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FAR FIELD COMPUTATIONAL BOUNDARY CONDITIONS

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1. Introduction

Far field computational boundary conditions are used in order to limit the domain of the independent variables in the numerical approximation of differential equations. This restriction of the domain is necessry if it originally is infinite or too large for practical computations. A well designed computational far field boundary condition can limit the size of the domain substantially without changing the solution too much. The effect is a reduction in storage and computing time.

There are applications from almost all fields of numerical solution of partial differential equations. For example, when a short time weather forcast is needed in some area, the calculation can be restricted to a small part of the full atmosphere. Similarly, in seismology, it is practical to compute the propagation of seismic waves in a limited volume rather than in the whole earth. The restricting boundary conditions should here have the property of avoiding reflections in the artificial boundaries.

A common type of problems is the solution of differential equations in exterior domains. One example is the flowfield around an airplane. Another is the electromagnetic field outside an object. The limiting conditions at infinity can here often be replaced by boundary conditions quite close to the bodies.

Research in the development and analysis of this class of boundary conditions for different types of differential equations and different applications has been very active during the last decade, see e.g.[1]-[8]. In the literature these boundary conditions are also called absorbing, artificial, radiation and transparent.

The far field computational boundry conditions should satisfy the following three requirements:

(a) The boundary conditions should, together with the differential equation and sometimes also together with other boundary conditions, form a well-posed problem.

(b) They should also be well satisfied by a relevant class of solutions to the original problem.

(c) It should be possible to implement the far field computational boundry conditions efficiently on the computer.

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The first two properties guarantee that the error between the original solution and the solution over the restricted domain with the new boundary condition is small.

We shall discuss three classes of problems in this paper. The first is hyperbolic equations. The purpose of the computational boundary conditions is to avoid reflections of waves in the artificial boundaries. This can be achieved rather well with local conditions based on microlocal analysis. The second class is elliptic problems. The boundary conditions are here either local or global in the form of integral equations. Finally ideas from the two classes of problems above will be combined into new far field computational boundary conditions for long time calculations and calculations to steady state.

2.Hyperbolic problems

Consider first the simple scalar wave equation in one space dimension.

$$u_{tt} = u_{xx} \quad , \quad -\infty < x < \infty, t > 0, \tag{2.1a}$$

$$u(x,0) = f(x), \tag{2.1b}$$

$$u_t(x,0) = g(x). \tag{2.1c}$$

Assume that f and g have support in the interval, $-1 < x < 1$, and that the solution is only of interest in this interval. We would like to replace the problem (2.1) by following equations on a bounded interval,

$$v_{tt} = v_{xx} \quad , -1 < x < 1, t > 0, \tag{2.2a}$$

$$v(x,0) = f(x), \tag{2.2b}$$

$$v_t(x,0) = g(x), \tag{2.2c}$$

$$B_{-1}v(x = -1, t) = 0, \tag{2.2d}$$

$$B_1 v(x = 1, t) = 0. \tag{2.2e}$$

For the ideal choice of boundary operators B_{-1} and B_1 the error $u(x,t) - v(x,t)$ should be minimal. The standard Dirichlet and Neumann boundary conditions are not appropriate. For example with $B_{-1} = \partial/\partial x$ and initial values corresponding to a left travelling wave $u(x,t) = F(t+x)$, the solution of (2.2) is,

$$v(x,t) = F(t+x) + F(t-x-2), \quad 0 < t < 2.$$

There is a strong artificial refelction in the left boundary.

All left travelling waves are annihilated by the operator B_{-1} below and all right travelling waves are annihilated by B_1,

$$B_{-1} = \frac{\partial}{\partial x} - \frac{\partial}{\partial t}, \tag{2.3a}$$

$$B_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}. \tag{2.3b}$$

The condition (b) in the introduction is satisfied and condition (a) follows from a simple energy estimate. The operators (2.3) are also easy to implement numerically, (c).

The far field computational boundary conditions (2.2b), (2.2e), (2.3) can be derived from a decomposition of the original wave operator,

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right) u. \tag{2.4}$$

For more than one space dimension this is not so straight forward. The desired decomposition for the wave equation,

$$u_{tt} = u_{xx} + u_{yy},$$

should have the form,

$$\left(\frac{\partial^2}{\partial x} - \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\right)\right) u = \left(\frac{\partial}{\partial x} - L_{-1}\right)\left(\frac{\partial}{\partial x} + L_1\right) u. \tag{2.5}$$

If we restrict the full space to the strip $-1 < x < 1, -\infty < y < \infty$ the conditions at $x = -1$ and $x = 1$ would then be,

$$B_{-1}v = \left(\frac{\partial}{\partial x} - L_{-1}\right) v(x = -1, y, t) = 0, \tag{2.6a}$$

$$B_1 v = \left(\frac{\partial}{\partial x} + L_1\right) v(x = 1, y, t) = 0. \tag{2.6b}$$

Our trouble with (2.6) is that there are no local operators L_{-1} and L_1 which satisfy (2.5),

We can derive L_{-1} (and L_1) in the form of pseudo-differential operators such that superpositions of plane waves,

$$v = e^{i(\sqrt{\omega^2 - k^2}x + ky + \omega t)}, \qquad |k| < |\omega|,$$

incident on the boundary at $x = -1$ are annihilated,

$$B_{-1}v = \frac{\partial v(x = -1, y, t)}{\partial x} + \int\int e^{(i\omega t + ky)} i\sqrt{\omega^2 - k^2}\hat{v}(-1, k, \omega) d\omega dk = 0. \tag{2.7}$$

The function v is the Fouriertransform of v with the dual variables k and $\omega, k < \omega$. Analogous operators are easily derived for $x = 1$ and for y-boundaries.

In order to be practical (point c) B_{-1} should be approximated by a local operator. This corresponds to approximating $\sqrt{\omega^2 - k^2}$ by rational functions since then the equation (2.7) can be expressed as a linear combinations of derivatives. Waves which are close to normal incidence are the most destructive if they are reflected. They penetrate deeper into the computational domain after a fixed time. It is thus reasonable to assume that $|k|/|\omega|$ is small. The approximation $\sqrt{\omega^2 - k^2} = \omega$ produces the operators (2.3). The second order Taylor expansion,

$$\sqrt{\omega^2 - k^2} \approx \omega - \frac{k^2}{2\omega}, \tag{2.8}$$

gives the boundary conditions,

$$v_{xt} - v_{tt} + 0.5 v_{yy} = 0, \quad x = -1, \tag{2.9a}$$

$$v_{xt} + v_{tt} - 0.5 v_{yy} = 0, \quad x = 1. \tag{2.9b}$$

This approach can be generalized in different directions. Boundary conditions can be derived for variable coefficient problems based on variable coefficient decompositions of the from (2.5). Far field computational boundary condition for systems are also possible. Instead of a product decomposition (2.5), the symbol of the hyperbolic operator is decomposed based on a separation of eigenvalues, see [4].

Well-posedness does not follow automatically from a derivation of this type. The boundary conditions (2.9) together with the wave equation are well-posed. This can be proved by using normal mode analysis . Higher order Taylor expansions give, however, rise to ill-posed problems, see [4]. Padé approximations of the square root in (2.7) have to be used.

3. Elliptic problems

For the hyperbolic problems above, local conditions were derived such that the boundary equations were well satisfied by the solution if the support of the data were inside the computational domain. The design principle was to avoid reflections of waves in the artificial boundaries. The influence from outside has to be konwn beforehand and is added through inhomogeneous terms. Local correctness is a reasonable principle for hyperbolic problems.

Another useful design principle is to assume constant coefficients outside the computational domain. The outside solution is then reduced to a problem in the boundary which is coupled to the interior solution. This procedure leads in general

to global boundary conditions and as we shall see below it is natural for elliptic problems.

It is also possible to have local far field computational boundary conditions for elliptic equtions. These conditions can be based on the behavior of the solution at infinty.

Consider as an example the Laplace equation

$$\Delta u = f(x,y,z), \tag{3.1a}$$

$$u(x,y,z) = O(r^{-1}), \quad r = (x^2+y^2+z^2)^{1/2}, r \to \infty. \tag{3.1b}$$

We shall assume that f has support inside the computational domain, $r < R$. In practice the domain , $r < R$, may contain an object for which we want to solve the exterior problem with given boundry values. We can also have variable coefficients in the computational domain.

Following [1], we use a far field expansion of the solution,

$$\sum_{j=0}^{\infty} r^{-(j+1)} F_j(\phi,\theta) \tag{3.2}$$

where (ϕ,θ) are the angular variables. The differential operators

$$B_R^{(m)} = \prod_{j=1}^{m} \left(\frac{\partial}{\partial r} + \frac{2j-1}{r} \right), \tag{3.3}$$

annihilate the first m terms in (3.2). The computational boundary conditions which replace (3.1b) are thus

$$B_R^{(m)} u = 0, \quad r = R. \tag{3.4}$$

Since we anyway need an implicit method to solve (3.1) in the computational domain, the extra cost for replacing the local boundary condition (3.4) by a global relation is not too demanding.

Let K be boundary integral operator mapping boundary values to normal derivatives for the exterior problem. For a problem of the type (3.1) this would mena,

$$K_g = -\frac{\partial w}{\partial n},$$

where w is the solution of (3.1) for $r > R$ and $w = g$ at $r = R$.

The global boundary condition instead of the local approximations (3.4) can then be formulated,

$$\frac{\partial v}{\partial n} + Kv = 0, \quad r = R. \tag{3.5}$$

For some special cases the operator K is known explicitly. If the artificial boundary is a hyperplane or a sphere, K can be decomposed by Fourier integrals or Fourier series. In general the boundary conditions can be expressed in a variational formulation. This formulation can be the basis for a numerical finite element and boundary integral method, see [7].

4. Far field boundary conditions for computations over long time

In this last section we shall combine ideas from sections 2 and 3. The goal is still to satisfy the points (a), (b) and (c) in the introduction. We shall assume the differential equation to be hyperbolic and let us be interested in calculations over a long time or even to steady state. Our relevant set of solutions in (b) should then contain the transient phase for which the conditions in section 2 were designed and steady solutions of the type discussed in section 3. For the details see [3].

Let us illustrate this discussion by the simple dispersive wave equation,

$$u_{tt} - u_{xx} + u = f(x), \quad -1 < x < 1, \quad t > 0. \tag{4.1}$$

Assume that f has compact support in $-1 < x < 1$ and that we want computational boundary conditions at $x = -1$ and $x = 1$. From section 2 we have the first order transient conditions,

$$u_x - u_t = 0 \ , \quad x = -1, \tag{4.2a}$$

$$u_x + u_t = 0 \ , \quad x = 1. \tag{4.2b}$$

The effect of the lower order term in (4.1) is only visible in the higher order boundary conditions. The relevant square root in (2.7) should then be expanded in terms of ω^{-1}. The result for a second order approximation is,

$$u_{xt} - u_{tt} - 0.5u = 0 \ , \quad x = -1, \tag{4.3a}$$

$$u_{xt} + u_{tt} + 0.5u = 0 \ , \quad x = 1. \tag{4.3b}$$

The correct steady solution of (4.1) satisfies,

$$u_x - u = 0 \ , \quad x = -1, \tag{4.4a}$$

$$u_x + u = 0 \ , \quad x = 1. \tag{4.4b}$$

The steady solution decays exponentially with x as $|x| \to \infty$. The equation (4.4) correspond to the conditions in section 3. Here even the "global condition" becomes local since we have only one space dimension.

The simplest new combined boundary condition for the scalar wave equation extends the relation (3.5) to the from,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial n} + Ku = 0, \tag{4.5}$$

in the computational boundary. For the one dimensional problem (4.1) we get,

$$u_x - u_t - u = 0 \quad , \quad x = -1, \tag{4.6a}$$

$$u_x + u_t + u = 0 \quad , \quad x = 1. \tag{4.6b}$$

It is easy to see that the steady solution is satisfied by (4.6) but not by (4.2). Waves from the interior will be strongly reflected by the steady conditions (4.4). Only weak reflections will occur in the artificial boundaries when applying (4.2), (4.3) and the new compromise condition (4.6). With (4.4) there is no convergence to the steady solution as $t \to \infty$. For (4.2) there is convergence but to the wrong solution.

The convergence, as $t \to \infty$, in the (4.6) case can be proved by the energy method. For the general theory, see [3].

Higher order computational boundary conditions of this type can be derived by letting a hyperbolic boundary operator of high order act on the steady boundary conditions.

It is of course not practical to solve the Laplace equation by computing the wave equation solution over long time. We are only using this model because it is simple and easy to analyze. For nonlinear steady problems in computational fluid dynamics this type of extension to a time dependent hyperbolic calulation is a standard procedure.

Let the linearized form of a hyperbolic system in two space variables be,

$$u_t + Au_x + Bu_y = f(x,y) \quad , \quad -1 < x, y < 1, \quad t > 0.$$

The first order far field computational boundary condition of the type discussed in section 2 has the form, $PTu(x = -1, y, t) = 0$, for the boundary at $x = -1$. The matrix T transforms A to diagonal form and P is a projection onto the space corresponding to positive eigenvalues of A, [4]. Let $Su(x = -1, y, t) = 0$ be a far field boundary condition for the steady state. The operator S is in general nonlocal and if u is periodic in y it can be written explicitly via a Fourier series expansion, [5]. The new type of boundary condition which is analogous to (4.5) is then,

$$P\left(\frac{\partial}{\partial t}T + S\right) u(x = -1, y, t) = 0. \tag{4.7}$$

In [5] down-stream computational boundary conditions for the Enler equations describing steady channel flow are studied. The conditions are nonlocal and of the type discussed in section 3. In this particular application the density is given in terms of the Fourier modes of the velocity normal to the channel. This defines our operator S above. The steady boundary conditions are applied to a dependent calculation to reach steady state. The boundary conditions are effective in that they make it possible to choose a small computational domain. The convergence is, however, slower than for characteristic boundary conditions on the form $PTu = 0$, above.

In a recent experiment with Gustafsson and Ferm we have used the new boundary conditions of the form (4.7) for the channel flow problem in [5]. These computations show the same accuracy as with the steady boundary conditions and almost the same convergence rate to steady state as with the characteristic boundary conditions.

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