \le K\}$ is a mesh function on $\Omega_\tau$, denote

$$w^{k-1/2} = (w^k + w^{k-1})/2, \qquad \delta_t w^{k-1/2} = (w^k - w^{k-1})/\tau.$$

The difference scheme constructed by Sun for (1.9) with (1.8) in [4] is as follows

$$\begin{aligned}
&\tfrac{1}{2}(x_{i-1/2}\delta_t u_{i-1/2}^{k-1/2} + x_{i+1/2}\delta_t u_{i+1/2}^{k-1/2})\\
&\quad = \delta_x(x_i \delta_x u_i^{k-1/2}) + x_i \epsilon h \sum_{j=1}^{M} x_{j-1/2}\delta_t u_{j-1/2}^{k-1/2}\\
&\quad + \tfrac{1}{2}(x_{i-1/2}\Phi_{i-1/2}^{k-1/2} + x_{i+1/2}\Phi_{i+1/2}^{k-1/2}), \quad 1 \le i \le M-1, 1 \le k \le K,\\
&u_0^{k-1/2} = \tfrac{1}{2}[f(t_k) + f(t_{k-1})], \quad 1 \le k \le K,\\
&x_{M-1/2}\delta_t u_{M-1/2}^{k-1/2} = \tfrac{2}{h}[\tfrac{b}{2}(g(t_k) + g(t_{k-1})) - x_{M-1/2}\delta_x u_{M-1/2}^{k-1/2} - b u_M^{k-1/2}]\\
&\quad + x_{M-1/2}\epsilon h \sum_{j=1}^{M} x_{j-1/2}\delta_t u_{j-1/2}^{k-1/2} + x_{M-1/2}\Phi_{M-1/2}^{k-1/2}, \qquad 1 \le k \le K,\\
&u_i^0 = \phi(x_i), \qquad 0 \le i \le M.
\end{aligned} \tag{1.10}$$

## 2. The Derivation of the Difference Scheme with a Slight Difference from that in [4]

Let

$$v = x\frac{\partial u}{\partial x}.$$

Then (1.9) with (1.8) is equivalent to

$$\begin{aligned}
&x\tfrac{\partial u}{\partial t} = \tfrac{\partial v}{\partial x} + x\epsilon \int_a^b \rho \tfrac{\partial u}{\partial t}(\rho, t) d\rho + x\Phi(x,t), \quad a < x < b,\ 0 < t \le T,\\
&\tfrac{1}{x} v = \tfrac{\partial u}{\partial x}, \quad a < x < b,\ 0 < t \le T,\\
&u(x,0) = \phi(x), \qquad a \le x \le b,\\
&u(a,t) = f(t), \quad v(b,t) + bu(b,t) = bg(t), \quad 0 < t \le T.
\end{aligned} \tag{2.1}$$

For (2.1), we construct the difference scheme as follows:

$$x_{i-1/2}\delta_t u_{i-1/2}^{k-1/2} = \delta_x v_{i-1/2}^{k-1/2} + x_{i-1/2}\epsilon h \sum_{j=1}^{M} x_{j-1/2}\delta_t u_{j-1/2}^{k-1/2} + x_{i-1/2}\Phi_{i-1/2}^{k-1/2},$$
$$1 \le i \le M,\ 1 \le k \le K, \tag{2.2.1}$$

$$\frac{1}{x_{i-1/2}} v_{i-1/2}^k = \delta_x u_{i-1/2}^k, \qquad 1 \le i \le M, \quad 0 \le k \le K, \tag{2.2.2}$$

$$u_i^0 = \phi(x_i), \qquad 0 \le i \le M, \tag{2.2.3}$$

$$u_0^k = f(t_k), \quad 1 \le k \le K; \qquad v_M^k + b u_M^k = bg(t_k), \quad 0 \le k \le K. \tag{2.2.4}$$

At the $k-$th time level, (2.2) is regarded as a system of linear algebraic equations with respect to the unknows $\{u_i^k, v_i^k, 0 \le i \le M\}$. (2.2.2) and (2.2.4) are slightly different from those in [4].

Similarly to the proof of Theorem 1 in [4], we have

**Theorem 1.** *The difference scheme (2.2) is equivalent to (1.10) and*

$$\begin{aligned}
&v_i^{k-1/2} = x_{i+1/2}\delta_x u_{i+1/2}^{k-1/2} - \tfrac{1}{2}h x_{i+1/2}[\delta_t u_{i+1/2}^{k-1/2} - \epsilon h \sum_{j=1}^{M} x_{j-1/2}\delta_t u_{j-1/2}^{k-1/2} - \Phi_{i+1/2}^{k-1/2}],\\
&\qquad 0 \le i \le M-1, 1 \le k \le K,\\
&v_M^{k-1/2} + b u_M^{k-1/2} = \tfrac{b}{2}(g(t_k) + g(t_{k-1})), \qquad 1 \le k \le K,\\
&v_M^0 + b u_M^0 = bg(t_0),\\
&\tfrac{1}{x_{i-1/2}} v_{i-1/2}^0 = \delta_x u_{i-1/2}^0, \qquad 1 \le i \le M.
\end{aligned}$$

## 3. A Prior Estimate

**Lemma 3.1.** *If* $u = \{u_i \mid 0 \le i \le M\}$ *and* $u_0 = 0$, *then*

$$\|u\|_\infty \le \sqrt{\ln \frac{b}{a}}\; \|\delta_x u\|.$$

*Proof.* Noticing

$$h \sum_{j=1}^{M} \frac{1}{x_{j-1/2}} \le \int_a^b \frac{1}{x} dx = \ln \frac{b}{a},$$

and using Cauchy-Schwartz inequality, we have

$$\begin{aligned}(u_i)^2 &= (h \sum_{j=1}^{i} \delta_x u_{j-1/2})^2 \\ &\le (h \sum_{j=1}^{i} \tfrac{1}{x_{j-1/2}})[h \sum_{j=1}^{i} x_{j-1/2}(\delta_x u_{j-1/2})^2] \\ &\le (h \sum_{j=1}^{M} \tfrac{1}{x_{j-1/2}})[h \sum_{j=1}^{M} x_{j-1/2}(\delta_x u_{j-1/2})^2] \\ &\le (\ln \tfrac{b}{a})\|\delta_x u\|^2, \qquad 1 \le i \le M.\end{aligned}$$

It follows easily that

$$\|u\|_\infty \le \sqrt{\ln \frac{b}{a}}\; \|\delta_x u\|.$$

**Lemma 3.2.** *Let* $\{u_i^k \mid 0 \le i \le M, 0 \le k \le K\}$ *be the solution of the following difference scheme*

$$x_{i-1/2}\delta_t u_{i-1/2}^{k-1/2} = \delta_x v_{i-1/2}^{k-1/2} + x_{i-1/2}\epsilon h \sum_{j=1}^{M} x_{j-1/2}\delta_t u_{j-1/2}^{k-1/2} + p_{i-1/2}^{k-1/2},$$
$$1 \le i \le M, 1 \le k \le K, \tag{3.1.1}$$
$$\frac{1}{x_{i-1/2}} v_{i-1/2}^k = \delta_x u_{i-1/2}^k + q_{i-1/2}^k, \qquad 1 \le i \le M, 0 \le k \le K, \tag{3.1.2}$$
$$u_i^0 = 0, \qquad 0 \le i \le M, \tag{3.1.3}$$
$$u_0^k = 0, \qquad 1 \le k \le K; \qquad v_M^k + b u_M^k = 0, \qquad 0 \le k \le K. \tag{3.1.4}$$

*If (1.5) is valid and* $\tau \le \frac{2}{3}$, *then, for* $0 \le k \le K$, *we have*

$$\|u^k\|_\infty^2 \le 2\left(\ln \frac{b}{a}\right)\left\{\exp(\frac{3}{2}T)\left[\|q^0\|^2 + \tau \sum_{m=1}^{k}(\frac{1}{2\alpha}\|p^{m-1/2}\|_*^2 + \|\delta_t q^{m-1/2}\|^2)\right] + \|q^k\|^2\right\}, \tag{3.2}$$

*where*

$$\alpha = 1 - \frac{1}{2}(b^2 - a^2)\max\{\epsilon, 0\}. \tag{3.3}$$

*Proof.* Taking the difference quotient in time for (3.1.2), we have

$$\frac{1}{x_{i-1/2}}\delta_t v_{i-1/2}^{k-1/2} = \delta_t \delta_x u_{i-1/2}^{k-1/2} + \delta_t q_{i-1/2}^{k-1/2}, \qquad 1 \le i \le M, 1 \le k \le K. \tag{3.4}$$

Multiplying (3.1.1) by $2\delta_t u_{i-1/2}^{k-1/2}$ and (3.4) by $2v_{i-1/2}^{k-1/2}$, then adding the results, we obtain

$$\begin{array}{rl} & 2x_{i-1/2}(\delta_t u_{i-1/2}^{k-1/2})^2 + \frac{1}{x_{i-1/2}}\frac{1}{\tau}[(v_{i-1/2}^k)^2 - (v_{i-1/2}^{k-1})^2] = 2\delta_x(v\delta_t u)_{i-1/2}^{k-1/2} \\ & +2x_{i-1/2}(\delta_t u_{i-1/2}^{k-1/2})\epsilon h \sum\limits_{j=1}^{M} x_{j-1/2}\delta_t u_{j-1/2}^{k-1/2} + 2p_{i-1/2}^{k-1/2}\delta_t u_{i-1/2}^{k-1/2} + 2v_{i-1/2}^{k-1/2}\delta_t q_{i-1/2}^{k-1/2} \\ \leq & 2\delta_x(v\delta_t u)_{i-1/2}^{k-1/2} + 2x_{i-1/2}(\delta_t u_{i-1/2}^{k-1/2})\epsilon h \sum\limits_{j=1}^{M} x_{j-1/2}\delta_t u_{j-1/2}^{k-1/2} \\ & +2\alpha x_{i-1/2}(\delta_t u_{i-1/2}^{k-1/2})^2 + \frac{1}{2\alpha x_{i-1/2}}(p_{i-1/2}^{k-1/2})^2 + \frac{1}{x_{i-1/2}}(v_{i-1/2}^{k-1/2})^2 + x_{i-1/2}(\delta_t q_{i-1/2}^{k-1/2})^2, \end{array} \tag{3.5}$$

where

$$\delta_x(v\delta_t u)_{i-1/2}^{k-1/2} = (v_i^{k-1/2}\delta_t u_i^{k-1/2} - v_{i-1}^{k-1/2}\delta_t u_{i-1}^{k-1/2})/h.$$

Multiplying (3.5) by $h$ and summing up for $i$ from 1 to $M$, we have

$$\begin{array}{l} 2\|\delta_t u^{k-1/2}\|^2 + \frac{1}{\tau}(\|v^k\|_*^2 - \|v^{k-1}\|_*^2) \leq 2[(v\delta_t u)_M^{k-1/2} - (v\delta_t u)_0^{k-1/2}] \\ +2\epsilon\left(h\sum\limits_{j=1}^{M} x_{j-1/2}\delta_t u_{j-1/2}^{k-1/2}\right)^2 \\ +2\alpha\|\delta_t u^{k-1/2}\|^2 + \frac{1}{2\alpha}\|p^{k-1/2}\|_*^2 + \|v^{k-1/2}\|_*^2 + \|\delta_t q^{k-1/2}\|^2 \end{array} \tag{3.6}$$

When $\epsilon \leq 0$, $\alpha = 1$. It follows from (3.6) that

$$\begin{array}{rl} & \frac{1}{\tau}(\|v^k\|_*^2 - \|v^{k-1}\|_*^2) \\ \leq & 2[(v\delta_t u)_M^{k-1/2} - (v\delta_t u)_0^{k-1/2}] + \frac{1}{2\alpha}\|p^{k-1/2}\|_*^2 + \|v^{k-1/2}\|_*^2 + \|\delta_t q^{k-1/2}\|^2 \end{array} \tag{3.7}$$

When $\epsilon > 0$, $\alpha = 1 - \frac{1}{2}(b^2 - a^2)\epsilon > 0$. Using $h\sum\limits_{j=1}^{M} x_{j-1/2} = \int_a^b x dx = \frac{b^2-a^2}{2}$ and Cauchy-Schwarz inequality, we have

$$\begin{array}{rl} & \left(h\sum\limits_{j=1}^{M} x_{j-1/2}\delta_t u_{j-1/2}^{k-1/2}\right)^2 \\ \leq & \left(h\sum\limits_{j=1}^{M} x_{j-1/2}\right)\left(h\sum\limits_{j=1}^{M} x_{j-1/2}(\delta_t u_{j-1/2}^{k-1/2})^2\right) = \frac{b^2-a^2}{2}\|\delta_t u^{k-1/2}\|^2. \end{array} \tag{3.8}$$

Inserting (3.8) into the right hand side of (3.6), we have

$$\begin{array}{l} 2\|\delta_t u^{k-1/2}\|^2 + \frac{1}{\tau}(\|v^k\|_*^2 - \|v^{k-1}\|_*^2) \leq 2[(v\delta_t u)_M^{k-1/2} - (v\delta_t u)_0^{k-1/2}] + 2\epsilon\frac{b^2-a^2}{2}\|\delta_t u^{k-1/2}\|^2 \\ +2\left[1 - \frac{1}{2}(b^2 - a^2)\epsilon\right]\|\delta_t u^{k-1/2}\|^2 + \frac{1}{2\alpha}\|p^{k-1/2}\|_*^2 + \|v^{k-1/2}\|_*^2 + \|\delta_t q^{k-1/2}\|^2, \end{array}$$

or,

$$\begin{array}{rl} & \frac{1}{\tau}(\|v^k\|_*^2 - \|v^{k-1}\|_*^2) \\ \leq & 2[(v\delta_t u)_M^{k-1/2} - (v\delta_t u)_0^{k-1/2}] + \frac{1}{2\alpha}\|p^{k-1/2}\|_*^2 + \|v^{k-1/2}\|_*^2 + \|\delta_t q^{k-1/2}\|^2 \end{array} \tag{3.9}$$

Comparing (3.7) and (3.9), they are the same. Using the second equality of (3.1.4), we have

$$\begin{array}{rl} & \frac{1}{\tau}(\|v^k\|_*^2 - \|v^{k-1}\|_*^2) \\ \leq & -\frac{b}{\tau}[(v_M^k)^2 - (v_M^{k-1})^2] + \frac{1}{2\alpha}\|p^{k-1/2}\|_*^2 + \|v^{k-1/2}\|_*^2 + \|\delta_t q^{k-1/2}\|^2 \end{array}$$

When $\tau \leq \frac{2}{3}$, it gives that

$$\|v^k\|_*^2 + b(v_M^k)^2 \leq (1 + \frac{3}{2}\tau)[\|v^{k-1}\|_*^2 + b(v_M^{k-1})^2] + \frac{3}{2}\tau(\frac{1}{2\alpha}\|p^{k-1/2}\|_*^2 + \|\delta_t q^{k-1/2}\|^2).$$

Then, applying discrete Gronwall inequality and noticing $v_M^0 = -bu_M^0 = 0$, we have

$$\|v^k\|_*^2 \le \exp(\frac{3}{2}k\tau)\left[\|v^0\|_*^2 + \tau\sum_{m=1}^{k}(\frac{1}{2\alpha}\|p^{m-1/2}\|_*^2 + \|\delta_t q^{m-1/2}\|^2)\right] \tag{3.10}$$

It follows from (3.1.2) ( when $k=0$) and (3.1.3) that

$$\frac{1}{x_{i-1/2}}(v_{i-1/2}^0)^2 = x_{i-1/2}(q_{i-1/2}^0)^2, \qquad 1 \le i \le M.$$

Summing up for $i$, we obtain

$$\|v^0\|_*^2 = \|q^0\|^2.$$

Substituting this equality into the right hand side of (3.10), we have

$$\|v^k\|_*^2 \le \exp(\frac{3}{2}k\tau)\left[\|q^0\|^2 + \tau\sum_{m=1}^{k}(\frac{1}{2\alpha}\|p^{m-1/2}\|_*^2 + \|\delta_t q^{m-1/2}\|^2)\right] \tag{3.11}$$

From (3.1.2), we have

$$\begin{aligned}(\delta_x u_{i-1/2}^k)^2 &= (\tfrac{1}{x_{i-1/2}}v_{i-1/2}^k - q_{i-1/2}^k)^2\\ &= (\tfrac{1}{x_{i-1/2}}v_{i-1/2}^k)^2 - 2\tfrac{1}{x_{i-1/2}}v_{i-1/2}^k q_{i-1/2}^k + (q_{i-1/2}^k)^2\end{aligned}$$

Multiplying it by $x_{i-1/2}$ and using

$$-2v_{i-1/2}^k q_{i-1/2}^k \le \frac{1}{x_{i-1/2}}(v_{i-1/2}^k)^2 + x_{i-1/2}(q_{i-1/2}^k)^2,$$

we have

$$x_{i-1/2}(\delta_x u_{i-1/2}^k)^2 \le 2\frac{1}{x_{i-1/2}}(v_{i-1/2}^k)^2 + 2x_{i-1/2}(q_{i-1/2}^k)^2, \qquad 1 \le i \le M.$$

Summing up for $i$, we obtain

$$\|\delta_x u^k\|^2 \le 2\|v^k\|_*^2 + 2\|q^k\|^2. \tag{3.12}$$

Inserting (3.11) into the right hand side of (3.12) and using Lemma 3.1, the result needed follows. The proof ends.

## 4. $L_\infty$ Convergence

Denote

$$C^{k,l}(\overline{Q}) = \{u \mid \frac{\partial^m u}{\partial x^m}, \frac{\partial^n u}{\partial t^n} \in C(\overline{Q}), m \le k, n \le l\}, \qquad Q = (a,b)\times(0,T).$$

We have the following convergence result.

**Theorem 2.** *Suppose (1.9) with (1.8) have smooth solution $u(x,t) \in C^{4,3}(\overline{Q})$. Let $\{u_i^k, 0 \le i \le M, 0 \le k \le K\}$ be the solution of (1.10) with the step sizes $h$ and $\tau$. In addition, suppose (1.5) is valid and $\tau \le \frac{2}{3}$. Then there exists a constant $c > 0$ independent of $h$ and $\tau$ such that*

$$|u(x_i,t_k) - u_i^k| \le c(h^2+\tau^2), \quad 0 \le i \le M, \quad 0 \le k \le K. \tag{4.1}$$

*Proof.* With the help of Theorem 1, it suffices to prove that the solution of (2.1) convergences to the solution of (2.2) with the convergence order $O(h^2+\tau^2)$ in $L_\infty$ norm. Denote the errors by:

$$\widetilde{u}_i^k = u(x_i,t_k) - u_i^k, \qquad \widetilde{v}_i^k = v(x_i,t_k) - v_i^k.$$

Using Taylor expansion and the theory of numerical intergration, we obtain the error equation of the difference scheme (2.2) as follows:

$$\begin{array}{l} x_{i-1/2}\delta_t\widetilde{u}_{i-1/2}^{k-1/2} = \delta_x\widetilde{v}_{i-1/2}^{k-1/2} + x_{i-1/2}\epsilon h\sum\limits_{j=1}^{M} x_{j-1/2}\delta_t\widetilde{u}_{j-1/2}^{k-1/2} + e_{i-1/2}^{k-1/2}, \\ \qquad\qquad\qquad 1\le i\le M, 1\le k\le K, \\ \frac{1}{x_{i-1/2}}\widetilde{v}_{i-1/2}^k = \delta_x\widetilde{u}_{i-1/2}^k + s_{i-1/2}^k, \quad 1\le i\le M, \quad 0\le k\le K, \\ \widetilde{u}_i^0 = 0, \quad 0\le i\le M, \\ \widetilde{u}_0^k = 0, \quad 1\le k\le K; \quad \widetilde{v}_M^k + b\widetilde{u}_M^k = 0, \quad 0\le k\le K, \end{array} \tag{4.2}$$

where $e_{i-1/2}^{k-1/2}, s_{i-1/2}^k$ are the truncation errors of (2.2) and there exists a constant $c_0$ independent of $h$ and $\tau$ such that

$$|e_{i-1/2}^{k-1/2}| \le c_0(h^2+\tau^2), \quad |s_{i-1/2}^k| \le c_0(h^2+\tau^2), \quad |\delta_t s_{i-1/2}^{k-1/2}| \le c_0(h^2+\tau^2). \tag{4.3}$$

If $u(x,t)\in C^{6,5}(\overline{Q})$, then $e_{i-1/2}^{k-1/2}, s_{i-1/2}^k$ can be written as

$$\begin{array}{l} e_{i-1/2}^{k-1/2} = (h^2+\tau^2)P_1(x_{i-1/2},t_{k-1/2};\frac{\tau}{h}) + O(h^4+\tau^4), \\ s_{i-1/2}^k = (h^2+\tau^2)P_2(x_{i-1/2},t_k;\frac{\tau}{h}) + O(h^4+\tau^4), \\ P_1(x,t;\frac{\tau}{h}) = \frac{(\tau/h)^2}{1+(\tau/h)^2}\left[\frac{1}{24}xu_{ttt} - \frac{1}{8}v_{xtt} - \frac{1}{24}\epsilon x\int_a^b \rho u_{ttt}(\rho,t)d\rho\right] \\ \quad + \frac{1}{1+(\tau/h)^2}\left[\frac{1}{8}xu_{xxt} - \frac{1}{24}v_{xxx} - \frac{1}{12}\epsilon x\int_a^b \frac{\partial^2}{\partial\rho^2}(\rho u_t(\rho,t))d\rho + \frac{1}{4}\epsilon x\int_a^b \frac{\partial^2 u}{\partial\rho\partial t}d\rho\right], \\ P_2(x,t;\frac{\tau}{h}) = \frac{1}{1+(\tau/h)^2}\left(\frac{1}{8x}v_{xx} - \frac{1}{24}u_{xxx}\right). \end{array} \tag{4.4}$$

The last inequality of (4.3) holds because $\delta_t s_{i-1/2}^{k-1/2}$ may be regarded as the truncation error of the difference scheme

$$\frac{1}{x_{i-1/2}}\delta_t v_{i-1/2}^{k-1/2} = \delta_t\delta_x u_{i-1/2}^{k-1/2}$$

approximating to the equation

$$\frac{1}{x}v_t = u_{xt}$$

at the point $(x_{i-1/2},t_{k-1/2})$. In addition, we have

$$\delta_t s_{i-1/2}^{k-1/2} = (h^2+\tau^2)\frac{\partial P_2}{\partial t}(x_{i-1/2},t_{k-1/2};\frac{\tau}{h}) + O(h^4+\tau^4). \tag{4.5}$$

By Lemma 3.2 and using (4.3), we obtain

$$\|\widetilde{u}^k\|_\infty \le c(h^2+\tau^2), \quad 0\le k\le K,$$

where

$$c^2 = 2\left(\ln\frac{b}{a}\right)c_0^2\left\{\exp(\frac{3}{2}T)\left[\frac{b^2-a^2}{2} + T(\frac{1}{2\alpha}\ln\frac{b}{a} + \frac{b^2-a^2}{2})\right] + \frac{b^2-a^2}{2}\right\}.$$

The proof ends.

## 5. Extrapolation Method

Consider the extrapolation algorithm of the difference scheme (1.10). we have the following theorem.

**Theorem 3.** *Suppose (1.9) with (1.8) have smooth solution $u(x,t) \in C^{6,5}(\overline{Q})$. Let $u_i^k(h,\tau)$ be the difference solution of (1.10) with the step sizes $h$ and $\tau$. If the following nonlocal parabolic differential equation:*

$$\begin{array}{l} \frac{\partial \overline{u}}{\partial t} = \frac{\partial^2 \overline{u}}{\partial x^2} + \frac{1}{x}\frac{\partial \overline{u}}{\partial x} + \epsilon \int_a^b \rho \frac{\partial \overline{u}}{\partial t}(\rho,t)d\rho + \frac{1}{x}P_1 + \frac{1}{x}\frac{\partial}{\partial x}(xP_2), \\ \qquad\qquad 0 < a < x < b, 0 < t \le T, \\ \overline{u}(x,0) = 0, \quad a \le x \le b, \\ \overline{u}(a,t) = 0, \quad \frac{\partial \overline{u}}{\partial x}(b,t) + \overline{u}(b,t) = -P_2(b,t;\frac{\tau}{h}), \quad 0 < t \le T, \end{array} \tag{5.1}$$

*has solution $\overline{u}(x,t;\frac{\tau}{h}) \in C^{4,3}(\overline{Q})$, then*

$$u(x_i,t_k) = u_i^k(h,\tau) + \overline{u}(x_i,t_k;\frac{\tau}{h})(h^2+\tau^2) + O(h^4+\tau^4), \quad 0 \le i \le M, 0 \le k \le K, \tag{5.2}$$

*and*

$$u(x_i,t_k) = \frac{4}{3}u_{2i}^{2k}(\frac{h}{2},\frac{\tau}{2}) - \frac{1}{3}u_i^k(h,\tau) + O(h^4+\tau^4), \quad 0 \le i \le M, 0 \le k \le K. \tag{5.3}$$

*Proof.* Let

$$\overline{v} = x(\frac{\partial \overline{u}}{\partial x} + P_2)$$

then (5.1) is equivalent to:

$$\begin{array}{l} x\frac{\partial \overline{u}}{\partial t} = \frac{\partial \overline{v}}{\partial x} + x\epsilon \int_a^b \rho \frac{\partial \overline{u}}{\partial t}(\rho,t)d\rho + P_1, \quad 0 < a < x < b, 0 < t \le T, \\ \frac{1}{x}\overline{v} = \frac{\partial \overline{u}}{\partial x} + P_2, \quad a < x < b, 0 < t \le T, \\ \overline{u}(x,0) = 0, \quad a \le x \le b, \\ \overline{u}(a,t) = 0, \quad \overline{v}(b,t) + b\overline{u}(b,t) = 0, \quad 0 < t \le T \end{array} \tag{5.4}$$

If $\{\overline{u}_i^k \mid 0 \le i \le M, 0 \le k \le K\}$ is the solution of the following scheme:

$$\begin{array}{l} x_{i-1/2}\delta_t \overline{u}_{i-1/2}^{k-1/2} = \delta_x \overline{v}_{i-1/2}^{k-1/2} + x_{i-1/2}\epsilon h \sum\limits_{j=1}^{M} x_{j-1/2}\delta_t \overline{u}_{j-1/2}^{k-1/2} \\ \qquad +P_1(x_{i-1/2},t_{k-1/2};\frac{\tau}{h}), \qquad 1 \le i \le M, 1 \le k \le K, \\ \frac{1}{x_{i-1/2}}\overline{v}_{i-1/2}^k = \delta_x \overline{u}_{i-1/2}^k + P_2(x_{i-1/2},t_k;\frac{\tau}{h}), \quad 1 \le i \le M, \quad 0 \le k \le K, \\ \overline{u}_i^0 = 0, \quad 0 \le i \le M, \\ \overline{u}_0^k = 0, \quad 1 \le k \le K; \quad \overline{v}_M^k + b\overline{u}_M^k = 0, \quad 0 \le k \le K. \end{array} \tag{5.5}$$

By Theorem 2, we obtain

$$\overline{u}(x_i,t_k;\frac{\tau}{h}) = \overline{u}_i^k(h,\tau) + O(h^2+\tau^2), \quad 0 \le i \le M, \ 0 \le k \le K. \tag{5.6}$$

Denote

$$U_i^k = \tilde{u}_i^k - (h^2+\tau^2)\overline{u}_i^k(h,\tau), \qquad V_i^k = \tilde{v}_i^k - (h^2+\tau^2)\overline{v}_i^k(h,\tau).$$

Multiplying (5.5) by $(h^2+\tau^2)$ and subtracting the results from (4.2) and using (4.4)-(4.5), we obtain

$$\begin{array}{l} x_{i-1/2}\delta_t U_{i-1/2}^{k-1/2} = \delta_x V_{i-1/2}^{k-1/2} + x_{i-1/2}\epsilon h \sum\limits_{j=1}^{M} x_{j-1/2}\delta_t U_{j-1/2}^{k-1/2} + \overline{p}_{i-1/2}^{k-1/2}, \\ \qquad 1 \le i \le M, 1 \le k \le K, \\ \frac{1}{x_{i-1/2}}V_{i-1/2}^k = \delta_x U_{i-1/2}^k + \overline{q}_{i-1/2}^k, \quad 1 \le i \le M, 0 \le k \le K, \\ U_i^0 = \phi(x_i), \quad 0 \le i \le M, \\ U_0^k = 0, \quad 1 \le k \le K; \quad V_M^k + bU_M^k = 0, \quad 0 \le k \le K, \end{array} \tag{5.7}$$

where

$$|\overline{p}_{i-1/2}^{k-1/2}| = O(h^4+\tau^4), \qquad |\overline{q}_{i-1/2}^{k}| = O(h^4+\tau^4), \qquad |\delta_t \overline{p}_{i-1/2}^{k-1/2}| = O(h^4+\tau^4).$$

By Lemma 3.2, we obtain

$$U_i^k = O(h^4+\tau^4), \quad 1 \le i \le M, \quad 1 \le k \le K,$$

or,

$$u(x_i,t_k) - u_i^k(h,\tau) - (h^2+\tau^2)\overline{u}_i^k(h,\tau) = O(h^4+\tau^4).$$

Applying (5.6), we have

$$u(x_i,t_k) = u_i^k(h,\tau) + \overline{u}(x_i,t_k;\frac{\tau}{h})(h^2+\tau^2) + O(h^4+\tau^4). \tag{5.8}$$

This is (5.2). Similarly, we have

$$u(x_i,t_k) = u_{2i}^{2k}(\frac{h}{2},\frac{\tau}{2}) + \overline{u}(x_i,t_k;\frac{\frac{\tau}{2}}{\frac{h}{2}})\left((\frac{h}{2})^2+(\frac{\tau}{2})^2\right) + O\left((\frac{h}{2})^4+(\frac{\tau}{2})^4\right). \tag{5.9}$$

Multiplying (5.9) by $\frac{4}{3}$ and (5.8) by $\frac{1}{3}$, then subtracting the results, we obtain

$$u(x_i,t_k) = \frac{4}{3}u_{2i}^{2k}(\frac{h}{2},\frac{\tau}{2}) - \frac{1}{3}u_i^k(h,\tau) + O(h^4+\tau^4).$$

This is (5.3). The proof ends.

## 6. Numerical Example

Compute by difference scheme (1.10) the following problem [4].

$$\begin{array}{l}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} + \int_1^2 0.5\rho \frac{\partial u}{\partial t}(\rho,t)d\rho \\
\qquad +(e^x - \frac{1}{2}e^2)\cos t - e^x(1+\frac{1}{x})\sin t, 1 < x < 2, 0 < t \le 1, \\
u(1,t) = e\sin t, \qquad \frac{\partial u}{\partial x}(2,t) + u(2,t) = 2e^2 \sin t, \qquad 0 < t \le 1, \\
u(x,0) = 0, \qquad 1 \le x \le 2,
\end{array}$$

whose exact solution is $u(x,t) = e^x \sin t$. Some numerical solutions are listed at the following tables. It can be seen that the absolute errors decrease about four times as the stepsizes decrease two times and that the difference solution is greatly improved by extrapolation method.

Table 1 Result before extrapolation

| $(x,t)$ | | (0.3,1.0) | (0.5,1.0) | (0.7,1.0) | (0.9,1.0) |
|---|---|---|---|---|---|
| Exact solution | | 3.0878067 | 3.7712113 | 4.6061679 | 5.6259862 |
| $h=0.1$ | Difference solution | 3.0865600 | 3.7702488 | 4.6059132 | 5.6271263 |
| $\tau=0.1$ | Absolute error | -1.0467e-3 | -9.625e-4 | -2.547e-4 | 1.1401e-3 |
| $h=0.05$ | Difference solution | 3.0873456 | 3.7711510 | 4.6061518 | 5.6260672 |
| $\tau=0.05$ | Absolute error | -2.611e-4 | -2.411e-4 | -6.42e-5 | 2.84e-4 |
| $h=0.025$ | Difference solution | 3.0875414 | 3.7702488 | 4.6059132 | 5.6271263 |
| $\tau=0.025$ | Absolute error | -6.53e-5 | -6.03e-5 | -1.61e-5 | 7.10e-5 |

Table 2 Result after extrapolation $\frac{4}{3}u_{2i}^{2k}(0.05,0.05) - \frac{1}{3}u_i^k(0.1,0.1)$

| $(x,t)$ | (0.3,1.0) | (0.5,1.0) | (0.7,1.0) | (0.9,1.0) |
|---|---|---|---|---|
| Extrapolation solution | 3.08760747 | 3.77121072 | 4.60616727 | 5.62598488 |
| Absolute error | 7.9e-7 | 6.0e-7 | 6.4e-7 | 1.31e-6 |

Table 3 Result after extrapolation $\frac{4}{3}u_{2i}^{2k}(0.025, 0.025) - \frac{1}{3}u_i^k(0.05, 0.05)$

| $(x,t)$ | (0.3,1.0) | (0.5,1.0) | (0.7,1.0) | (0.9,1.0) |
|---|---|---|---|---|
| Extrapolation solution | 3.08760667 | 3.77121131 | 4.60616787 | 5.62598612 |
| Absolute error | -1.0e-8 | -1.0e-8 | -3.0e-8 | -6.0e-8 |

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