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THE CALAHAN METHOD FOR PARABOLIC EQUATIONS WITH TIME-DEPENDENT COEFFICIENTS*

WU WEI (吴 微)

(*Jilin University, Changchun, China*)

Abstract

A modified Calahan method for parabolic equations with time-dependent coefficients is presented. It is shown that the convergence order is $O(h^{r+1}+k^3)$ while the convergence order obtained in [1] for a standard Calahan method is only $O(h^{r+1}+k^2)$.

§ 1. Introduction

Sammon[1] considered the single step methods (including fully discrete finite element methods) for parabolic equations with time-dependent coefficients (PETC). The convergence order he obtained is $O(h^{r+1}+k^{\min(2,q)})$ under the restriction $k\leqslant Ch^2$ where h and k are respectively the step lengths of space and time, while $O(h^{r+1}+k^q)$ is the optimal order that we usually get for parabolic equations with time-independent coefficients. It was asserted in [1] that this estimate cannot be generally improved. It is then an interesting problem whether we could give some schemes for the PETC with convergence order better than $O(h^{r+1}+k^{\min(2,q)})$ in some special cases. As a trial in this respect, we consider in this paper a modified Calahan method ($q=3$ in this case, see [2]) for the PETC. The convergence order we shall show is optimal, $O(h^{r+1}+k^3)$, under the same restriction $k\leqslant Ch^2$ as in [1].

We shall describe in this section the problem to be dealt with and give some notations and definitions. In Sections 2 and 3 we shall do some preparative work and in the last section we shall give our main result, the convergence order estimate.

Let Ω be a bounded region in R^N with smooth boundary Γ and $t^*>0$ a constant, $T=[0, t^*]$. We consider the following PETC,

$$\frac{du}{dt}=Au, \quad (x, t)\in\Omega\times T, \tag{1.1}$$

$$u|_\Gamma=0, \quad t\in T, \tag{1.2}$$

$$u|_{t=0}=u^0, \quad x\in\Omega, \tag{1.3}$$

where
$$A=A(t)=\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(a_{ij}(x, t)\frac{\partial}{\partial x_j}\right), \quad a_{ij}=a_{ij}(x, t)$$
are prescribed smooth functions in $\overline{\Omega}\times T$, $\overline{\Omega}=\Omega+\Gamma$; $u^0=u^0(x)$ is a given smooth function in $\overline{\Omega}$.

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Let $H^s=H^s(\Omega)$ $(s\geqslant 0)$ be Sobolev spaces of degree s with norm $\|\cdot\|_s$ and H_0^1 the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_1$; $(\ ,\ )$ denotes the inner product in $L^2=H^0$. We also use $(\ ,\ )$ to denote the dual pair (cf. [5]) of H_0^1 and its dual space $(H_0^1)'=H^{-1}$ normed by

$$\|v\|_{-1}=\sup_{\substack{w\in H_0^1\\ \|w\|=1}}(v,\ w).$$

Let $S_h^r(h\geqslant 0)$ be a family of finite dimensional subspaces of H_0^1 with the approximation property

(H1) $$\inf_{v_h\in S_h}(\|v-v_h\|_0+h\|v-v_h\|_1)\leqslant Ch^{r+1}\|v\|_{r+1},\quad \forall v\in H^{r+1}\cap H_0^1,$$

where and bellow we always use C to denote generalized constants which are not necessarily the same at any two places.

Define

$$a(t;\ v,\ w)=\sum_{i,j=1}^N\Big(a_{ij}\frac{\partial v}{\partial x_j},\frac{\partial w}{\partial x_i}\Big),\ \text{for}\ v,\ w\in H_0^1,\ t\in T.$$

Take $k=t^*/M$ as the time step length and $t^s=sk$ for $s\geqslant 0$. The modified Calahan method for (1.1)—(1.3) is to find $\{U^n\}_{n=0}^M\subset S_h^r$ such that for $n=0,\ 1,\ \cdots,\ M-1$ and all $v_h\in S_h^r$,

$$(W^{n+1},\ v_h)+bka(t^{n+\frac{2}{3}};\ W^{n+1},\ v_h)=-ka(t^{n+b};\ U^n,\ v_h),\tag{1.4}$$

$$\begin{aligned}(Z^{n+1},\ v_h)+bka(t^{n+\frac{2}{3}};\ Z^{n+1},\ v_h)&=-ka(t^{n+b};\ U^n,\ v_h)\\ &+cka(t^{n+1};\ U^n,\ v_h)-a(t^n;\ U^n,\ v_h)+a(t^{n+\frac{2}{3}};\ W^{n+1},\ v_h),\end{aligned}\tag{1.5}$$

$$U^{n+1}=U^n+W^{n+1}+\frac{1}{4}Z^{n+1},\tag{1.6}$$

$$U^0=\tilde{u}^0,\tag{1.7}$$

where $b=\frac{1}{2}\Big(1+\frac{1}{3}\sqrt{3}\Big)$, $c=\frac{2}{3}\sqrt{3}$; $\{W^n\}_{n=1}^M$, $\{Z^n\}_{n=1}^M\subset S_h^r$; $\tilde{u}^0$ is an approximation of u^0 in S_h^r satisfying

(H2) $$\|u^0-\tilde{u}^0\|_0+h\|u^0-\tilde{u}^0\|_1\leqslant Ch^{r+1}\|u^0\|_{r+1}.$$

Throughout this paper we shall always assume that the conditions (H1), (H2) and the following hypotheses (H3) and (H4) are valid.

(H3)
$$\begin{cases}1)\ a_{ij}=a_{ji},\ \text{for}\ (x,\ t)\in\overline{\Omega}\times T;\\ 2)\ \sum\limits_{i,j=1}^N a_{ij}r_ir_j\geqslant C\sum\limits_{i=1}^N r_i^2,\ \text{for}\ (r_1,\ \cdots,\ r_N)\in R^N,\ (x,\ t)\in\overline{\Omega}\times T;\\ 3)\ \dfrac{\partial^s a_{ij}}{\partial t}\in C^0(\overline{\Omega}\times T),\quad \text{for}\ i,\ j=1,\ 2,\ \cdots,\ N;\ s=0,\ 1,\ \cdots,\ 4;\\ 4)\ (1.1)\text{—}(1.3)\ \text{possesses a unique solution}\ u\ \text{and}\\ \qquad \|u\|_{p,\infty}=\sup\limits_{t\in T}\|u(t)\|_p<\infty,\ p=\max(r+4,\ 8);\\ 5)\ \|v\|_{s+2}\leqslant C\|Av\|_s,\ \text{for}\ v\in H^{s+2}\cap H_0^1,\ t\in T,\ s\geqslant -1.\end{cases}$$

(H4) $$k\leqslant Ch^2.$$

Remark 1. Condition 5) of (H3) is valid when Γ and a_{ij}'s are sufficiently smooth (cf. [6]), and this implies that A^{-1}: $H^s \to H^{s+2} \cap H_0^1$ is well-defined,

$$\|A^{-1}v\|_{s+2} \leq C\|v\|_s, \quad \text{for} \quad v \in H^s,\ s \geq -1,\ t \in T. \tag{1.8}$$

We denote $f(t^q, x)$ by f^q for $q \geq 0$ and

$$A^q_{t^s} = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\left(\frac{\partial^s a_{ij}}{\partial t^s}\right)^q \frac{\partial}{\partial x_j}\right), \quad 0 \leq s \leq 4;$$

$$A^q_{t^0} = A^q.$$

In virtue of hypothesis (H3) they are well-defined and bounded in the sense that

$$\|A^q_{t^s} v\|_p \leq C\|v\|_{p+2} \quad \text{for} \quad v \in H^{p+2},\ p \geq 0,\ q \geq 0,\ 0 \leq s \leq 4. \tag{1.9}$$

Moreover, noticing (1.8) we know that by integrating by parts we can write, for $v, w \in H_0^1$,

$$a(t;\ v,\ w) = -(Av,\ w), \quad \text{for} \quad t \in T, \tag{1.10}$$

and

$$a_t(t;\ v,\ w) = \sum_{i,j=1}^{N}\left(\frac{\partial a_{ij}}{\partial t}\ \frac{\partial v}{\partial x_j},\ \frac{\partial w}{\partial x_i}\right) = -(A_t v,\ w), \quad \text{for} \quad t \in T. \tag{1.11}$$

By virtue of (H3) we have

$$|a(t;\ v,\ w)| + |a_t(t;\ v,\ w)| \leq C\|v\|_1\|w\|_1, \quad \text{for} \quad t \in T;\ v,\ w \in H_0^1, \tag{1.12}$$

$$a(t;\ v,\ v) \geq C\|v\|_1^2, \quad \text{for} \quad t \in T,\ v \in H_0^1. \tag{1.13}$$

By the Taylor expansion we can write for $p,\ q \geq 0$

$$A^q = \sum_{j=0}^{s-1} \frac{1}{j!}(q-p)^j k^j A^p_{t^j} + \frac{1}{s!}(q-p)^s k^s A^{\bar{p}}_{t^s}; \tag{1.14}$$

$$a(t^q;\ v,\ w) - a(t^p;\ v,\ w) = (q-p)k a_t(t^{\bar{p}};\ v,\ w), \quad \text{for} \quad v,\ w \in H^1, \tag{1.15}$$

where $\bar{p}$'s denote numbers between p and q.

To end this section we point out that (1.4)—(1.6) possesses a unique solution $\{U^n\}_{n=1}^M$ for given $\tilde{u}^0$, since (1.4)—(1.6) is equivalent to a series of positive definite systems of linear equations.

§ 2. Elliptic Approximation

Let $V = V(t)$: $t \to S_h^r$ be the elliptic approximation of u, the solution of (1.1)—(1.3), namely

$$a(t;\ V-u,\ v_h) = 0, \quad \text{for} \quad v_h \in S_h^r,\ t \in T. \tag{2.1}$$

It is well known (cf. [7]) that we have the following error estimates

$$\|V-u\|_0 + h\|V-u\|_1 \leq Ch^{r+1}\|u\|_{r+1,\infty}, \quad \text{for} \quad t \in T, \tag{2.2}$$

$$\begin{aligned}\|V_t - u_t\|_0 &\leq Ch^{r+1}\|u_t\|_{r+1,\infty} = Ch^{r+1}\|Au\|_{r+1,\infty} \\ &\leq Ch^{r+1}\|u\|_{r+3,\infty}, \quad \text{for} \quad t \in T,\end{aligned} \tag{2.3}$$

where

$$\|u\|_{s,\infty} = \sup_{t \in T}\|u(t)\|_s.$$

Let $\{W_h^n\}_{n=1}^M \subset S_h^r$ be the solutions of the following equations

$$(W_h^{n+1},\ v_h) + bka(t^{n+\frac{2}{3}};\ W_h^{n+1},\ v_h) = -ka(t^{n+b};\ V^n,\ v_h),$$
$$\text{for} \quad v_h \in S_h^r,\ n = 0,\ 1,\ \cdots,\ M-1. \tag{2.4}$$

Let $\{W_*^n\}_{n=1}^M \subset H_0^1$ satisfy

$$(W_*^{n+1}, v)+bka(t^{n+\frac{2}{3}}; W_*^{n+1}, v) = -ka(t^{n+b}; u^n, v),$$
$$\text{for} \quad v\in H_0^1,\ n=0, 1, \cdots, M-1. \tag{2.5}$$

Equation (2.5) can be rewritten in operator form

$$(I-bkA^{n+\frac{2}{3}})W_*^{n+1}=kA^{n+b}u^n, \text{ in } H^{-1}. \tag{2.6}$$

Using (H3) we know that for given $v\in H^s$ the equation

$$(I-bkA)W=v, \quad \text{in } H^{-1}$$

has a unique solution $W\in H_0^1\cap H^{s+2}$, where $t\geqslant 0$, $k>0$, $s\geqslant -1$ (cf. [6]), that is, $(I-bkA)^{-1}$: $H^s\to H^{s+2}\cap H_0^1$ are well defined operators and we can deduce that $\|(I-bkA)^{-1}v\|_{s+2}\leqslant C(k)\|v\|_s$. But what is useful for us is the following regularity estimate.

Lemma 1. *For all $v\in H^{s+1}$, $0\leqslant s\leqslant r+1$, $t\in T$, $k>0$, we have*

$$\|(I-bkA)^{-1}v\|_s\leqslant\begin{cases} C\|v\|_s, & \textit{for even numbers,}\\ C\|v\|_{s+1}, & \textit{for odd numbers.}\end{cases} \tag{2.7}$$

Proof. Setting $W=(I-bkA)^{-1}v\in H_0^1$ for $v\in H^{-1}$, we have from (1.10) and (1.13) that

$$\begin{aligned}\|v\|_0=\|(I-bkA)w\|_0&=\sup_{\bar v\in H^0,\|\bar v\|_0=1}((I-bkA)w, \bar v)\\ &\geqslant((I-bkA)w, w/\|w\|_0)=[(w, w)+bka(t; w, w)]/\|w\|_0\\ &\geqslant\|w\|_0=\|(I-bkA)^{-1}v\|_0.\end{aligned} \tag{2.8}$$

Furthermore, for even number $s=2p$ $(p\geqslant 1)$, we have by (1.8) that

$$\|(I-bkA)^{-1}v\|_{2p}=\|A^{-1}(I-bkA)^{-1}Av\|_{2p}\leqslant C\|(I-bkA)^{-1}Av\|_{2(p-1)}.$$

Hence, by induction we conclude by (2.8) that

$$\|(I-bkA)^{-1}v\|_{2p}\leqslant C\|Av\|_{2(p-1)}\leqslant C\|v\|_{2p}.$$

For odd number $s=2p-1$ we simply get (2.7) in terms of

$$\|(I-bkA)^{-1}v\|_{2p-1}<\|(I-bkA)^{-1}v\|_{2p}\leqslant C\|v\|_{2p}.$$

The proof is then completed.

Lemma 2. *For all $0\leqslant n<M$ we have*

$$\|W_*^{n+1}-W_h^{n+1}\|_1\leqslant Ck^{\frac{1}{2}}h^{r+1}\|u\|_{r+4,\infty}. \tag{2.9}$$

Proof. Subtracting (2.4) from (2.5) we get by (2.1) and (1.15) that for $v_h\in S_h^r$

$$\begin{aligned}(W_*^{n+1}-W_h^{n+1}, v_h)&+bka(t^{n+\frac{2}{3}}; W_*^{n+1}-W_h^{n+1}, v_h)\\ &=-ka(t^{n+b}; u^n-V^n, v_h)\\ &=-ka(t^n; u^n-V^n, v_h)-bk^2a_t(t^{\bar n}; u^n-V^n, v_h)\\ &=-bk^2a_t(t^{\bar n}; u^n-V^n, v_h),\end{aligned}$$

where and bellow we always use $\bar n\in(n, n+1)$ to denote constants which are not necessarily the same at any two places.

Then, it follows that

$$(W_*^{n+1}-W_h^{n+1},\ W_*^{n+1}-W_h^{n+1})+bka(t^{n+\frac{2}{3}};\ W_*^{n+1}-W_h^{n+1},\ W_*^{n+1}-W_h^{n+1})$$
$$=(W_*^{n+1}-W_h^{n+1},\ W_*^{n+1}-v_h)+bka(t^{n+\frac{2}{3}};\ W_*^{n+1}-W_h^{n+1},\ W_*^{n+1}-v_h)$$
$$-bk^2a_t(t^n;\ u^n-V^n,\ v_h-W_h^{n+1}). \tag{2.10}$$

In virtue of (2.10) and (1.12), (1.13) we have

$$\|W_*^{n+1}-W_h^{n+1}\|_0^2+k\|W_*^{n+1}-W_h^{n+1}\|_1^2$$
$$\leqslant C\inf_{v_h\in S_h^r}\{\|W_*^{n+1}-W_h^{n+1}\|_0\|W_*^{n+1}-v_h\|_0$$
$$+k\|W_*^{n+1}-W_h^{n+1}\|_1\|W_*^{n+1}-v_h\|_1$$
$$+k^2\|u^n-V^n\|_1(\|W_*^{n+1}-v_h\|_1+\|W_*^{n+1}-W_h^{n+1}\|_1)\}.$$

This implies by (H1), (H4) and (2.2) that

$$\|W_*^{n+1}-W_h^{n+1}\|_0+k^{\frac{1}{2}}\|W_*^{n+1}-W_h^{n+1}\|_1$$
$$\leqslant C\{k^{\frac{3}{2}}h^r\|u\|_{r+1,\infty}+h^{r+1}\|W_*^{n+1}\|_{r+1}\}\leqslant Ckh^{r+1}\|u\|_{r+4,\infty},$$

where we have used the fact that by Lemma 1,

$$\|W_*^{n+1}\|_{r+1}=k\|(I-bkA^{n+\frac{2}{3}})^{-1}A^{n+b}u^n\|_{r+1}$$
$$\leqslant Ck\|A^{n+b}u^n\|_{r+2}\leqslant Ck\|u^n\|_{r+4}\leqslant Ck\|u\|_{r+4,\infty}.$$

Now, Setting $D_tV^n=V^{n+1}-V^n$, $\theta^n=V^n-u^n$, we write the equality

$$(D_tV^n,\ v_h)+bka(t^{n+\frac{2}{3}};\ D_tV^n,\ v_h)+ka(t^{n+b};\ V^n,\ v_h)$$
$$-\frac{1}{4}Ck[a(t^{n+1};\ V^n,\ v_h)-a(t^n;\ V^n,\ v_h)$$
$$+a(t^{n+\frac{2}{3}};\ W^{n+1},\ v_h)]=(\Delta^n,\ v_h),\quad \text{for } v_h\in S_h^r,\ 0\leqslant n\leqslant M-1, \tag{2.11}$$

where $\Delta^n=\Delta_1^n+\Delta_2^n$ and

$$(\Delta_1^n,\ v_h)=(D_t\theta^n,\ v_h)+bka(t^{n+\frac{2}{3}};D_t\theta^n,\ v_h)$$
$$+ka(t^{n+b};\ \theta^n,\ v_h)-\frac{1}{4}Ck[a(t^{n+1};\ \theta^n,\ v_h)$$
$$-a(t^n;\ \theta^n,\ v_h)+a(t^{n+\frac{2}{3}};\ W_h^{n+1}-W_*^{n+1},\ v_h)], \tag{2.12}$$

$$(\Delta_2^n,\ v_h)=\Big(D_tu^n-bkA^{n+\frac{2}{3}}D_tu^n-kA^{n+b}u^n$$
$$+\frac{1}{4}Ck[(A^{n+1}-A^n)u^n+A^{n+\frac{2}{3}}W_*^{n+1}],\ v_h\Big). \tag{2.13}$$

Lemma 3. *For all $v\in S_h^r$, $0\leqslant n\leqslant M-1$, we have*

$$|(\Delta^n,\ v_h)|\leqslant Ck(k^3+h^{r+1})\|u\|_{p,\infty}(\|v_h\|_0+k^{\frac{1}{2}}\|v_h\|_1), \tag{2.14}$$

where $p=\max(8,\ r+4)$.

Proof. By (2.3) we have

$$\|D_t\theta^n\|_0=\int_{t^n}^{t^{n+1}}\|\theta_t\,dt\|_0\leqslant\int_{t^n}^{t^{n+1}}\|\theta_t\|_0\,dt$$
$$\leqslant Ckh^{r+1}\|u_t\|_{r+1,\infty}=Ckh^{r+1}\|Au\|_{r+1,\infty}\leqslant Ckh^{r+1}\|u\|_{p,\infty}. \tag{2.15}$$

By (2.1) and (1.15), (1.11) it follows that

$$\left|a(t^{n+\frac{2}{3}};\ D_t\theta^n,\ v_h)\right| = \left|a(t^{n+1};\ \theta^{n+1},\ v_h) - \frac{1}{3}ka_t(t^n;\ \theta^{n+1},\ v_h)\right.$$

$$\left. - a(t^n;\ \theta^n,\ v_h) - \frac{2}{3}ka_t(t^n;\ \theta^n,\ v_h)\right|$$

$$\leqslant Ck(\|\theta^{n+1}\|_1 + \|\theta^n\|_1)\|v_h\|_1 \leqslant Ckh^r\|u\|_{r+1,\infty}\|v_h\|_1 \leqslant Ckh^r\|u\|_{p,\infty}\|v_h\|_1. \tag{2.16}$$

Similarly

$$\left|a(t^{n+b};\ \theta^n,\ v_h) - \frac{1}{4}C[a(t^{n+1};\ \theta^n,\ v_h) - a(t^n;\ \theta^n,\ v_h)]\right|$$

$$= \left|bka_t(t^n;\ \theta^n,\ v_h) - \frac{1}{4}Cka_t(t^n;\ \theta^n,\ v_h)\right|$$

$$\leqslant Ck\|\theta^n\|_1\|v_h\|_1 \leqslant Ckh^r\|u\|_{p,\infty}\|v_h\|_1. \tag{2.17}$$

Using Lemma 2, we know that

$$|a(t^{n+\frac{2}{3}};\ W_h^{n+1} - W_*^{n+1},\ v_h)| \leqslant C\|W_h^{n+1} - W_*^{n+1}\|_1\|v_h\|_1 \leqslant Ck^{\frac{1}{2}}h^{r+1}\|u\|_{p,\infty}\|v_h\|_1. \tag{2.18}$$

Combining (2.15)—(2.18) we get from (1.12) that

$$(\Delta_1^n,\ v_h) \leqslant Ckh^{r+1}\|u\|_{p,\infty}(\|v_h\|_0 + k^{\frac{1}{2}}\|v_h\|_1). \tag{2.19}$$

Next, we estimate Δ_2^n. Under the hypothesis (H3) we can write by virtue of (1.1) that for $t\in T$

$$Au = u_t, \tag{2.20}$$

$$Au_{tt} = (Au)_{tt} - A_{tt}u - 2A_tu_t = u_{ttt} - A_{tt}u - 2A_tu_t \tag{2.21}$$

and so on. By these equalities and (1.9) we have

$$\left\|\frac{\partial^s u}{\partial t^s}\right\|_{8-2s,\infty} \leqslant C\|u\|_{8,\infty} \leqslant C\|u\|_{p,\infty}, \quad \text{for} \quad 0\leqslant s\leqslant 4. \tag{2.22}$$

By (2.6) we have

$$-A^{n+1}W_*^{n+1} = -kA^{n+\frac{2}{3}}(I - bkA^{n+\frac{2}{3}})^{-1}A^{n+b}u^n$$

$$= \frac{1}{b}(I - (I - bkA^{n+\frac{2}{3}})^{-1})A^{n+b}u^n. \tag{2.23}$$

For $v\in H^6$, $t\in T$, we can write

$$(I - bkA)^{-1}v = (I + bkA + b^2k^2AA + (I - bkA)^{-1}b^3k^3AAA)v. \tag{2.24}$$

Therefore, by setting $p = n + \frac{1}{2}$ in (1.14) and using (2.20)—(2.24) and (2.7) it is easy to deduce that

$$\|\Delta_2^n\|_0 = \left\|(I - bkA^{n+\frac{2}{3}})D_tu^n\right.$$

$$\left. + \left\{-kA^{n+b} + \frac{1}{4}Ck[A^{n+1} - A^n + kA^{n+\frac{2}{3}}(I - bkA^{n+\frac{2}{3}})^{-1}A^{n+b}]\right\}u^n\right\|_0$$

$$\leqslant Ck^4\sum_{s=0}^{4}\left\|\frac{\partial^s u}{\partial t^s}\right\|_{8-2s,\infty} \leqslant Ck^4\|u\|_{8,\infty}. \tag{2.25}$$

Now, (2.14) holds by (2.19) and (2.25).

§ 3. On the Discrete Elliptic Operator A_h

Define the projection operator P_h: $H^{-1}\to S_h^r$ satisfying

$$(P_h v-v,\ w_h)=0, \quad \text{for} \quad w_h\in S_h^r. \tag{3.1}$$

Let $A_h=P_hAP_h$. Then we have (cf. [4])

$$-(A_h v_h,\ w_h)=a(t;\ v_h,\ w_h), \quad \text{for} \quad v_h,\ w_h\in S_h^r,\ t\in T. \tag{3.2}$$

Lemma 4. *For $t\in T$, $v_h\in S_h^r$, we have*

$$\|(I-bkA_h)^{-1}v_h\|_1\leqslant C\|v_h\|_1, \tag{3.3}$$

$$\|(I-bkA_h)^{-1}v_h\|_0\leqslant \|v_h\|_0. \tag{3.4}$$

Proof. Notice that by (3.2) the linear operator $-A_h$ is positive and symmetric in S_h^r so that there exist $\lambda_i>0$, $v_i\in S_h^r$ for $i=1,\ 2,\ \cdots,\ n$ (n is the dimension of S_h^r) satisfying

$$-A_h v_i=\lambda_i v_i,\ (v_i,\ v_j)=\delta_{ij}, \quad \text{for} \quad 1\leqslant i,\ j\leqslant n, \tag{3.5}$$

and for any $v_h\in S_h^r$

$$v_h=\sum_{i=1}^n (v_h,\ v_i)v_i. \tag{3.6}$$

Then, we can write

$$\begin{aligned}\|v_h\|_1^2&\leqslant Ca(t;\ v_h,\ v_h)=-C(A_h v_h,\ v_h)\\ &=C\left(\sum_{i=1}^n -A_h v_i(v_h,\ v_i),\ \sum_{i=1}^n (v_h,\ v_i)v_i\right)\\ &=C\left(\sum_{i=1}^n \lambda_i v_i(v_h,\ v_i),\ \sum_{i=1}^n (v_h,\ v_i)v_i\right)=C\sum_{i=1}^n \lambda_i(v_h,\ v_i)^2.\end{aligned} \tag{3.7}$$

Similarly, noticing $\|v_h\|_1^2\geqslant Ca(t;\ v_h,\ v_h)$ we can write

$$\sum_{i=1}^n \lambda_i(v_h,\ v_i)^2\leqslant C\|v_h\|_1^2. \tag{3.8}$$

Then, by (3.7), (3.8) we have

$$\begin{aligned}\|(I-bkA_h)^{-1}v_h\|_1^2&\leqslant C\sum_{i=1}^n \lambda_i((I-bkA_h)^{-1}v_h,\ v_i)^2\\ &=C\sum_{i=1}^n \lambda_i((1+bk\lambda_i)^{-1}v_i,\ v_i)^2(v_h,\ v_i)^2\\ &=C\sum_{i=1}^n \lambda_i(1+bk\lambda_i)^{-2}(v_h,\ v_i)^2\\ &\leqslant C\sum_{i=1}^n \lambda_i(v_h,\ v_i)^2\leqslant C\|v_h\|_1^2.\end{aligned}$$

Therefore (3.3) holds.

It is easy to show (3.4) by using the same argument as in (2.8). The proof is ended.

Define

$$L_h^q=A_h^q(I-bkA_h^q)^{-1}A_h^q, \quad \text{for} \quad q\geqslant 0,$$

where $A_h^q=P_hA^qP_h$. Obviously L_h^q's are symmetric and positive in S_h^r since $-A_h^q$'s are. Therefore S_h^r becomes a Hilbert space equipped with inner product

$$\langle v_h, w_h \rangle_q = (L_h^q v_h, w_h)$$

and the corresponding norm

$$|v_h|_q^2 = \langle v_h, v_h \rangle_q.$$

Lemma 5. *For $v_h \in S_h^r$, $q, p > 0$ with $|q-p| \leqslant C$, we have*

$$|v_h|_q^2 \leqslant \frac{1}{bk} a(t^q; v_h, v_h), \tag{3.9}$$

$$|v_h|_q^2 - |v_h|_p^2 \leqslant C \|v_h\|_1^2. \tag{3.10}$$

Proof. It is easy to see that

$$L_h^q = -\frac{1}{bk} A_h^q - \frac{1}{b^2k^2} I + \frac{1}{b^2k^2}(I - bkA_h^q)^{-1}. \tag{3.11}$$

Therefore, by (3.4) we have

$$\begin{aligned} |v_h|_q^2 = (L_h^q v_h, v_h) &= -\frac{1}{bk}(A_h^q v_h, v_h) - \frac{1}{b^2k^2}(v_h, v_h) + \frac{1}{b^2k^2}((I - bkA_h^q)^{-1} v_h, v_h) \\ &\leqslant \frac{1}{bk} a(t_h^q; v_h, v_h) - \frac{1}{b^2k^2} \|v_h\|_0^2 + \frac{1}{b^2k^2} \|(I - bkA_h^q)^{-1} v_h\|_0 \|v_h\|_0 \\ &\leqslant \frac{1}{bk} a(t^q; v_h, v_h). \end{aligned}$$

This implies (3.9).

Besides, we can write

$$(I - bkA_h^q)^{-1} - (I - bkA_h^p)^{-1} = (I - bkA_h^q)^{-1}(bkA_h^q - bkA_h^p)(I - bkA_h^p)^{-1}. \tag{3.12}$$

Then, by (3.12), (3.2), (1.5), (1.12) and (3.3), it follows that

$$\begin{aligned} &|((I - bkA_h^q)^{-1} v_h, v_h) - ((I - bkA_h^p)^{-1} v_h, v_h)| \\ &= |((I - bkA_h^q)^{-1}(bkA_h^q - bkA_h^p)(I - bkA_h^p)^{-1} v_h, v_h)| \\ &= |bk((A_h^q - A_h^p)(I - bkA_h^p)^{-1} v_h, (I - bkA_h^q)^{-1} v_h)| \\ &= |bk^2(q-p) a_t(t^{\bar{p}}; (I - bkA_h^p)^{-1} v_h, (I - bkA_h^q)^{-1} v_h)| \\ &\leqslant Ck^2 \|(I - bkA_h^p)^{-1} v_h\|_1 \|(I - bkA_h^q)^{-1} v_h\|_1 \leqslant Ck^2 \|v_h\|_1^2. \end{aligned} \tag{3.13}$$

In virtue of (3.11) and (3.13) we get (3.10) since by (1.15)

$$\begin{aligned} |v_h|_q^2 - |v_h|_p^2 &= -\frac{1}{bk}((A_h^q - A_h^p) v_h, v_h) \\ &\quad + \frac{1}{b^2k^2}(((I - bkA_h^q)^{-1} - (I - bkA_h^p)^{-1}) v_h, v_h) \\ &\leqslant C a_t(t^{\bar{p}}; v_h, v_h) + C \|v_h\|_1^2 \leqslant C \|v_h\|_1^2. \end{aligned}$$

This completes the proof.

§ 4. Error Estimate

By (1.4) $\times \frac{3}{4}$ + (1.5) $\times \frac{1}{4}$, we get form (1.6)

$$(D_tU^n, v_h)+bka(t^{n+\frac{2}{3}}; D_tU^n, v_h)+ka(t^{n+b}; U^n, v_h)$$
$$-\frac{1}{4}Ck[a(t^{n+1}; U^n, v_h)-a(t^n; U^n, v_h)+a(t^{n+\frac{2}{3}}; W^{n+1}, v_h)]=0. \tag{4.1}$$

Set $e^n=V^n-U^n$ and subtract (4.1) form (2.11). Then it follows by choosing $v_h=e^{n+1}+e^n$ that

$$(D_te^n, e^{n+1}+e^n)+bka(t^{n+\frac{2}{3}}; D_te^n, e^{n+1}+e^n)$$
$$+ka(t^{n+b}; e^n, e^{n+1}+e^n)-\frac{1}{4}Ck[a(t^{n+1}; e^n, e^{n+1}+e^n)$$
$$-a(t^n; e^n, e^{n+1}+e^n)+a(t^{n+\frac{2}{3}}, W_h^{n+1}-W^{n+1}, e^{n+1}+e^n)]$$
$$=(\Delta^n, e^{n+1}+e^n). \tag{4.2}$$

Lemma 6. *For* $0\leqslant n<M$, *we have*

$$a(t^{n+b}; e^n, e^{n+1}+e^n)-\frac{1}{4}Ca(t^{n+\frac{2}{3}}; W_h^{n+1}-W^{n+1}, e^{n+1}+e^n)$$
$$\geqslant\frac{1}{2}a(t^n; e^n, e^n)-\frac{1}{2}a(t^{n+1}; e^{n+1}, e^{n+1})$$
$$+\frac{1}{8}Ck[|e^n|_n^2-|e^{n+1}|_{n+1}^2]-Ck(\|e^{n+1}\|_1^2+\|e^n\|_1^2). \tag{4.3}$$

Proof. First, it is easy to show that for $t\in T$ and $v, w\in H_0^1$,

$$a(t; v, w)\leqslant\frac{1}{2}a(t; v, v)+\frac{1}{2}a(t; w, w), \tag{4.4}$$

and for $q\geqslant 0$; $v_h, w_h\in S_h^r$,

$$\langle v_h, w_h\rangle_q\leqslant\frac{1}{2}|v_h|_q^2+\frac{1}{2}|w_h|_q^2. \tag{4.5}$$

Hence, in view of (1.15), (4.4) and (1.12) it follows that

$$a(t^{n+b}; e^n, e^{n+1}+e^n)\geqslant a(t^n; e^n, e^n)-bka_t(t^{\bar{n}}; e^n, e^n)$$
$$-\frac{1}{2}a(t^{n+b}; e^n, e^n)-\frac{1}{2}a(t^{n+b}; e^{n+1}, e^{n+1})$$
$$=a(t^n; e^n, e^n)-bka_t(t^{\bar{n}}; e^n, e^n)-\frac{1}{2}a(t^n; e^n, e^n)$$
$$-\frac{1}{2}bka_t(t^{\bar{n}}; e^n, e^n)-\frac{1}{2}a(t^{n+1}; e^{n+1}, e^{n+1})$$
$$+\frac{1}{2}(1-b)ka_t(t^{\bar{n}}; e^{n+1}, e^{n+1})\geqslant\frac{1}{2}a(t^n; e^n, e^n)$$
$$-\frac{1}{2}a(t^{n+1}; e^{n+1}, e^{n+1})-Ck(\|e^n\|_1^2+\|e^{n+1}\|_1^2). \tag{4.6}$$

Next, by (1.4) and (2.4) we can write, by (3.2),

$$W_h^{n+1}-W^{n+1}=k(I-bkA_h^{n+\frac{2}{3}})^{-1}A_h^{n+b}e^n.$$

Then, in view of the definitions of L_h^q and A_h^q it follows that

$$-a\left(t^{n+\frac{2}{3}};\ W_h^{n+1}-W^{n+1},\ e^{n+1}+e^n\right)$$
$$=k\left(A_h^{n+\frac{2}{3}}(I-bkA_h^{n+\frac{2}{3}})^{-1}A_h^{n+b}e^n,\ e^{n+1}+e^n\right)$$
$$=k\left(A_h^{n+\frac{2}{3}}e^n,\ (I-bkA_h^{n+\frac{2}{3}})^{-1}A_h^{n+\frac{2}{3}}(e^{n+1}+e^n)\right)$$
$$+\left(b-\frac{2}{3}\right)k^2a_t\left(t^n,\ e^n,\ (I-bkA_h^{n+\frac{2}{3}})^{-1}A_h^{n+\frac{2}{3}}(e^{n+1}+e^n)\right)$$
$$\geqslant k\langle e^n, e^{n+1}+e^n\rangle_{n+\frac{2}{3}}-Ck\|e^n\|_1\|(I-bkA_h^{n+\frac{2}{3}})^{-1}A_h^{n+\frac{2}{3}}(e^{n+1}+e^n)\|_1. \tag{4.7}$$

But by Lemma 5 and (4.5),

$$\langle e^n, e^{n+1}+e^n\rangle_{n+\frac{2}{3}}=|e^n|^2_{n+\frac{2}{3}}+\langle e^n, e^{n+1}\rangle_{n+\frac{2}{3}}$$
$$\geqslant|e^n|^2_{n+\frac{2}{3}}-\frac{1}{2}\left(|e^n|^2_{n+\frac{2}{3}}+|e^{n+1}|^2_{n+\frac{2}{3}}\right)$$
$$=\frac{1}{2}|e^n|^2_{n+\frac{2}{3}}-\frac{1}{2}|e^{n+1}|^2_{n+\frac{2}{3}}$$
$$\geqslant\frac{1}{2}|e^n|^2_n-\frac{1}{2}|e^{n+1}|^2_{n+1}-C\left(\|e^{n+1}\|_1^2+\|e^n\|_1^2\right). \tag{4.8}$$

By Lemma 4 we have

$$\|(I-bkA_h^{n+\frac{2}{3}})^{-1}bkA_h^{n+\frac{2}{3}}(e^{n+1}+e^n)\|_1$$
$$=\|[(I-bkA_h^{n+\frac{2}{3}})^{-1}-I](e^{n+1}+e^n)\|_1$$
$$\leqslant\|(I-bkA^{n+\frac{2}{3}})^{-1}(e^{n+1}+e^n)\|_1+\|e^{n+1}+e^n\|_1$$
$$\leqslant C\|e^{n+1}+e^n\|_1\leqslant C\left(\|e^n\|_1+\|e^{n+1}\|_1\right). \tag{4.9}$$

From (4.7)—(4.9) we get

$$-a\left(t^{n+\frac{2}{3}};\ W_h^{n+1}-W^{n+1},\ e^{n+1}+e^n\right)$$
$$\geqslant\frac{1}{2}k\left(|e^n|^2_n-|e^{n+1}|^2_{n+1}\right)-Ck\left(\|e^{n+1}\|_1^2+\|e^n\|_1^2\right). \tag{4.10}$$

Finally, by combining (4.6) and (4.10) we obtain (4.3).

Now, having done so much preparative work, we can show the following error estimate.

Theorem. *Under the hypotheses* (H1)—(H4), *we have for* $0\leqslant n\leqslant M$ *and sufficiently small* k

$$\|u^n-U^n\|_0+k^{\frac{1}{2}}\|u^n-U^n\|_1\leqslant C(h^{r+1}+k^3)\|u\|_{p,\infty}, \tag{4.11}$$

where $p=\max(r+4,\ 8)$.

Proof. Notice that

$$(D_te^n,\ e^{n+1}+e^n)=\|e^{n+1}\|_0^2-\|e^n\|_0^2, \tag{4.12}$$

$$a\left(t^{n+\frac{2}{3}};\ D_te^n,\ e^{n+1}+e^n\right)=a\left(t^{n+\frac{2}{3}};\ e^{n+1},\ e^{n+1}\right)-a\left(t^{n+\frac{2}{3}};\ e^n,\ e^n\right)$$
$$\geqslant a(t^{n+1};\ e^{n+1},\ e^{n+1})-a(t^n;\ e^n,\ e^n)-Ck\left(\|e^{n+1}\|_1^2+\|e^n\|_1^2\right), \tag{4.13}$$

$$|a(t^{n+1};\ e^n,\ e^{n+1}+e^n)-a(t^n;\ e^n,\ e^{n+1}+e^n)|$$
$$=|ka_t(t^n;\ e^n,\ e^{n+1}+e^n)|\leqslant Ck\left(\|e^{n+1}\|_1^2+\|e^n\|_1^2\right). \tag{4.14}$$

Then, using (4.3) and (4.12)—(4.14) in (4.2) we have for $0\leqslant n<M$

$$\|e^{n+1}\|_0^2-\|e^n\|_0^2+k\left(b+\frac{1}{2}\right)[a(t^{n+1};\ e^{n+1},\ e^{n+1})-a(t^n;\ e^n,\ e^n)]$$
$$+\frac{1}{8}Ck[|e^n|_n^2-|e^{n+1}|_{n+1}^2]-Ck^2(\|e^{n+1}\|_1^2+\|e^n\|_1^2)\leqslant(\varDelta^n,\ e^{n+1}+e^n).$$

Hence, adding the above inequalities from $n=0$ to $n=m-1(<M)$, we have

$$\|e^m\|_0^2+\left(b-\frac{1}{2}\right)ka(t^m;\ e^m,\ e^m)-\frac{1}{8}Ck^2|e^m|_m^2$$
$$\leqslant Ck^2\sum_{n=0}^{m-1}(\|e^{n+1}\|_1^2+\|e^n\|_1^2)+\sum_{n=0}^{m-1}(\varDelta^n,\ e^{n+1}+e^n)$$
$$+\|e^0\|_0^2+\left(b-\frac{1}{2}\right)ka(t^0;\ e^0,\ e^0). \tag{4.15}$$

But using (3.9) we know that

$$\left(b-\frac{1}{2}\right)a(t^m;\ e^m,\ e^m)-\frac{1}{8}Ck|e^m|_m^2\geqslant\left(b-\frac{1}{2}-\frac{C}{8b}\right)a(t^m;\ e^m,\ e^m)$$
$$=\frac{1}{2(3+\sqrt{3})}a(t^m;\ e^m,\ e^m)\geqslant C\|e^m\|_1^2. \tag{4.16}$$

On the other hand, by (H2), (H4) and (2.2) we have

$$\|e^0\|_0^2+Ck\|e^0\|_1^2=\|V^0-\tilde{u}^0\|_0^2+Ck\|V^0-\tilde{u}^0\|_1^2$$
$$\leqslant C(\|V^0-u^0\|_0^2+\|u^0-\tilde{u}^0\|_0^2+k\|V^0-u^0\|_1^2+k\|u^0-\tilde{u}^0\|_1^2)$$
$$\leqslant C(h^{r+1}+k^{\frac{1}{2}}h^r)^2\|u^0\|_{r+1,\infty}^2\leqslant Ch^{2(r+1)}\|u\|_{p,\infty}^2. \tag{4.17}$$

Therefore, by (4.15)—(4.17) and Lemma 3 we deduce for sufficiently small k

$$\|e^m\|_0^2+k\|e^m\|_1^2\leqslant Ck\sum_{n=1}^{m-1}(\|e^n\|_0^2+k\|e^n\|_1^2)+C(h^{r+1}+k^3)^2\|u\|_{p,\infty}^2.$$

Finally, using the discrete Gronwall lemma, we get for $0\leqslant m\leqslant M$

$$\|e^m\|_0^2+k\|e^m\|_1^2\leqslant C(h^{r+1}+k^3)^2\|u\|_{p,\infty}^2.$$

This implies (4.10) immediately.

Remark 2. For odd number r, (4.11) holds with $p=\max(r+3,\ 8)$ by noticing (2.7).

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