ERROR ESTIMATES FOR THE FINITE ELEMENT SOLUTIONS OF SOME VARIATIONAL INEQUALITIES WITH NONLINEAR MONOTONE OPERATOR*

WANG LIE-HENG[1)] (王烈衡)
(*Computing Center, Academia Sinica*)

Abstract

In this paper, we estimate the error of the linear finite element solutions of the obstacle problem and the unilateral problem with monotone operator. We obtained $O(h)$ error bound for the obstacle problem and $O(h^{3/4})$ error bound for the unilateral problem. And if the solution u^* of the unilateral problem possesses more smoothness, then $O(h)$ error bound can be obtained in the same way as [2].

1. Introduction

In Brezzi, Hager and Raviart[2], the error estimates for the linear finite element solutions of the obstacle problem and the unilateral problem with linear V-elliptic operator have been obtained. Their results are the following: $O(h)$ error bounds for both the obstacle and unilateral problems with linear finite elements. Now in this paper, we obtained the same result for the obstacle problem with nonlinear monotone operator. For the unilateral problem with nonlinear monotone operator, we obtained $O(h^{3/4})$ error bound just as [5], and if the solution u^* of the unilateral problem possesses more smoothness, then $O(h)$ error bound holds in the same way as [2].

Let Ω denote a bounded convex open subset of $\mathbb{R}^2$, $\partial\Omega$ denote the boundary of Ω. Let $H^m(\Omega)$ be the usual Sobolev space [1] consisting of real value functions defined on Ω with derivatives through order m in $L^2(\Omega)$; the norm on $H^m(\Omega)$ is denoted by $\|\cdot\|_{m,\Omega}$. Let V be a Hilbert space with norm $\|\cdot\|$ and V' be the dual of V with norm $\|\cdot\|_*$, the pairing between V and V' be denoted by $\langle\cdot, \cdot\rangle$.

Let T be a (generally nonlinear) mapping

$$T: V \mapsto V',$$

which possesses the two following properties (c. f. [3]):

(i) The mapping T is uniformly monotone, i. e., there exists a positive constant $\alpha>0$, such that

$$\langle Tu-Tv,\ u-v\rangle \geqslant \alpha\|u-v\|^2,\ \forall u,\ v\in V. \tag{1.1}$$

(ii) The mapping T is Lipschitz-continuous for bounded arguments in the sense

* Received January 18, 1982.

1) This work was done during the author stay in Italy supported by CNR through Istituto per le Applicazioni del Calcolo of Roma.

that, for any ball $B(0;r)=\{v\in V:\|v\|\leq r\}$, there exists a constant $\Gamma(r)$ such that

$$\|Tu-Tv\|_*\leq\Gamma(r)\cdot\|u-v\|,\ \forall u,\ v\in B(0;r). \tag{1.2}$$

We state a well known result of interpolation [3, p. 124]: Let $v\in H^{k+1}(\Omega^h)$, v^I be the piecewise linear interpolation of v on Ω^h, and Ω^h be a regular triangulation[3], then

$$\|v-v^I\|_{m,\Omega^h}\leq Ch^{k+1-m}|v|_{k+1,\Omega},\ \text{for } k=1,\ m=0,\ 1, \tag{1.3}$$

$$\|v-v^I\|_{1,\Omega^h}\leq C|v|_{1,\Omega},\ \forall v\in H^1(\Omega), \tag{1.3'}$$

where C is a constant independent of h and v.

2. The Obstacle Problem

Let us introduce some other notations:

$$a(u,\ v)=\int_\Omega\left[a_1(u,\ \nabla u)\frac{\partial v}{\partial x_1}+a_2(u,\ \nabla u)\frac{\partial v}{\partial x_2}+a_0(u,\ \nabla u)v\right]dx_1\,dx_2, \tag{2.1}$$

$$\langle f,\ v\rangle=\int_\Omega f\cdot v\,dx_1dx_2, \tag{2.2}$$

$$K=\{v\in H^1(\Omega):v\geq\psi \text{ a. e. in } \Omega,\ v|_{\partial\Omega}=g\}. \tag{2.3}$$

Let us assume that $a_i(\xi_0,\ \xi_1,\ \xi_2)\in H^1(\mathbb{R}^3)$, $(i=0,\ 1,\ 2)$ and $a_{ij}=\dfrac{\partial}{\partial\xi_j}a_i(\xi)$, and

$$\sum_{i,j=0}^{2}a_{ij}(\xi)\eta_i\eta_j\geq\alpha\|\eta\|^2,\ \|\eta\|^2=\sum_{i=0}^{2}\eta_i^2,\ \forall\xi,\ \eta\in\mathbb{R}^3, \tag{2.4}$$

$$|\eta^T[a_{ij}(v,\ \nabla v)]_{ij}\eta|\leq\Gamma(r)\|\eta\|^2,\ \forall v\in H^1(\Omega);\ \|v\|_{1,\Omega}\leq r,\ \eta\in\mathbb{R}^3. \tag{2.5}$$

Then there exists an operator T,

$$T:H^1(\Omega)\mapsto(H^1(\Omega))',$$

defined by

$$a(u,\ v)=\langle Tu,\ v\rangle. \tag{2.6}$$

We can find that the mapping T defined above possesses two properties (i) and (ii) in section 1.

Let L denote an operator defined by

$$Lu=-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\,a_i(u,\ \nabla u)+a_0(u,\ \nabla u). \tag{2.7}$$

The obstacle problem is to find $u^*\in K$, such that

$$a(u^*,\ v-u^*)\geq\langle f,\ v-u^*\rangle,\quad\forall v\in K,$$

or we can write it in the another form: to find $u^*\in K$, such that

$$\langle Tu^*,\ v-u^*\rangle\geq\langle f,\ v-u^*\rangle,\quad\forall v\in K. \tag{2.8}$$

If $f\in L^2(\Omega)$, $\psi\in H^2(\Omega)$, g is the restriction to $\partial\Omega$ of an $H^2(\Omega)$ function and $g\geq\psi$ on $\partial\Omega$, then the existence and unique of the solution of the problem (2.8) are insured by classical result[5].

If the solution $u^*\in H^2(\Omega)$, $\psi\in H^2(\Omega)$, $f\in L^2(\Omega)$ and $Lu^*\in L^2(\Omega)$, then the following differential forms holds[7]:

$$\begin{cases}Lu^*-f\geq0,\ (Lu^*-f)(u^*-\psi)=0,\ u^*\geq\psi \text{ a. e. in } \Omega,\\ u^*|_{\partial\Omega}=g.\end{cases} \tag{2.9}$$

We now consider the approximate problem of the problem (2.8) by linear finite element. Let Ω^h denote a polygon inscribed in Ω and also denote a regular triangulation[3]. Let u^I, ψ^I denote the piecewise linear interpolations of u^* and ψ on Ω^h respectively. Let V^h denote a space of the piecewise linear continuous functions defined on Ω^h. Let

$$K^h=\{v^h\in V^h: v^h\geqslant\psi^I \text{ at the nodes on } \Omega^h,\ v^h=g \text{ at the nodes on } \partial\Omega\}. \tag{2.10}$$

Then the approximate problem is that to find $u^h\in K^h$ such that

$$\langle Tu^h,\ v^h-u^h\rangle_{\Omega^h}\geqslant\langle f,\ v^h-u^h\rangle_{\Omega^h},\ \forall v^h\in K^h, \tag{2.11}$$

where the index Ω^h denotes the integration on Ω^h.

We have the error estimate as follows:

Theorem 1. *If $f\in L^2(\Omega)$, $\psi\in H^2(\Omega)$, $g\in H^2(\Omega)$, $g\geqslant\psi$ on $\partial\Omega$ and the solution $u^*\in H^2(\Omega)$ of the problem* (2.8), *$Lu^*\in L^2(\Omega)$, then*

$$\|u^h-u^*\|_{1,\Omega^h}=O(h).$$

Proof. Since u^* and u^h are the solutions of the problems (2.8) and (2.11) respectively, and with use of Green's formula we have

$$\begin{aligned}\langle Tu^*-Tu^h,\ u^*-u^h\rangle_{\Omega^h}&=\langle Tu^*-Tu^h,\ u^*-u^I\rangle_{\Omega^h}\\&\quad+\langle Tu^*,\ u^I-u^h\rangle_{\Omega^h}-\langle f,\ u^I-u^h\rangle_{\Omega^h}\\&\quad-\langle Tu^h,\ u^I-u^h\rangle_{\Omega^h}+\langle f,\ u^I-u^h\rangle_{\Omega^h}\\&\leqslant\langle Tu^*-Tu^h,\ u^*-u^I\rangle_{\Omega^h}+\langle Tu^*-f,\ u^I-u^h\rangle_{\Omega^h}\\&=\langle Tu^*-Tu^h,\ u^*-u^I\rangle_{\Omega^h}+\int_{\Omega^h}(Lu^*-f)(u^I-u^h)dx_1dx_2\\&\quad+\int_{\partial\Omega^h}\frac{\partial u^*}{\partial n}(u^I-u^h)ds\\&=\langle Tu^*-Tu^h,\ u^*-u^I\rangle_{\Omega^h}+\int_{\Omega^h}(Lu^*-f)(u^I-u^h)dx_1\,dx_2,\end{aligned} \tag{2.12}$$

where in the inequality we have used (2.11), and in the last equality we have used the fact that both of u^I and u^h are piecewise linear functions and $u^I=u^h=g$ at the nodes on $\partial\Omega$; and $\partial u^*/\partial n$ is defined as follows:

$$\frac{\partial u^*}{\partial n}=\sum_{i=1}^{2}a_i(u^*,\ \nabla u^*)\cos(n,\ x_i), \tag{2.13}$$

where n is the external unit normal on $\partial\Omega$.

Let us first estimate the second term on the right side of (2.12), we have

$$\begin{aligned}\int_{\Omega^h}(Lu^*-f)(u^I-u^h)dx_1\,dx_2&=\int_{\Omega^h}(Lu^*-f)[(u^I-\psi^I)-(u^*-\psi)]dx_1\,dx_2\\&\quad+\int_{\Omega^h}(Lu^*-f)(u^*-\psi)dx_1\,dx_2+\int_{\Omega^h}(Lu^*-f)(\psi^I-u^h)dx_1\,dx_2\\&\leqslant\int_{\Omega^h}(Lu^*-f)[(u^I-\psi^I)-(u^*-\psi)]dx_1\,dx_2\\&\leqslant\|Lu^*-f\|_{0,\Omega}\|(u^I-\psi^I)-(u^*-\psi)\|_{0,\Omega^h}\leqslant Ch^2\|u^*-\psi\|_{2,\Omega},\end{aligned} \tag{2.14}$$

since (2.9), so $\int_{\Omega^h}(Lu^*-f)(u^*-\psi)dx_1\,dx_2=0$; and since ψ^I, u^h are piecewise linear on Ω^h, and $u^h\geqslant\psi^I$ at the nodes of Ω^h, so $u^h\geqslant\psi^I$ on Ω^h, and since (2.9): $Lu^*-f\geqslant0$ a. e. in Ω^h, so

$$\int_{\Omega^h} (Lu^* - f)(\psi^I - u^h)\,dx_1dx_2 \leqslant 0;$$

and in the last inequality of (2.14), we have used the error estimate (1.3).

Let us now estimate the first term on the right side of (2.12). From the property (ii) of the mapping T and the error estimate (1.3), we have

$$\begin{aligned} |\langle Tu^* - Tu^h,\ u^* - u^I\rangle_{\Omega^h}| &\leqslant \|Tu^* - Tu^h\|_* \|u^* - u^I\|_{1,\Omega^h} \\ &\leqslant \Gamma(\|u^*\|_{1,\Omega} + \|u^h\|_{1,\Omega^h})\|u^* - u^h\|_{1,\Omega^h}\|u^* - u^I\|_{1,\Omega^h} \\ &\leqslant \Gamma(\|u^*\|_{1,\Omega} + \|u^h\|_{1,\Omega^h}) h\|u^*\|_{2,\Omega}\cdot\|u^* - u^h\|_{1,\Omega^h}. \end{aligned} \tag{2.15}$$

Now we prove that the solution u^h of the approximate problem (2.11) is bounded independently of h. Let

$$\varphi(x) = \max\{g(x),\ \psi(x)\}. \tag{2.16}$$

Since $g,\ \psi\in H^1(\Omega)$, and by a well known result of Lewy and Stampacchia[4], then we have $\varphi\in H^1(\Omega)$.

Let φ^I denote the piecewise linear interpolation of φ on Ω^h, if P_i is an any given node on Ω^h, then we have $\varphi^I(p_i) \geqslant \psi^I(p_i)$; and since $g(x) \geqslant \psi(x)$, $\forall x\in\partial\Omega$, then we have $\varphi^I = g$, at the node on $\partial\Omega$. Thus $\varphi^I\in K^h$. From the properties (i), (ii) of the mapping T and (2.11), we have

$$\begin{aligned} \alpha\|u^h - \varphi^I\|^2_{1,\Omega^h} &\leqslant \langle Tu^h - T\varphi^I,\ u^h - \varphi^I\rangle_{\Omega^h} \\ &= \langle Tu^h,\ u^h - \varphi^I\rangle_{\Omega^h} - \langle T\varphi^I,\ u^h - \varphi^I\rangle_{\Omega^h} \\ &\leqslant \langle f,\ u^h - \varphi^I\rangle_{\Omega^h} - \langle T\varphi^I,\ u^h - \varphi^I\rangle_{\Omega^h} \\ &\leqslant (\|f\|_{0,\Omega} + \|T\varphi^I\|_*)\cdot\|u^h - \varphi^I\|_{1,\Omega^h}, \end{aligned} \tag{2.17}$$

and then

$$\|u^h - \varphi^I\|_{1,\Omega^h} \leqslant \frac{1}{\alpha}(\|f\|_{0,\Omega} + \|T\varphi^I\|_*). \tag{2.18}$$

From (1.2), we have

$$\|T\varphi^I - T\varphi\|_* \leqslant \Gamma(\|\varphi^I\|_{1,\Omega^h} + \|\varphi\|_{1,\Omega})\cdot\|\varphi^I - \varphi\|_{1,\Omega^h}. \tag{2.19}$$

And from the error estimate (1.3′), we have

$$\|\varphi^I - \varphi\|_{1,\Omega^h} \leqslant C\|\varphi\|_{1,\Omega}. \tag{2.20}$$

From (2.19) and (2.20), we can see that there exists $C = \text{const.} > 0$, such that

$$\|\varphi^I\|_{1,\Omega^h},\ \|T\varphi^I\|_* \leqslant C. \tag{2.21}$$

And from (2.18), we have

$$\|u^h\|_{1,\Omega^h} \leqslant C. \tag{2.22}$$

Thus from (2.12)—(2.15) and (2.22), we have

$$\begin{aligned} \alpha\|u^* - u^h\|^2_{1,\Omega^h} &\leqslant \langle Tu^* - Tu^h,\ u^* - u^h\rangle_{\Omega^h} \\ &\leqslant Ch\cdot\|u^* - u^h\|_{1,\Omega^h} + C'h^2. \end{aligned} \tag{2.23}$$

Thus the proof is completed.

3. The Unilateral Problem

In this section, let us assume that Ω is a convex polygon in plane with the boundary $\partial\Omega$. The unilateral problem is that to find $u^*\in K$, such that

$$\langle Tu^*, v-u^*\rangle \geqslant \langle f, v-u^*\rangle, \quad \forall v\in K, \tag{3.1}$$

where

$$K=\{v\in H^1(\Omega): v\geqslant g \text{ a. e. on } \partial\Omega\}. \tag{3.2}$$

Let the mapping T be defined in section 2, then the existence and uniqueness of the problem (3.1) are insured by [5]. If the solution $u^*\in H^2(\Omega)$, $f\in L^2(\Omega)$, $g\in C^0(\partial\Omega)$, $Lu^*\in L^2(\Omega)$, then we can prove the following differential form as in [7]:

$$\begin{cases} Lu^*-f=0 \quad \text{a. e. in } \Omega, \\ u^*\geqslant g, \ \dfrac{\partial u^*}{\partial n}\geqslant 0 \text{ and } \dfrac{\partial u^*}{\partial n}(g-u^*)=0 \quad \text{a. e. on } \partial\Omega. \end{cases} \tag{3.3}$$

The approximate problem by linear finite element method is the following: to find $u^h\in K^h$, such that

$$\langle Tu^h, v^h-u^h\rangle_{\Omega^h} \geqslant \langle f, v^h-u^h\rangle_{\Omega^h}, \ \forall v^h\in K^h, \tag{3.4}$$

where

$$K^h=\{v^h\in V^h: v^h\geqslant g \text{ at the nodes on } \partial\Omega^h\}. \tag{3.5}$$

Since we assume that the domain Ω is a polygon, so $\Omega=\Omega^h$, $\partial\Omega=\partial\Omega^h$, thus we will not write the subscript Ω^h in (3.4) below.

We have the following error estimate:

Theorem 2. *If $f\in L^2(\Omega)$, g is the restriction to $\partial\Omega$ of a function in $H^2(\Omega)$, and the solution $u^*\in H^2(\Omega)$ of the problem (3.1), $Lu^*\in L^2(\Omega)$, then*

$$\|u^*-u^h\|_{1,\Omega}=O(h^{3/4}).$$

Proof. Since u^* and u^h are the solutions of the problems (3.1) and (3.4) respectively, using Green's formula, we have

$$\begin{aligned} \langle Tu^*-Tu^h, u^*-u^h\rangle &= \langle Tu^*-Tu^h, u^*-u^I\rangle \\ &\quad +\langle Tu^*, u^I-u^h\rangle-\langle f, u^I-u^h\rangle-\langle Tu^h, u^I-u^h\rangle+\langle f, u^I-u^h\rangle \\ &\leqslant \langle Tu^*-Tu^h, u^*-u^I\rangle+\langle Tu^*-f, u^I-u^h\rangle \\ &= \langle Tu^*-Tu^h, u^*-u^I\rangle+\int_\Omega (Lu^*-f)(u^I-u^h)\,dx_1\,dx_2 \\ &\quad +\int_{\partial\Omega} \frac{\partial u^*}{\partial n}(u^I-u^h)\,ds \\ &= \langle Tu^*-Tu^h, u^*-u^I\rangle+\int_{\partial\Omega} \frac{\partial u^*}{\partial n}(u^I-u^h)\,ds. \end{aligned} \tag{3.6}$$

In the last equality of (3.6), we have used (3.3), and u^I, $\partial u^*/\partial n$ etc. are the same as in the section 2.

Let us first estimate the first term on the right of (3.6). From the error estimate of interpolation (1.3) and the property (ii) of the mapping T, we have

$$\begin{aligned} |\langle Tu^*-Tu^h, u^*-u^I\rangle| &\leqslant \|Tu^*-Tu^h\|_*\|u^*-u^I\|_{1,\Omega} \\ &\leqslant \Gamma(\|u^*\|_{1,\Omega}+\|u^h\|_{1,\Omega})\|u^*-u^h\|_{1,\Omega}\cdot\|u^*-u^I\|_{1,\Omega} \\ &\leqslant Ch\cdot\Gamma(\|u^*\|_{1,\Omega}+\|u^h\|_{1,\Omega})\|u^*\|_{2,\Omega}\|u^*-u^h\|_{1,\Omega}. \end{aligned} \tag{3.7}$$

We also need to prove that $\|u^h\|_{1,\Omega}\leqslant C=\text{const.}>0$, $\forall h$. But this proof is the same as the proof of (2.29), and simpler than the proof of (2.29). Because of g^I, the piecewise linear interpolation of g of $H^2(\Omega)$, belongs to K^h, so we can replace φ^I in the proof of Theorem 1 by g^I. Then we have

$$|\langle Tu^*-Tu^h,\ u^*-u^I\rangle|\leqslant Ch\|u^*-u^h\|_{1,\Omega}. \tag{3.8}$$

Next let us estimate the boundary integration on the right side of (3.6) as follows:

$$\begin{aligned}\int_{\partial\Omega}\frac{\partial u^*}{\partial n}(u^I-u^h)ds&=\int_{\partial\Omega}\frac{\partial u^*}{\partial n}[(u^I-g^I)-(u^*-g)]ds\\&+\int_{\partial\Omega}\frac{\partial u^*}{\partial n}(u^*-g)ds+\int_{\partial\Omega}\frac{\partial u^*}{\partial n}(g^I-u^h)ds\\&\leqslant\int_{\partial\Omega}\frac{\partial u^*}{\partial n}[(u^I-g^I)-(u^*-g)]ds,\end{aligned} \tag{3.9}$$

since (3.3), so $\int_{\partial\Omega}\frac{\partial u^*}{\partial n}(u^*-g)ds=0$; and since g^I and u^h are piecewise linear functions, and $u^h\geqslant g^I$ at the nodes on $\partial\Omega$, so $g^I-u^h\leqslant 0$ on $\partial\Omega$, and taking account of $\partial u^*/\partial n\geqslant 0$ on $\partial\Omega$ in (3.3), then we have

$$\int_{\partial\Omega}\frac{\partial u^*}{\partial n}(g^I-u^h)ds\leqslant 0.$$

Thus from (3.9), with use of the trace Theorem[1] and the error estimate of interpolation (1.3), we have

$$\begin{aligned}\int_{\partial\Omega}\frac{\partial u^*}{\partial n}(u^I-u^h)ds&\leqslant\int_{\partial\Omega}\frac{\partial u^*}{\partial n}[(u^I-g^I)-(u^*-g)]ds\\&\leqslant C\|u^*\|_{1,\partial\Omega}\|(u^*-g)^I-(u^*-g)\|_{0,\partial\Omega}\\&\leqslant C\|u^*\|_{2,\Omega}\|(u^*-g)^I-(u^*-g)\|_{0,\Omega}^{1/2}\cdot\|(u^*-g)^I-(u^*-g)\|_{1,\Omega}^{1/2}\\&\leqslant Ch^{3/2}\|u^*\|_{2,\Omega}\|u^*-g\|_{2,\Omega}.\end{aligned} \tag{3.10}$$

Taking account of $u^*\in H^2(\Omega)$, $g\in H^2(\Omega)$ and the property (i) of the mapping T, we have, from (3.6), (3.8) and (3.10),

$$\alpha\|u^*-u^h\|_{1,\Omega}^2\leqslant\langle Tu^*-Tu^h,\ u^*-u^h\rangle\leqslant Ch\|u^*-u^h\|_{1,\Omega}+Ch^{3/2}. \tag{3.11}$$

Thus the proof is completed.

Remark 1. If the domain Ω is a convex bounded open set in plane with boundary sufficiently smooth, and $u^*\in W^1_\infty$(near $\partial\Omega$), $g\in W^1_\infty$ (near $\partial\Omega$), then in the same way as [2], it can be obtained as in [2] that

$$\|u^*-u^h\|_{1,\Omega}=O(h).$$

Remark 2. For the obstacle problem in the section 2, H. D. Mittelmann[8] has proved the same order of error bound under different hypotheses. In [8] the author treated the error estimate for domain Ω not necessarily convex but the following hypothesises are needed and assumed: the coefficients $a_i(\xi)\in C^1(\mathbb{R}^3)$ and the solution $u^*\in W^{1,\infty}(\Omega)$, which in fact assumes that the mapping T is uniformly Lipschtz continuous, and $g\in W^{1,\infty}(\partial\Omega)$, which is an extra hypothesis for the non-convex domain Ω.

References

[1] R. A. Adams, Sobolev spaces, New York, Academic Press, 1975.

[2] F. Brezzi, W. W. Hager, P. A. Raviart, Error estimates for the finite element solution of variational inequalities. Part I. Primal theory. *Numer. Math.*, 28, 1977, 431—443.

[3] P. G. Ciarlet, The finite element method for elliptic problems, Amsterdam, New York, Oxford, 1978.

[4] H. Lewy, G. Stampacchia, On the regularity of the solution of a variational inequality, *Comm. Pure Appl. Math.*, Vol. 22, 1969, 153—188.

[5] J. L. Lions, G. Stampacchia, Variational inequalities, *Comm. Pure Appl. Math.*, Vol. 20, 1967, 493—519.

[6] F. Scarpini, M. A. Vivaldi, Error estimates for the approximation of some unilateral problems, *R. A. I. R. O. Anal. Numer.*, 2(1977), 197—208.

[7] Huang Hong-ci, Wang Lie-heng, On the equivalency between variational inequality and differential form for the obstacle problem, *Mathematica Numerica Sinica*, Vol. 4, 1982, 436—439.

[8] H. D. Mittelmann, On the approximate solution of nonlinear variational inequalities, *Numer. Math.*, 29, 1978, 451—462.