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ON THE PROBLEM OF BEST RATIONAL APPROXIMATION WITH INTERPOLATING CONSTRAINTS (I)* [1)]

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Abstract

The aim of this conjoint paper is to discuss the problem of best rational approximation with interpolating constraints. In Part I ,we give the necessary and sufficient conditions for the existence of the best rational approximations and establish some characterizatoin theorems for such approximations. The problems of uniqueness, the properties for the set of best approximations, strong uniqueness and continuity of the best approximation operator are considered in Part II. The results obtained in this paper are the completion and extension of those given in [1].

§1. Introduction

Let m, n and t be nonnegative integers (i.e. $m, n, t \in \mathbf{N}$),

$$a \le x_0 < x_1 < \cdots < x_s \le b,$$

and $\{k_0, k_1, \cdots, k_s\} \subset \mathbf{N}$ such that

$$0 \le k_i \le t, \ i = 0, 1, \cdots, s; \ k \equiv \sum_{i=0}^{s}(k_i + 1) \le m + n.$$

Let

$$\mathbf{H}_i = \{P : P(x) = \sum_{j=0}^{i} a_j x^j, \ a_j \in R\},$$

$$\mathbf{R}(m, n) = \{P/Q : P \in \mathbf{H}_m, Q \in \mathbf{H}_n, P/Q \ irreducible\}.$$

For $g \in \mathbf{C}^t[a, b]$, we define the set $\mathbf{R}_1(m, n)$ to be

$$\mathbf{R}_1(m, n) = \{R \in \mathbf{R}(m, n) : (R - g)^{(j)}(x_i) = 0, \ i = 0, 1, ..., s; \ j = 0, 1, ..., k_i\}.$$

The problem that we wish to consider in this paper is: For a given $f \in \mathbf{C}[a, b]$, find an element R from $\mathbf{R}_1(m, n)$, such that

$$\|f - R\| = \inf_{T \in \mathbf{R}_1(m,n)} \|f - T\| \equiv d_1(f) \tag{1.1}$$

where $\|\cdot\|$ denotes the Chebyshev norm in $\mathbf{C}[a, b]$.

The set $\mathbf{R}_1(m, n)$ is called the interpolating restricted range. The cardinalities about this set are discussed in [2]. We assume in this paper that $\mathbf{R}_1(m, n) \cap \mathbf{C}[a, b] \ne \phi$ and that the denominators of the elements in $\mathbf{R}_1(m, n) \cap \mathbf{C}[a, b]$ are positive on $[a, b]$.

Next we study the solvability of problem (1.1) and the characterizations of its solutions.

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§2. Existence

For the given $g \in \mathbf{C}^t[a,b]$, we introduce first the following sets:

$$\mathbf{L}(m,n) = \{(P,Q) \in \mathbf{H}_m \times \mathbf{H}_n : (P-Qg)^{(j)}(x_i) = 0,$$
$$i = 0,1,...,s;\ j = 0,1,...,k_i\}$$

$$\mathbf{R}_0(m,n) = \{P/Q \in \mathbf{R}(m,n) : \exists q \ \ s.t.\ (qP,qQ) \in \mathbf{L}(m,n)\}.$$

Then we introduce the following problem which is related to the problem (1.1): Find $R \in \mathbf{R}_0(m,n)$ such that

$$\|f-R\| = \inf\{\|f-T\| : T \in \mathbf{R}_0(m,n)\} \equiv d_0(f). \tag{2.1}$$

It is easy to deduce that $\mathbf{R}_1(m,n) \subset \mathbf{R}_0(m,n)$, therefore

$$d_0(f) \le d_1(f). \tag{2.2}$$

Next we discuss the solvabilities of problems (1.1) and (2.1) and the relationship between them.

Example 1. Let $m=0$, $n=1$, $[a,b]=[0,1]$, $s=0$, $k_0=0$, $x_0=0$, and $g=1$. Then we have

$$\mathbf{R}_1(m,n) = \{\frac{1}{ax+1} : a \in R\}, \quad \mathbf{R}_0(m,n) = \mathbf{R}_1(m,n) \cup \{0\}.$$

a) Take $f = -2x+1$, then (1.1) is unsolvable. The reason for this is, for any $R \in \mathbf{R}_1(m,n)$, $\|f-R\| > 1$, and $R_k = \dfrac{1/k}{x+1/k} \in \mathbf{R}_1(m,n)$ such that $\|f-R_k\| \longrightarrow 1$. Therefore $d_1(f) = 1$. However , problem (2.1) is solvable and $R=0$ is one of its solutions.

b) Take $f = C$ (constant), then for any $a \ge 0$, $R = 1/(ax+1)$ is a solution of problem (1.1) if $C \le 1/2$, while $R=0$ is the unique solution of (2.1) if $C < 1/2$.

c) Take $f = C, C > 1/2$ and $C \neq 1$, then the two problems have the same, and infinitely many, solutions.

This example illustrates that the solvability of (1.1) is a complicated problem compared with the problem of ordinary best rational approximation. For problem (2.1), however, we have the following determinate conclusion.

Theorem 1. *Problem* (2.1) *is solvable for any given* $f \in \mathbf{C}[a,b]$.

Proof. Similarly to the proof given in, [3, p.155], there exists a sequence

$$\{R_k = P_k/Q_k, k=1,2,\cdots\} \subset \mathbf{R}_0(m,n)$$

such that

$$P_k \longrightarrow \hat{P},\ Q_k \longrightarrow \hat{Q},\ \|Q_k\| = \|\hat{Q}\| = 1 \quad \|P/Q - f\| = d_0(f), \tag{2.3}$$

where P/Q is the irreducible form of $\hat{P}/\hat{Q}$. Now we shall show that $P/Q \in \mathbf{R}_0(m,n)$.

By the definition of $\mathbf{R}_0(m,n)$, there exist q_k such that

$$(\hat{P}_k, \hat{Q}_k) \equiv q_k(P_k, Q_k) \in \mathbf{L}(m,n),$$

and $w(x) \equiv \prod_{i=0}^{s}(x-x_i)^{k_i+1}$ can be divided exactly by q_k. Since q_k's are bounded above , we may assume $q_k \longrightarrow q$ without loss of generality. Then we have from (2.3) that

$$\hat{P}_k \longrightarrow q\hat{P},\ \hat{Q}_k \longrightarrow q\hat{Q}.$$

Therefore

$$\hat{P}_k^{(j)}(x_i) \longrightarrow (q\hat{P})^{(j)}(x_i), \quad \hat{Q}_k^{(j)}(x_i) \longrightarrow (q\hat{Q})^{(j)}(x_i),$$
$$i = 0, 1, ..., s;\ j = 0, 1, ..., k_i.$$

From

$$(\hat{P}_k - g\hat{Q}_k)^{(j)}(x_i) = 0,\ i = 0, 1, ..., s;\ j = 0, 1, ..., k_i,$$

it follows that

$$(q\hat{P} - gq\hat{Q})^{(j)}(x_i) = 0,\ i = 0, 1, ..., s;\ j = 0, 1, ..., k_i,$$

i.e., $(q\hat{P}, q\hat{Q}) \in \mathbf{L}(m, n)$. Then $P/Q \in \mathbf{R}_0(m, n)$. It implies that P/Q is a solution of problem (2.1).

Now we can consider the solvability of problem (1.1).

Lemma 1. *Let $R = P/Q \in \mathbf{R}_1(m, n)$. Then*

$$\mathrm{card}(\mathbf{R}_1(m, n)) = 1$$

if and only if $m + n < k + d(R)$, where card $(\cdot)$ *denotes the cardinality of a set ,and $d(R) = \min\{m - \partial P,\ n - \partial Q\}$.*

Proof. Sufficiency. Let $R_1 = P_1/Q_1 \in \mathbf{R}_1(m, n)$. Then it is easy to deduce that

$$(P_1Q - PQ_1)^{(j)}(x_i) = 0, \quad i = 0, 1, ..., s;\ j = 0, 1, ..., k_i.$$

This means that $P_1Q - PQ_1$ has at least k zeros. Since

$$\partial(P_1Q - PQ_1) \leq m + n - d(R) < k,$$

so $P_1Q - PQ_1 = 0$, and then $R_1 = R$.

Necessity can be derived directly from the theorem established in [2].

Let

$$X = \{x_i : i = 0, ..., s\},$$
$$Y(R) = \{y \in [a, b] : |\ f(y) - R(y)\ | = \|f - R\|\}, \quad R \in \mathbf{C}[a, b],$$
$$\mathbf{R}_0^f(m, n) = \{R \in \mathbf{R}_0(m, n) :\ \|f - R\| \leq d_1(f)\}.$$

Then one has from Theorem 1 that $\mathbf{R}_0^f(m, n) \neq \phi$.

The points in $Y(R)$ are called *deviation points*, and those in $Y(R) \cap X$ are called *neutral deviation points*.

Theorem 2. *Let $f \in \mathbf{C}[a, b]$. Then problem (1.1) is solvable if and only if one of the following two conditions is satisfied:*

(i) *$\exists\ R \in \mathbf{R}_1(m, n)$, such that $Y(R) \cap X \neq \phi$ or $m + n < k + d(R)$.*

(ii) *There does not exist such an R which satisfies*

a) *$R = P/Q \in \mathbf{R}_0^f(m, n) \setminus \mathbf{R}_1(m, n)$,*

b) *$\exists j_i$ $(0 \leq i \leq s)$, each of which is an even integer for $x_i \in (a, b)$, sush that*

$$\left(P \prod_{i=0}^{s} (x - x_i)^{j_i}, Q \prod_{i=0}^{s} (x - x_i)^{j_i}\right) \in \mathbf{L}(m, n).$$

Proof. If condition (i) is valid, then by Proposition 1 in [1] or Corollary 3 and Lemma 1 in this paper , the theorem holds. If problem (1.1) is unsolvable under condition (ii), then, similarly to the proof of Theorem 1 , there exists $R_j = P_j/Q_j \in \mathbf{R}_1(m,n)$, such that

$$P_j \longrightarrow P,\ Q_j \longrightarrow Q,\ \|Q_j\| = \|Q\| = 1,$$

$$\|R_j - f\| \longrightarrow d_1(f),$$

and

$$\|f - \hat{P}/\hat{Q}\| = d_1(f),$$

where $\hat{P}/\hat{Q}$ is the irreducible form of P/Q. Since (1.1) has no solution , we have $\hat{P}/\hat{Q} \notin \mathbf{R}_1(m,n)$. Besides, it is easy to show that $(P,Q) \in \mathbf{L}(m,n)$; then $\hat{P}/\hat{Q} \in \mathbf{R}_0(m,n)$. Thus $\hat{P}/\hat{Q}$ satisfies condition a) of the theorem.

Since $Q_j > 0$ on $[a,b]$, if $x_i \in (a,b)$ is a zero of Q, its multiplicity must be an even integer. Hence, from the boundedness of P/Q , x_i is also a zero of P with at least the same multiplicity. Then condition b) is satisfied by $\hat{P}/\hat{Q}$. This is a contradiction. Therefore (1.1) is solvable.

If (1.1) has a solution , we shall prove that condition (ii) must be true without condition (i) being valid. This proof needs the characterization theorem. We leave it to §3.

Corollary 1. Suppose that the interpolating conditions are imposed on one or both end points of the interval [a,b], and condition (i) is not true. Then problem (1.1) is solvable if and only if the solution of (2.1) is in $\mathbf{R}_1(m,n)$. Furthermore, if (1.1) is solvable, then the two problems have the same solutions.

Example 2. Take $[a,b] = [0,1], g = f, f \in \mathbf{C}^t[a,b] \setminus \mathbf{R}_1(m,n)$. Find $R \in \mathbf{R}(m,n)$, such that

$$R^{(j)}(0) = f^{(j)}(0), \quad j = 0,1,\dots,k-1,$$

$$\|f - R\| = \min .$$

We can see that R is a kind of approximation between ordinary best rational approximation and Padé approximation. In practice it may be a more suitable approximation of f for some special purposes. The conditions of Corollary 1 are satisfied in this case.

Corollary 2. If condition b) in Theorem 2 is not valid for any $R = P/Q \in (\mathbf{R}_0(m,n) \setminus \mathbf{R}_1(m,n)) \cap \mathbf{C}[a,b]$, then problem (1.1) is solvable for any $f \in \mathbf{C}[a,b]$.

Example 3. Take $[a,b] = [-1,1]$, $g = 1$, $m = 0$, $n = 1$, $x_0 = 0$, $s = 0$, $k_0 = 0$. Then

$$\mathbf{R}_1(m,n) = \{\frac{1}{ax+1} : a \in R\}, \quad \mathbf{R}_0(m,n) = \mathbf{R}_1(m,n) \cup \{0\}.$$

Since $\mathbf{R}_0(m,n) \setminus \mathbf{R}_1(m,n) = \{0\}$, and $R = 0$ does not satisfy condition b) of Theorem 2, problem (1.1) is solvable for any $f \in \mathbf{C}[a,b]$.

§3. Characterization

For $R \in \mathbf{R}_1(m,n) \cap \mathbf{C}[a,b]$, define

$$\mathbf{P} - R\mathbf{Q} = \{P - RQ : (P,Q) \in \mathbf{L}(m,n)\},$$

then $\mathbf{P} - R\mathbf{Q}$ is a linear subspace of $\mathbf{C}[a,b]$.

Theorem 3. *Let $R \in \mathbf{R}_1(m,n) \cap \mathbf{C}[a,b]$, $f \in \mathbf{C}[a,b] \setminus \mathbf{R}_1(m,n)$. Then R is a best approximation of f in $\mathbf{R}_1(m,n)$ if and only if*

$$\inf_{y \in Y(R)} (f(y) - R(y))\phi(y) \leq 0, \ \forall \phi \in \mathbf{P} - R\mathbf{Q}.$$

Proof. Sufficiency. If $R = P/Q$ is not a solution of (1.1), there exists $R^* = P^*/Q^* \in \mathbf{R}_1(m,n)$, such that $\|R^* - f\| < \|f - R\|$. Let $\phi = P^* - RQ^*$. Then $\phi \in \mathbf{P} - R\mathbf{Q}$. Let $\sigma(y) = sign(f - R)(y)$. Then

$$\sigma(y)(f - R^*)(y) \leq \|f - R^*\| < \|f - R\| = \sigma(y)(f - R)(y), \ y \in Y(R).$$

Therefore $\sigma(y)(R^* - R)(y) > 0$. That is, $(f - R)(y)\phi(y) > 0$, $\forall y \in Y(R)$. Since $Y(R)$ is a closed set, $\inf_{y \in Y(R)} (f - R)(y)\phi(y) > 0$. This is a contradiction.

Necessity. Suppose there exists $\phi = P^* - RQ^* \in \mathbf{P} - R\mathbf{Q}$, such that

$$\inf_{y \in Y(R)} (f - R)(y)\phi(y) > 0.$$

Let

$$R_\lambda(x) = \frac{P + \lambda P^*}{Q + \lambda Q^*}.$$

Then we have from Lemma 2 of [2] that $R_\lambda \in \mathbf{R}_1(m,n)$ when λ is sufficiently small. Now we shall show that $\lambda > 0$ can be chosen small enough, so that

$$\|f - R_\lambda\| < \|f - R\|.$$

Let

$$X_1 = \{x \in [a,b] : (f - R)(x)\phi(x) \leq 0\}.$$

Then X_1 is a closed set, and $X_1 \cap Y(R) = \phi$. Therefore, if we let

$$\sup_{x \in X_1} |f(x) - R(x)| = \mu,$$

then $\mu < \|f - R\|$. Thus $\lambda > 0$ can be taken so small that

$$\|R - R_\lambda\| < \|f - R\| - \mu. \tag{3.1}$$

It follows that, if $x \in X_1$,

$$\begin{aligned} |f(x) - R_\lambda(x)| &\leq |f(x) - R(x)| + |R(x) - R_\lambda(x)| \\ &\leq \mu + \|R - R_\lambda\| < \|f - R\|. \end{aligned}$$

If $x \in [a,b] \setminus X_1$, since $(f - R)(x)\phi(x) > 0$, we have

$$\sigma(x)(R - R_\lambda)(x) = -\lambda \frac{\sigma(x)\phi(x)}{Q(x) + \lambda Q^*(x)} < 0. \tag{3.2}$$

Then the two terms on the right side of the equation

$$|f(x) - R_\lambda(x)| = |\sigma(x)(f - R)(x) + \sigma(x)(R - R_\lambda)(x)|$$

have opposite signs. It follows from (3.1) and (3.2) that

$$|f(x) - R_\lambda(x)| < \max\{|f(x) - R(x)|, |R(x) - R_\lambda(x)|\} \leq \|f - R\|.$$

Hence $|f(x) - R_\lambda(x)| < \|f - R\|$ for all $x \in [a, b]$, and then $\|f - R_\lambda\| < \|f - R\|$.

Corollary 3. Let $R \in \mathbf{R}_1(m, n) \cap \mathbf{C}[a, b]$, $f \in \mathbf{C}[a, b] \setminus \mathbf{R}_1(m, n)$. Then R is a best approximation of f in $\mathbf{R}_1(m, n)$ if and only if for any $R_1 \in \mathbf{R}_1(m, n)$ there exists $y \in Y(R)$, such that

$$|f(y) - R_1(y)| \geq \|f - R\|.$$

The Alternation Theorem given in [1] can be derived from Theorem 3. In order to carry out the proof, we now introduce

Lemma 2. *Let $R = P/Q \in \mathbf{R}_1(m, n) \cap \mathbf{C}[a, b]$. Then*

(i) *If $m + n < k + d(R)$,*

$$\mathbf{P} - R\mathbf{Q} = \{0\}.$$

(ii) *If $m + n \geq k + d(R)$,*

$$\mathbf{P} - R\mathbf{Q} = \frac{w}{Q}\mathbf{H}_{m+n-d(R)-k}.$$

Proof. Conclusion (i) can be derived from the proof of Lemma 1. Next we prove (ii). Let

$$\phi = P_0 - RQ_0 \in \mathbf{P} - R\mathbf{Q}, \ (P_0, Q_0) \in \mathbf{L}(m, n).$$

Since $QP_0 - PQ_0$ can be divided exactly by w, we have

$$\phi = \frac{QP_0 - PQ_0}{Q} = \frac{w}{Q}p, \quad p \in \mathbf{H}_{m+n-d(R)-k}.$$

On the contrary, let $p(x) \in \mathbf{H}_{m+n-d(R)-k}$. Since P and Q are coprime to each other, there exist polynomials $P_0 \in \mathbf{H}_m, Q_0 \in \mathbf{H}_n$ (see [1]), such that

$$pw = P_0Q - PQ_0,$$

that is, $pw/Q = P_0 - RQ_0$. This means that $(P_0, Q_0) \in \mathbf{L}(m, n)$. Hence

$$\frac{w}{Q}p \in \mathbf{P} - R\mathbf{Q}.$$

Theorem 4. *Let $R \in \mathbf{R}_1(m, n) \cap \mathbf{C}[a, b]$, $f \in \mathbf{C}[a, b] \setminus \mathbf{R}_1(m, n)$. Then R is a best approximation of f in $\mathbf{R}_1(m, n)$ if and only if one of the following three conditions is satisfied*

(i) $Y(R) \cap X \neq \phi$.

(ii) $m + n < k + d(R)$.

(iii) *$r = f - R$ alternates at least $m + n - d(R) - k + 2$ times according to w (the meaning of alternating according to w can be seen in [1]).*

Proof. Sufficiency. If condition (i) or (ii) is valid, then R is a solution of (1.1) by Lemma 1 and Corollary 3. Now assume that (iii) is true and we shall prove that R is a best approximation of f. On the contrary, there is $\phi = wp/Q \in \mathbf{P} - R\mathbf{Q}$ from Theorem 3, such that

$$(f - R)(y)\phi(y) = w(y)(f - R)(y)p(y)/Q(y) > 0, \quad \forall y \in Y(R). \tag{3.3}$$

It follows from condition (iii) that $p(y)$ alternates at least $m+n-d(R)-k+2$ times. That is, $p(x)$ has at least $m+n-d(R)-k+1$ zeros. Hence $p=0$ by noting that $\partial(p) \leq m+n-d(R)-k$. This contradicts (3.3).

Necessity. If (i) and (ii) do not hold , then for any $p \in \mathbf{H}_{m+n-d(R)-k}$, we have from Theorem 3 that

$$\inf_{y \in Y(R)} w(y)(f-R)p(y) \leq 0. \tag{3.4}$$

Let

$$V = \{v(y) \in \mathbf{R}^{m+n-d(R)-k+1} :$$
$$v(y) = w(y)(f-R)(y)(1, y, ..., y^{m+n-d(R)-k})^T, y \in Y(R)\}.$$

Then $V \subset \mathbf{R}^{m+n-d(R)-k+1}$ is a compact set , and relation (3.4) implies that the inequality

$$\langle v, c \rangle > 0, v \in V$$

is unsolvable. Then by the theorem on linear inequalities and the Carathéodory theorem(see [3],p.17-19) there exist $t_i \in Y(R)$, such that

$$a \leq t_1 < t_2 < \cdots < t_l \leq b, \ l \leq m+n-d(R)-k+2,$$

$$\sum_{i=1}^{l} \lambda_i v(t_i) = 0, \quad \sum_{i=1}^{l} \lambda_i = 1, \ \lambda_i \geq 0.$$

Let $a_i = \lambda_i w(t_i)(f-R)(t_i)$. Then the relation above can be rewritten as

$$\sum_{i=1}^{l} a_i (1, t_i, \cdots, t_i^{m+n-d(R)-k})^T = 0.$$

Since $\{x^j\}_0^p$ is a Haar system , we have

$$l = m+n-d(R)-k+2, \ |a_i| > 0 \ \text{ and } a_i's \text{ change sign with } i \text{ alternately.}$$

Therefore $w(t_i)(f-R)(t_i)$ change sign with i alternately, and then the theorem is proved.

In order to complete the proof of Theorem 2 , we introduce the following distinct facts without proof.

Lemma 3. *Let $t_0 \leq t_1 \leq \cdots \leq t_l$, and let p be a polynomial. Suppose that*

$$p(t_i) = (-1)^i \lambda_i \quad (i = 0, 1, \cdots, l), \ \textit{and } \lambda_i\textit{'s do not change sign with } i \tag{3.5}$$

and

$$p(t_i) = p'(t_i) = \cdots = p^{(k_i)}(t_i) = 0, \quad \textit{if there are } k_i (>0)\ t\textit{'s coinciding with } t_i.$$

Then p has at least l zeros (counting multiplicity) on $[t_0, t_l]$, and if $\lambda_0 \neq 0$ (or $\lambda_l \neq 0$, or $\lambda_0 \lambda_l \neq 0$), these zeros are contained in the interval $(t_0, t_l]$ (or $[t_0, t_l)$, or (t_0, t_l)).

Now we are in a position to finish the proof of Theorem 2.

Suppose that (1.1) has a solution R and there exists $R_1 = P_1/Q_1$ which satisfies conditions a) and b) of Theorem 2 . We shall come to a contradiction. Let $q = \prod_{i=0}^{s} (x - x_i)^{j_i}$. Then by using

$$\|f - R_1\| \leq \|f - R\|$$

and Theorem 4, there exist t_i such that $a \le t_1 < t_2 < \cdots < t_{m+n-d(R)-k+2} \le b$,

$$w(t_i)(R_1 - R)(t_i) = w(t_i)(R_1 - f + f - R)(t_i) = (-1)^i \lambda_i,$$

λ_i's do not change sign, $i = 1, 2, \cdots, m+n-d(R)-k+2$. Let $w_1(x) = w(x)/q(x)$. Then by Lemma 2

$$w(x)(R_1 - R)(x) = w(x)\frac{P_1(x)Q(x) - P(x)Q_1(x)}{Q_1(x)Q(x)} = \frac{ww_1p}{Q_1Q} = \frac{w_1^2qp}{Q_1Q},$$

where $p \in \mathbf{H}_{m+n-d(R)-k}$. Since j_i are even integers when $x_i \in (a,b)$, and $Y(R) \cap X = \phi$, we have

$$p(t_i) = (-1)^i \hat{\lambda}_i, \quad i = 1, 2, \cdots, m+n-d(R)-k+2,$$

and $\hat{\lambda}_i$'s do not change sign. It follows from Lemma 3 that p has at least $m+n-d(R)-k+1$ zeros, and hence $p = 0$ and $R = R_1$. This is a contradiction and then the theorem is proved.

Finally, we point out that Theorem 4 can be used to construct the Remes algorithm for computing the best rational approximation with interpolating constraints. It is a natural and convenient tactics to replace the interpolating constraints by linear ones. That is, we try to find a solution of problem (2.1). Therefore it is important for us to introduce problem (2.1) and to discuss the relationship between the two problems.

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