ASYMPTOTIC RADIATION CONDITIONS FOR REDUCED WAVE EQUATION*

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Abstract

In this note the exact non-local radiation condition and its local approximations at finite artificial boundary for the exterior boundary value problem of the reduced wave equation in 2 and 3 dimensions are discussed. Based on the asymptotic expansion of Hankel functions for large arguments, an approach for the construction of local approximations is suggested and gives expression of the normal derivative at spherical artificial boundary in terms of linear combination of Laplace-Beltrami operator and its iterates, i.e. tangential derivatives of even order exclusively. The resulting formalism is compatible with the usual variational principle and the finite element methodology and thus seems to be convenient in practical implementation.

Boundary value problems of P. D. E. involving infinite domain occur in many areas of applications, e. g., fluid flow around obstacles, coupling of structures with foundation and environment, scattering and radiation of waves and so on. For the numerical solution of this class of problems, the natural approach is to cut off an infinite part of the domain and to set up, at the computational boundary of the remaining finite domain, appropriate artificial boundary conditions. In the usual treatment, the latter is carried out, however, in an oversimplified way without sufficient justification. Along this line of approach, there is recent interest and progress leading to a latter understanding of the nature of the problem and several sequences of improved artificial boundary conditions. In the following we shall discuss briefly, for the reduced wave equation with spherical computational boundary, the exact integral boundary condition at the spherical computational boundary, and suggest, using asymptotic expansions of Hankel functions, a method for deriving a sequence of approximations of the non-local boundary operator by means of tangential differential operators on the boundary, in a form which is compatible with the variational form and the finite element method for the original problem.

I

The general solution of the 2-D reduced wave (Helmholtz) equation

$$\Delta_2 u+\omega^2 u=0, \quad \Delta_2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \tag{1.1}$$

in the exterior $\Omega_a=\{r>a\}$ to the circle $\Gamma_a=\{r=a\}$ of radius a satisfying the radiation condition at infinity

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$$u=O(r^{-1}),$$
$$u_r+i\omega u=o(r^{-\frac{1}{2}}) \quad \text{as} \quad r\to\infty \tag{1.2}$$

can be represented as a Fourier series

$$u(r,\ \theta)=\sum_{-\infty}^{\infty} A_n H_n^{(2)}(\omega r)e^{in\theta}, \tag{1.3}$$

where $H_n^{(2)}$ is the Hankel function of the 2nd kind of order n, and in particular,

$$u(a,\ \theta)=\sum_{-\infty}^{\infty} A_n H_n^{(2)}(\omega a)e^{in\theta}. \tag{1.4}$$

So (1.3) can be written as

$$u(r,\ \theta)=\sum_{-\infty}^{\infty}\left(\frac{H_n^{(2)}(\omega r)}{H_n^{(2)}(\omega a)}\right)A_n H_n^{(2)}(\omega a)e^{in\theta}.$$

(1.4) and (1.5) together give the Poisson integral formula

$$u=P\hat{u} \tag{1.5}$$

expressing the solution $u=u(r,\ \theta)$ in domain Ω_a in terms of its Dirichlet data $\hat{u}=u(a,\ \theta)$ on boundary Γ_a, or explicitly

$$u(r,\ \theta)=P(\theta)*u(a,\ \theta)=\int_0^{2\pi}P(\theta-\theta')u(a,\ \theta')d\theta',\quad 0\leqslant\theta\leqslant 2\pi,\ r>a, \tag{1.6}$$

where

$$P(\theta)=\frac{1}{2\pi}\sum_{-\infty}^{\infty}\left(\frac{H_n^{(2)}(\omega r)}{H_n^{(2)}(\omega a)}\right)e^{in\theta}, \tag{1.7}$$

$*$ denotes the circular convolution

$$f(\theta)*g(\theta)=\int_0^{2\pi}f(\theta-\theta')g(\theta')d\theta'.$$

Differentiation of (1.3) gives

$$u_r(r,\ \theta)=\sum_{-\infty}^{\infty}A_n\omega H_n^{(2)\prime}(\omega r)e^{in\theta} \tag{1.8}$$

and

$$u_r(a,\ \theta)=\sum_{-\infty}^{\infty}A_n\omega H_n^{(2)\prime}(\omega a)e^{in\theta}=\sum_{-\infty}^{\infty}\left(\omega\frac{H_n^{(2)\prime}(\omega a)}{H_n^{(2)}(\omega a)}\right)A_n H_n^{(2)}(\omega a)e^{in\theta}. \tag{1.9}$$

(1.4) and (1.9) together give the integral relation

$$\hat{u}_\nu=-\hat{u}_r=\boldsymbol{K}\hat{u} \tag{1.10}$$

expressing the solution's normal derivative $\hat{u}_\nu=-\hat{u}_r=u_r(a,\ \theta)$ (ν is directed to the exterior of the domain Ω_a), i.e. the Neumann data on Γ_a, of the solution u in terms of the corresponding Dirichlet data $\hat{u}$, or explicitly

$$-u_r(a,\ \theta)=K(\theta)*u(a,\ \theta)=\int_0^{2\pi}K(\theta-\theta')u(a,\ \theta')d\theta', \tag{1.11}$$

where

$$K(\theta)=\frac{1}{2\pi}\sum_{-\infty}^{\infty}\left(-\omega\frac{H_n^{(2)\prime}(\omega a)}{H_n^{(2)}(\omega a)}\right)e^{in\theta}. \tag{1.12}$$

The integral in (1.11) is highly singular of non-integrable type, it is to be understood in the sense of regularization of divergent integrals in the theory of distributions. $\boldsymbol{K}$ is in fact a pseudo-differential operator of order 1 on the boundary maniford Γ_a and defines a linear continuous map

$$K: H^s(\Gamma_a) \to H^{s-1}(\Gamma_a). \tag{1.13}$$

Thus the differential operator $\Delta_2+\omega^2$ in the domain Ω_a induces an operator K, called the canonical integral operator, on the boundary Γ_a. The canonical integral operator induced by elliptic operator preserves all the essential properties of the latter, and plays a crucial role in the reduction of elliptic problems to boundary integral equations[1,2].

Parallel to the association of the operator $\Delta_2+\omega^2$ in domain Ω_a with the bilinear functional

$$D(u, v)=\int_{\Omega_a}(\nabla u\cdot\nabla\bar{v}+\omega^2 u\bar{v})dx=\int_a^\infty r\,dr\int_0^{2\pi}\left(u_r\bar{v}_r+\frac{1}{r^2}u_\theta\bar{v}_\theta+\omega^2 u\bar{v}\right)d\theta, \tag{1.14}$$

there is also an association of the operator K on boundary Γ_a with the linear functional

$$\hat{D}(\varphi, \psi)=a\int_0^{2\pi}\bar{\psi}(\theta)d\theta\int_0^{2\pi}K(\theta-\theta')\varphi(\theta')d\theta, \tag{1.15}$$

which is inherently related to $D(u, v)$ by the equality

$$D(u, v)=\hat{D}(\hat{u}, \hat{v})$$

for every solutions u, v satisfying (1.1), (1.2).

From the facts above follows immediately the equivalence of the exterior boundary problem, the Neumann problem, say,

$$\Omega:\ \Delta_2 u+\omega^2 u=0;\ u=O(r^{-1}),\ u_r+i\omega u=o(r^{-1/2}),\quad r\to\infty, \tag{1.16}$$

$$\Gamma:\ u_\nu=f, \tag{1.17}$$

where Ω is the domain exterior to the bounded closed curve Γ, ν is the normal on Γ, directed to the exterior of Ω, to the reduced boundary value problem

$$\Omega'_a:\ \Delta_2 u+\omega^2 u=0, \tag{1.18}$$

$$\Gamma:\ u_\nu=f, \tag{1.19}$$

$$\Gamma_a:\ u_r=-Ku, \tag{1.20}$$

where Ω'_a is the remaining bounded part of the domain Ω by deleting the infinite part $\bar{\Omega}_a=\{r\geqslant a\}\subset\Omega$, Γ_a is the thereby created computational boundary, which, together with Γ, form the complete boundary of the reduced domain $\Omega'_a=\Omega\backslash\bar{\Omega}_a$. The variational formulations of (1.16)—(1.17) and (1.18)—(1.20) are

$$D(u, v)=\int_{\Omega=\Omega'_a\cup\bar{\Omega}_a}(\nabla u\cdot\nabla\bar{v}+\omega^2 u\bar{v})dx=\int_\Gamma f\bar{v}\,dx$$

for every test function u in Ω, and

$$D'(u, v)=\int_{\Omega'_a}(\nabla u\cdot\nabla\bar{v}+\omega^2 u\bar{v})dx+\hat{D}(\hat{u}, \hat{v})=\int_\Gamma f\bar{v}\,dx$$

for every test function u in Ω'_a, respectively.

Thus we see that, upon deletion of the infinite domain $\Omega_a=\{r>a\}$, (1.20) is the exact radiation condition to be imposed on the computational boundary Γ_a, which, on the one hand, accounts for the full interaction of the deleted and remaining parts, and, on the other hand, has a formalism compatible with the variational formulation and the finite element method (FEM) for solving the original problem.

The boundary operator K is non-local, so, after FEM discretization, it becomes a full matrix with storage requirement $O(N^2)$, N is the number of boundary degrees of freedom. In the present case of circular boundary due to the convolutional nature of the operator, the resulting matrix is circulant, requiring only $O(N)$ storage. However, due to the analytical complexity of the kernel, the computational effort is always expensive. So it is disirable to approximate the exact but non-local radiation condition by local, i.e., differential boundary conditions. Moreover, the approximate boundary conditions, like the exact one, should be put in a form which is compatible with the original variational formulation and the FEM, so, we look for the approximations in the following form

$$-u_r = K\hat{u} \sim -u_r = K_p\hat{u} = \sum_{q=0}^{p} (-1)^q \alpha_q \partial_\theta^{2q} u(a, \theta), \tag{1.21}$$

and correspondingly

$$\hat{D}(\hat{u}, \hat{v}) \sim \hat{D}_p(\hat{u}, \hat{v}) = a\int_0^{2\pi} \sum_{q=0}^{p} \alpha_q \partial_\theta^q u(a, \theta) \cdot \partial_\theta^q \bar{v}(a, \theta) d\theta. \tag{1.22}$$

To this end, for the case of large ωa, a heuristic approach, based on asymptotic expansions of Hankel functions for large arguments, is as follows. We start from the asymptotic series for large ωa of the Fourier coefficient of the kernel $K(\theta)$,

$$-\omega \frac{H_n^{(2)\prime}(\omega a)}{H_n^{(2)}(\omega a)} = i\omega \sum_{q=0}^{\infty} c_q(\lambda_n)\tau^q, \quad \tau = \frac{1}{2i\omega a}, \quad \lambda_n = n^2 - \frac{1}{4}, \tag{1.23}$$

where $c_q(\lambda)$ are polynomials in λ, and

$$\begin{aligned} &c_0(\lambda) = c_1(\lambda) = 1, \\ &c_2(\lambda) = 2\lambda, \quad c_3(\lambda) = -4\lambda, \\ &c_4(\lambda) = -2\lambda(\lambda - 6), \quad c_5(\lambda) = 16\lambda(\lambda - 3), \\ &c_6(\lambda) = 4\lambda(\lambda^2 - 28\lambda + 60), \quad c_7(\lambda) = -16\lambda(4\lambda^2 - 51\lambda + 90), \\ &\cdots\cdots\cdots \end{aligned} \tag{1.24}$$

A discussion will be given in section III. Truncate the series at finite term

$$\left(-\omega \frac{H_n^{(2)\prime}(\omega a)}{H_n^{(2)}(\omega a)}\right) \sim i\omega \sum_{q=0}^{p} c_q(\lambda_n)\tau^q =: K_p(\lambda_n), \quad \tau = \frac{1}{2i\omega a}. \tag{1.25}$$

Note that $\partial_\theta^2 = \Delta_1 =$ Laplace-Beltrami operator on the unit circle and

$$\lambda_n e^{in\theta} = \left(n^2 - \frac{1}{4}\right)e^{in\theta} = \left(-\Delta_1 - \frac{1}{4}\right)e^{in\theta},$$

$$c_q(\lambda_n)e^{in\theta} = c_q\left(-\Delta_1 - \frac{1}{4}\right)e^{in\theta},$$

$$K_p(\lambda_n)e^{in\theta} = K_p\left(-\Delta_1 - \frac{1}{4}\right)e^{in\theta},$$

we have then, from (1.4), (1.5)

$$\begin{aligned} -u_r(a, \theta) &= \sum_{n=-\infty}^{\infty} \left(-\omega \frac{H_n^{(2)\prime}(\omega a)}{H_n^{(2)}(\omega a)}\right) A_n H_n^{(2)}(\omega a) e^{in\theta} \\ &\sim \sum_{n=-\infty}^{\infty} K_p(\lambda_n) A_n H_n^{(2)}(\omega a) = K_p\left(-\Delta_1 - \frac{1}{4}\right)u(a, \theta). \end{aligned}$$

So we obtain for large ωa a family of approximate differential boundary conditions, called the asymptotic radiation conditions, in the desired form of (1.21):

$$\Gamma_a\colon\ -u_r=K_p u,\ \text{i. e.}\ -u_r(a,\ \theta)=K_p\left(-\Lambda_1-\frac{1}{4}\right)u(a,\ \theta),\qquad (1.26)$$
$$p=0,\ 1,\ 2,\ \cdots.$$

In particular,

$$(K_0)\qquad -u_r=K_0u=i\omega u,$$

$$(K_1)\qquad -u_r=K_1u=\left(i\omega+\frac{1}{2a}\right)u,$$

$$(K_2)\qquad -u_r=K_2u=\left(i\omega+\frac{1}{2a}+\frac{i}{8\omega a^2}\right)u+\frac{i}{2\omega a^2}\Lambda_1u,$$

$$(K_3)\qquad -u_r=K_3u=\left(i\omega+\frac{1}{2a}+\frac{i}{8\omega a^2}-\frac{1}{8\omega^2a^3}\right)u+\left(\frac{i}{2\omega a^2}-\frac{1}{2\omega^2a^3}\right)\Lambda_1u,$$

$$(K_4)\qquad -u_r=K_4u=K_3u-\frac{1}{16\omega^3a^4}\left(\frac{25}{8}u+13\Lambda_1u+2\Lambda_1^2u\right),$$

$$(K_5)\qquad -u_r=K_5u=K_4u+\frac{1}{32\omega^4a^5}(13u+56\Lambda_1u+16\Lambda_1^2u),$$

$$\cdots\cdots\cdots$$

We may compare this family of asymptotic radiation conditions with the family of absorbing boundary conditions of Engquist and Majda[3], based on the factorization of pseudo-differential operators,

$$(E_1)\qquad -u_r=\left(i\omega+\frac{1}{2a}\right)u,$$

$$(E_2)\qquad -u_r=\left(i\omega+\frac{1}{2a}\right)u-\left(-\frac{i}{2\omega a^2}+\frac{1}{2\omega^2a^3}\right)\partial_\theta^2u,$$

$$\cdots\cdots\cdots$$

and with the family of Bayliss and Turkel[4], based on the asymptotics of the solution of the wave equation

$$(B_1)\qquad B_1u=u_r+\left(i\omega+\frac{1}{2a}\right)u=0,$$

$$(B_2)\qquad B_2u=u_{rr}+\left(-2i\omega+\frac{3}{a}\right)u_r+\left(\frac{3i\omega}{a}-\omega^2+\frac{3}{4a^2}\right)u=0,$$

$$\cdots\cdots\cdots$$

$$(B_k)\qquad B_ku=\left(\frac{\partial}{\partial r}+i\omega+\frac{4k-3}{r}\right)B_{k-1}u=0,\quad k=2,\ 3,\ \cdots.$$

Note that K_0 is simply the formal Sommerfeld condition, K_1, E_1, B_1 are the same. For $i\geqslant 2$, the three sequences K_i, E_i, B_i diverge. For $i\geqslant 3$, E_i and B_i are not expressible in the required form. From the table of the polynomials $c_q(\lambda)$ we see that the differential operator K_{2p+1} has the same order as that of K_{2p} but with formally higher accuracy, so it is more preferable.

A remark on the integral boundary condition (1.20) $u_\nu=-Ku$ and its local approximation of the form (1.21) is in order. The former represents the complete coupling of the system based on Ω with its environment accross the interface $\Gamma=\partial\Omega$, it is a generalization of the conventional boundary condition of the third kind $u_\nu=-\alpha_0u$. The latter always expresses, in certain sense and in certain degree, the coupling of the system and its environment, for example, elastic coupling in elasticity, impedance coupling in electromagnetism, law of cooling, etc.; it is, however, merely the crudest approximation in the form (1.21). The next approximation $u_\nu = -\alpha_0 u + \alpha_1 \partial_\theta^2 u$, which represents the coupling much better and involves hardly and more additional effort in the FEM implementation, deserves attention. The coefficients c_1, in addition to c_0, should be theoretically predictable as well as experimentally determinable. This kind of improved approximation is expected to have potentially wide applications.

II

The treatment of 3-D case is analogous to that of 2-D, so it will be only briefly indicated. The general solution of reduced wave equation

$$\Delta_3 u + \omega^2 u = 0, \tag{2.1}$$

$$\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_2,$$

$$\Delta_2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

satisfying the radiation condition

$$u = O(r^{-2}), \quad u_r + i\omega u = o(r^{-1}) \tag{2.2}$$

in the exterior domain $\Omega_a = \{r > a\}$ of the sphere $\Gamma_a = \{r = a\}$ can be represented as a series in spherical functions

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm}\zeta^{(2)}_{n+\frac{1}{2}}(\omega r) Y_{nm}(\theta, \varphi), \tag{2.3}$$

where

$$\zeta^{(2)}_{n+\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}\, H^{(2)}_{n+\frac{1}{2}}(x), \qquad Y_{nm}(\theta, \varphi) = P^{(m)}_n(\cos\theta) e^{im\varphi},$$

$$P^{(m)}_n(x) = \frac{(1-x^2)^{\frac{m}{2}}}{2^n n!}\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n. \tag{2.4}$$

Then follows

$$u(a, \theta, \varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm}\zeta^{(2)}_{n+\frac{1}{2}}(\omega a) Y_{nm}(\theta, \varphi) = \hat{u}, \tag{2.5}$$

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(\frac{\zeta^{(2)}_{n+\frac{1}{2}}(\omega r)}{\zeta^{(2)}_{n+\frac{1}{2}}(\omega a)}\right) A_{nm}\zeta^{(2)}_{n+\frac{1}{2}}(\omega a) Y_{nm}(\theta, \varphi) = \boldsymbol{P}\hat{u}, \tag{2.6}$$

$$-u_r(a, \theta, \varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(-\omega\frac{\zeta^{(2)\prime}_{n+\frac{1}{2}}(\omega a)}{\zeta^{(2)}_{n+\frac{1}{2}}(\omega a)}\right) A_{nm}\zeta^{(2)}_{n+\frac{1}{2}}(\omega a) Y_{nm}(\theta, \varphi) = \boldsymbol{K}\hat{u}. \tag{2.7}$$

We want to approximate the canonical integral operator $\boldsymbol{K}$ by tangential differential operators, analogous to the form (1.20). Since

$$\frac{\zeta^{(2)\prime}_\nu(x)}{\zeta^{(2)}_\nu(x)} = -\frac{1}{2x} + \frac{H^{(2)\prime}_\nu(x)}{H^{(2)}_\nu(x)},$$

let

$$c_q^*(\lambda) = c_q(\lambda), \quad q \neq 1, \quad c_1^*(\lambda) = 1 + c_1(\lambda) = 2,$$

$c_q(\lambda)$ are the same polynomials in λ given by (1.24), see also section III. So we have

$$c_0^*(\lambda)=1, \quad c_1^*(\lambda)=2, \quad c_q^*(\lambda)=c_q,$$

$$\left(-\omega\frac{\zeta_{n+\frac{1}{2}}^{(2)\prime}(\omega a)}{\zeta_{n+\frac{1}{2}}^{(2)}(\omega a)}\right)=i\omega\left[\tau+\sum_{q=0}^{\infty}c_q(\lambda_{n+\frac{1}{2}})\tau^q\right]=i\omega\sum_{q=0}^{\infty}c_q^*(\lambda_{n+\frac{1}{2}})\tau^q,$$

$$\tau=\frac{1}{2i\omega a}, \qquad \lambda_{n+\frac{1}{2}}=\left(n+\frac{1}{2}\right)^2-\frac{1}{4}=n(n+1). \tag{2.8}$$

Take finite truncation on the L. H. S., we get the approximations

$$\left(-\omega\frac{\zeta_{n+\frac{1}{2}}^{(2)\prime}(\omega a)}{\zeta_{n+\frac{1}{2}}^{(2)}(\omega a)}\right)\sim i\omega\sum_{q=0}^{p}c_q^*(\lambda_{n+\frac{1}{2}})\tau^q=:K_p^*(\lambda_{n+\frac{1}{2}}), \quad \tau=\frac{1}{2i\omega a},$$

$$p=0,\ 1,\ 2,\ \cdots. \tag{2.9}$$

Using the eigenvalue property of the Laplace-Beltrami operator Λ_2 on the unit sphere

$$\lambda_{n+\frac{1}{2}}Y_{nm}(\theta,\ \varphi)=n(n+1)Y_{nm}(\theta,\ \varphi)=-\Lambda_2Y_{nm}(\theta,\ \varphi),$$

$$K_p^*(\lambda_{n+\frac{1}{2}})Y_{nm}(\theta,\ \varphi)=K_p^*(-\Lambda_2)Y_{mn}(\theta,\ \varphi),$$

we get

$$-u_r(a,\ \theta,\ \varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(-\omega\frac{\zeta_{n+\frac{1}{2}}^{(2)\prime}(\omega a)}{\zeta_{n+\frac{1}{2}}^{(2)}(\omega a)}\right)A_{nm}\zeta_{n+\frac{1}{2}}^{(2)}(\omega a)Y_{nm}(\theta,\ \varphi)$$

$$\sim\sum_{n=0}^{\infty}\sum_{m=-n}^{n}K_p^*(\lambda_{n+\frac{1}{2}})\,A_{nm}\,\zeta_{n+\frac{1}{2}}^{(2)}(\omega a)Y_{nm}(\theta,\ \varphi)$$

$$=K_p^*(-\Lambda_2)u(a,\ \theta,\ \varphi),$$

which leads to a family of asymptotic radiation conditions

$$-u_r(a,\ \theta,\ \varphi)=K_p\hat{u}=K_p^*(-\Lambda_2)u(a,\ \theta,\ \varphi), \quad p=0,\ 1,\ 2,\ \cdots, \tag{2.10}$$

K_p^* are differential operators as linear combinations of the Laplace-Beltrami operator Λ_2 and its iterates Λ_2^k exclusively. In particular,

(K_0^*) $\quad -u_r=i\omega u,$

(K_1^*) $\quad -u_r=\left(i\omega+\frac{1}{a}\right)u,$

(K_2^*) $\quad -u_r=\left(i\omega+\frac{1}{a}\right)u+\frac{i}{2\omega a^2}\Lambda_2u,$

(K_3^*) $\quad -u_r=\left(i\omega+\frac{1}{a}\right)u+\left(\frac{i}{2\omega a^2}-\frac{1}{2\omega^2a^3}\right)\Lambda_2u,$

(K_4^*) $\quad -u_r=\left(i\omega+\frac{1}{a}\right)u+\left(\frac{i}{2\omega a^2}-\frac{1}{2\omega^2a^3}-\frac{3i}{2\omega^3a^4}\right)\Lambda_2u+\frac{i}{8\omega^3a^4}\Lambda_2^2u,$

(K_5^*) $\quad -u_r=\left(i\omega+\frac{1}{a}\right)u+\left(\frac{i}{2\omega a^2}-\frac{1}{2\omega^2a^3}-\frac{3i}{2\omega^3a^4}+\frac{3}{2\omega^4a^5}\right)\Lambda_2u$

$$+\left(\frac{i}{8\omega^3a^4}+\frac{1}{2\omega^4a^5}\right)\Lambda_2^2u$$

....

III

Hankel function $H_\nu^{(2)}(x)$ of the 2nd kind of complex order ν has an asymptotic expansion, for large real argument x, written formally as (see, e.g., [5])

$$H_n^{(2)}(x)=\sqrt{\frac{2}{\pi x}}\, e^{-i\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}\sum_{k=0}^{\infty} b_k(\lambda_\nu)\tau^k, \quad \tau=\frac{1}{2ix},\ \lambda_\nu=\nu^2-\frac{1}{4},\ \mathrm{Re}\,\nu>-\frac{1}{2}, \tag{3.1}$$

$b_k(\lambda)$ is a polynomial of degree k in λ defined as

$$b_0(\lambda)=1,\ b_k(\lambda)=\frac{1}{k!}\prod_{j=1}^{k}(\lambda-j(j-1)), \qquad k=1,\ 2,\ \cdots, \tag{3.2}$$

so that

$$b_0(\lambda_\nu)=1,\ b_k(\lambda_\nu)=\frac{1}{k!}\prod_{j=1}^{k}\left(\nu^2-\left(j-\frac{1}{2}\right)^2\right), \qquad k=1,\ 2,\ \cdots.$$

From this we deduce the asymptotic expansion

$$\frac{H_\nu^{(2)\prime}(x)}{H_\nu^{(2)}(x)}=-i\sum_{k=0}^{\infty} c_k(\lambda_\nu)\tau^k, \qquad \tau=\frac{1}{2ix},\ \lambda_\nu=\nu^2-\frac{1}{4}, \tag{3.3}$$

where $c_k(\lambda)$ are polynomials in λ to be determined, we need only the cases $\nu=\pm n$ and $\nu=n+\frac{1}{2}$, $n=0,\ 1,\ 2,\ \cdots$, in sections I and II respectively.

Using the relation $H_\nu^{(2)\prime}(x)=\frac{1}{2}[H_{\nu-1}^{(2)}(x)-H_{\nu+1}^{(2)}(x)]$, and assuming $\mathrm{Re}\,\nu>\frac{1}{2}$, we obtain from (3.1) the asymptotic expansion

$$H_\nu^{(2)\prime}(x)=-i\sqrt{\frac{2}{\pi x}}\, e^{-i\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}\sum_{k=0}^{\infty} a_k(\lambda_\nu)\tau^k, \tag{3.4}$$

where

$$a_k(\lambda_\nu)=\frac{1}{2}\,[b_k(\lambda_{\nu-1})+b_k(\lambda_{\nu+1})]=\begin{cases}1, & k=0,\\ b_k(\lambda_\nu)-(2k-1)b_{k-1}(\lambda_\nu), & k\geqslant 1.\end{cases} \tag{3.5}$$

(3.4) and (3.5) are valid too for $\nu=0$ and $\nu=\frac{1}{2}$ by direct verification.

Setting now

$$\frac{H_\nu^{(2)\prime}(x)}{H_\nu^{(2)}(x)}=-i\frac{\sum_0^\infty a_k(\lambda_\nu)\tau^k}{\sum_0^\infty b_k(\lambda_\nu)\tau^k}=-i\sum_0^\infty c_k(\lambda_\nu)\tau^k,$$

we have

$$a_k=b_0c_k+b_1c_{k-1}+\cdots+b_kc_0, \qquad k=0,\ 1,\ 2,\ \cdots.$$

From which $c_k=c_k(\lambda)=c_k(\lambda_\nu)$ can be determined recursively

$$\begin{aligned}
&c_0=1,\\
&c_1=a_1-b_1=b_0=1,\\
&\cdots\cdots\cdots\\
&c_k=a_k-b_k-b_{k-1}c_1-b_{k-2}c_2-\cdots-b_1c_{k-1}\\
&\quad=(2k-1)b_{k-1}-b_{k-1}c_1-b_{k-2}c_2-\cdots-b_1c_{k-1}, \qquad k=2,\ 3,\ \cdots.
\end{aligned}$$

Thus far we have determined the asymptotic expansion of $\frac{H_\nu^{(2)\prime}(x)}{H_\nu^{(2)}(x)}$ for all $\nu=n$ and $\nu=n+\frac{1}{2}$, $n=0,\ 1,\ \cdots$. Since

$$\frac{H_{-n}^{(2)\prime}(x)}{H_{-n}^{(2)}(x)}=\frac{H_n^{(2)\prime}(x)}{H_n^{(2)}(x)},$$

so the expansions are determined for all $\nu=n=0,\ \pm1,\ \pm2,\ \cdots$, and $\nu=n+\frac{1}{2}$, $n=$

$0, 1, 2, \cdots$. $c_k = c_k(\lambda)$ for $k \leqslant 6$ are given in (1.24). For $k \geqslant 2$, all $c_k(\lambda)$ are polynomials in λ without constant term. Computational evidences suggest the conjecture that both $c_{2p}(\lambda)$ and $c_{2p+1}(\lambda)$ are polynomials of degree p in λ. This is so for all the practical useful (in sections I and II) cases, it has not yet been proved, however, for the general case. Estimates of the remainder C_p for the expansion (3.3)

$$\frac{H_\nu^{(2)\prime}(x)}{H_\nu^{(2)}(x)} = -i\left\{\sum_{k=0}^{p-1} c_k(\lambda_\nu)\tau^k + C_p\right\}$$

can be obtained indirectly from those of the remainder B_p for the expansion (3.1)

$$H_\nu^{(2)}(x) = \sqrt{\frac{2}{\pi x}}\, e^{-i\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)}\left\{\sum_{k=0}^{p-1} b_k(\lambda_\nu)\tau^k + B_p\right\},$$

for which one is referred to [5].

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