

## RESTRICTED ADDITIVE SCHWARZ METHOD FOR A KIND OF NONLINEAR COMPLEMENTARITY PROBLEM \*

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### Abstract

In this paper, a new Schwarz method called restricted additive Schwarz method (RAS) is presented and analyzed for a kind of nonlinear complementarity problem (NCP). The method is proved to be convergent by using weighted maximum norm. Besides, the effect of overlap on RAS is also considered. Some preliminary numerical results are reported to compare the performance of RAS and other known methods for NCP.

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### 1. Introduction

We consider the following finite-dimensional nonlinear complementarity problem (NCP): find  $u \in R^n$  such that

$$u \geq \phi, \quad F(u) \geq 0, \quad (u - \phi)^T F(u) = 0, \quad (1.1)$$

where  $\phi \in R^n$ ,  $F = Au + f(u)$  with an  $M$ -matrix  $A$ , and  $\partial f / \partial u \geq 0$ . This kind of problem can be arised from free boundary problem with nonlinear source terms, e.g. the diffusion problem involving Michaelis-Menten or second-order irreversible reactions, see, e.g., [8,15].

It is well known that domain decomposition method is a kind of very important method for PDEs since 1980's. It has many advantages, for example it is easy to be parallelized on parallel machines, and it is effective for large scale problem. Moreover, the convergence rate will not be deteriorated with the refinement of the mesh when it is applied to discretized differential equations. Researches in this field for the solution of variational inequalities and complementarity problems were also fruitful in the last decade. We refer the reader to [1,2,9-11,13,16,17,20-22] and the extensive references therein. In [6], a new variant Schwarz method called restricted additive Schwarz method (RAS) was proposed for general sparse linear systems. This method attracts much attention, since it reduces communication time while maintaining the most desirable used in practice. Up to now, much effort have been made to study the convergence theory for RAS, and extended RAS for some other kinds of linear systems, see, e.g., [4,5,7,9] and the references therein. [18] presented the RAS method for a kind of NCP, but

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the authors did not prove the convergence of the proposed method. The purpose of this paper is to extend the RAS method to NCP (1.1) and establish its convergence results. Moreover, we discuss the effect of overlap on proposed method.

The paper is organized as follows: in Section 2, we give some preliminaries and present the RAS method. In Section 3, we estimate the weighted max-norm bounds for iteration errors and establish global convergence theorem for RAS. In Section 4, we discuss the effect of overlap on the RAS method. In Section 5, we present some numerical results and give some conclusions in Section 6.

## 2. Restricted Additive Schwarz Method

In this section, we present the RAS method for solving problem (1.1). As in [6] and [9], we consider  $m$  nonoverlapping subspaces  $V_{i,0}$ ,  $i = 1, \dots, m$ , which are spanned by columns of the identity  $I$  over  $R^n$ . Let  $S = \{1, 2, \dots, n\}$  and let

$$S = \cup_{i=1}^m S_{i,0}$$

be a partition of  $S$  into  $m$  disjoint, nonempty subsets. Let  $\{S_{i,1}\}$  be the one-overlap partition of  $S$  which is obtained by adding those indices to  $S_{i,0}$  which correspond to nodes lying at distance 1 or less from those nodes corresponding to  $S_{i,0}$ . Using the idea recursively, we can define  $\delta$ -overlap partition of  $S$ ,  $S = \cup_{i=1}^m S_{i,\delta}$ , where  $S_{i,0} \subset S_{i,\delta}$ , with  $\delta$  level of overlaps with its neighboring subsets. Hence, we have a nested sequence of larger sets  $S_{i,\delta}$  with

$$S_{i,0} \subseteq S_{i,1} \subseteq S_{i,2} \subseteq \dots \subseteq S = \{1, 2, \dots, n\}. \quad (2.1)$$

For  $\delta > 0$ , the sets  $S_{i,\delta}$  are not necessarily pairwise disjoint, i.e., we have introduced overlap. This approach is adequate in discretizations of PDEs where the indices correspond to the nodes of the discretization mesh, and also is valid for problem (1.1).

Let  $n_{i,\delta} = |S_{i,\delta}|$  denote the cardinality of the set  $S_{i,\delta}$ . For each nested sequence from (2.1), we can find a permutation matrix  $\pi_i$  on  $\{1, 2, \dots, n\}$  with the property that for all  $\delta \geq 0$  we have  $\pi_i(S_{i,\delta}) = \{1, 2, \dots, n_{i,\delta}\}$ . Let  $R_{i,\delta} : R^n \rightarrow R^{n_{i,\delta}}$  be the restriction operator.  $R_{i,\delta}$  is an  $n_{i,\delta} \times n$  matrix with  $\text{rank}(R_{i,\delta}) = n_{i,\delta}$  whose rows are precisely those row  $j$  of the identity for which  $j \in S_{i,\delta}$ . Its transpose  $R_{i,\delta}^T : R^{n_{i,\delta}} \rightarrow R^n$  is a prolongation operator. Formally, such a matrix  $R_{i,\delta}$  can be expressed as

$$R_{i,\delta} = [I_{i,\delta} \quad O] \pi_i \quad (2.2)$$

with  $I_{i,\delta}$  the identity on  $R^{n_{i,\delta}}$ . We define the weighting matrices

$$E_{i,\delta} = R_{i,\delta}^T R_{i,\delta} = \pi_i^T \begin{bmatrix} I_{i,\delta} & O \\ O & O \end{bmatrix} \pi_i \in R^{n \times n} \quad (2.3)$$

and the subspaces

$$V_{i,\delta} = \text{range}(E_{i,\delta}), \quad i = 1, 2, \dots, m.$$

Note the inclusion  $V_{i,\delta} \subseteq V_{i,\delta'}$  for all  $\delta \leq \delta'$ , and in particular,  $V_{i,0} \subseteq V_{i,\delta}$  for all  $\delta \geq 0$ . Furthermore, the images of the bases of  $V_{i,\delta}$  under the prolongation operator  $R_{i,\delta}^T$  are linearly independent unit elements in  $R^n$ , and we can verify the images of  $R_{i,\delta}^T$  with the subspaces  $V_{i,\delta}$ . For matrix  $A$ , let  $A_{i,\delta} = R_{i,\delta} A R_{i,\delta}^T$  denote the restriction of  $A$  to  $V_{i,\delta}$ . By (2.2), we have matrix

$A_{i,\delta}$  is an  $n_{i,\delta} \times n_{i,\delta}$  principal submatrix of  $A$ . Let  $\pi \in R^{n \times n}$  be a permutation matrix, we denote by  $A_\pi = \pi A \pi^T$ . Noting that

$$A_{i,\delta} = R_{i,\delta} A R_{i,\delta}^T = [I_{i,\delta} \quad O] A_{\pi_i} [I_{i,\delta} \quad O]^T \in R^{n_{i,\delta} \times n_{i,\delta}}$$

is the  $n_{i,\delta} \times n_{i,\delta}$  submatrix of  $A_{\pi_i}$ , we can represent matrix  $A_{\pi_i}$  in the form

$$A_{\pi_i} = \begin{bmatrix} A_{i,\delta} & G_{i,\delta} \\ H_{i,\delta} & A_{i_c,\delta} \end{bmatrix}. \tag{2.4}$$

And it is convenient to represent vectors

$$u_{\pi_i} = \begin{bmatrix} u_{i,\delta} \\ u_{i_c,\delta} \end{bmatrix}, \quad \phi_{\pi_i} = \begin{bmatrix} \phi_{i,\delta} \\ \phi_{i_c,\delta} \end{bmatrix},$$

where  $u_{i,\delta} = R_{i,\delta} u \in R^{n_{i,\delta}}$  and  $\phi_{i,\delta} = R_{i,\delta} \phi \in R^{n_{i,\delta}}$ .

The natural partial ordering  $\leq$  between matrices  $A = (a_{ij})$ ,  $B = (b_{ij})$ , of the same size is defined componentwise, i.e.,  $A \leq B$  iff  $a_{ij} \leq b_{ij}$  for all  $i, j$ . If  $A \geq O$  we call  $A$  nonnegative. If all entries of  $A$  are positive, we say that  $A$  is positive and write  $A > O$ . This notation and terminology carries over to vectors as well.

**Definition 2.1.** ([3, 9]) Consider the splitting  $A = M - N \in R^{n \times n}$  with  $M$  nonsingular. This splitting is said to be

- (i) regular if  $M^{-1} \geq O$  and  $N \geq O$ ,
- (ii)  $M$ -splitting if  $M$  is an  $M$ -matrix and  $N \geq O$ .

Note that

$$A_{i_c,\delta} = [O \quad I_{i_c,\delta}] \pi_i A \pi_i^T [O \quad I_{i_c,\delta}]^T,$$

where  $I_{i_c,\delta}$  is the  $(n - n_{i,\delta}) \times (n - n_{i,\delta})$  identity matrix. Let

$$M_{i,\delta} = \pi_i^T \begin{bmatrix} A_{i,\delta} & O \\ O & A_{i_c,\delta} \end{bmatrix} \pi_i, \tag{2.5}$$

$N_{i,\delta} = M_{i,\delta} - A$ . Since  $A$  is an  $M$ -matrix and any principal submatrix of an  $M$ -matrix is also an  $M$ -matrix (see [3]), it is easy to verify that  $A = M_{i,\delta} - N_{i,\delta}$  is an  $M$ -splitting of  $A$ , and hence it is also a regular splitting.

It follows from (2.3) and (2.5) that

$$E_{i,\delta} M_{i,\delta}^{-1} = R_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta}. \tag{2.6}$$

In order to describe the RAS method, we introduce ‘‘restricted’’ operator  $\bar{R}_{i,\delta}$  as

$$\bar{R}_{i,\delta} = R_{i,\delta} E_{i,0} \in R^{n_{i,\delta} \times n}. \tag{2.7}$$

The image of  $\bar{R}_{i,\delta}^T = E_{i,0} R_{i,\delta}^T$  can be identified with  $V_{i,0}$ , so  $\bar{R}_{i,\delta}^T$  in the sense that the image of  $V_{i,\delta}$ , is restricted to its subspace  $V_{i,0}$ , the space from the nonoverlapping decomposition. Now, we can present the RAS method for (1.1). Let  $u^0$  be an initial approximation to the solution of problem (1.1). Generally, at step  $k$ , the RAS method consists of the following substeps.

**Algorithm 2.1 (Restricted Additive Schwarz Method).**

*Substep 1 (restriction):* Restrict the vector  $\phi - u^k$  as  $\phi^{k,i_\delta} = R_{i,\delta}(\phi - u^k)$ . For each  $i = 1, 2, \dots, m$ , solve in parallel the local problem of finding  $u^{k,i_\delta} \in R^n$  such that

$$u_{\pi_i}^{k,i_\delta} = \begin{bmatrix} u_{n_{i,\delta}}^{k,i_\delta} \\ u_{n_{i_c,\delta}}^{k,i_\delta} \end{bmatrix}, \begin{bmatrix} u_{n_{i,\delta}}^{k,i_\delta}, u_{i_c,\delta}^{k,i_\delta} \end{bmatrix}^T, u_{n_{i_c,\delta}}^{k,i_\delta} = u_{n_{i_c,\delta}}^k \text{ and}$$

$$u_{n_{i,\delta}}^{k,i_\delta} \geq \phi^{k,i_\delta}, \quad R_{i,\delta}F(u^k + R_{i,\delta}^T u_{n_{i,\delta}}^{k,i_\delta}) \geq 0, \tag{2.8a}$$

$$(u_{n_{i,\delta}}^{k,i_\delta} - \phi^{k,i_\delta})^T R_{i,\delta}F(u^k + R_{i,\delta}^T u_{n_{i,\delta}}^{k,i_\delta}) = 0. \tag{2.8b}$$

*Substep 2 (prolongation):* Prolongate the approximations of errors by

$$u_e^{k,i_\delta} = \bar{R}_{i,\delta}^T u_{n_{i,\delta}}^{k,i_\delta}, \quad i = 1, 2, \dots, m. \tag{2.9}$$

*Substep 3 (correction):* Correct  $u^k$  to get

$$u^{k+1} = u^k + \sum_{i=1}^m u_e^{k,i_\delta}. \tag{2.10}$$

### 3. Global Convergence of RAS

In this section, we establish the global convergence for RAS.

**Definition 3.1.** ([12]) Let  $\omega \in R^n$  be a positive vector. For a vector  $y \in R^n$ , the weighted max-norm is defined by

$$\|y\|_\omega = \max_{1 \leq j \leq n} \left| \frac{y_j}{\omega_j} \right|.$$

For a matrix  $A \in R^{n \times n}$ , the weighted max-norm is defined by

$$\|A\|_\omega = \sup_{\|y\|_\omega=1} \{\|Ay\|_\omega : y \in R^n\}.$$

Now, we present some useful lemmas.

**Lemma 3.1.** For any  $y \geq \phi$ ,  $z \geq \phi$ , and  $y \geq z$ , we have the following inequality:

$$A(y) - A(z) \leq F(y) - F(z).$$

Noting that  $F(u) = Au + f(u)$  and  $\partial f/\partial u \geq 0$ , we immediately get the above result.

**Lemma 3.2.** ([10]) Let  $P$  be a matrix,  $\omega$  be a positive vector, and  $\gamma$  be a positive scalar such that

$$P\omega \leq \gamma\omega. \tag{3.1}$$

Then  $\|P\|_\omega \leq \gamma$ . In particular,  $\|Px\|_\omega \leq \gamma\|x\|_\omega$  holds for all  $x$ . Moreover, if the strict inequality holds in (3.1), then we have  $\|Px\|_\omega < \gamma\|x\|_\omega$ .

**Lemma 3.3.** Let  $\bar{u}$  be the solution of (1.1) and the sequence  $\{u^k\}$  be generated by Algorithm 2.1. If  $u^k = \bar{u}$ , we have  $u_{n_{i,\delta}}^{k,i_\delta} = 0$  for all  $i \in \{1, \dots, m\}$ .

*Proof.* Since  $\bar{u} - \phi \geq 0$ ,  $F(\bar{u}) \geq 0$  and  $(\bar{u} - \phi)^T F(\bar{u}) = 0$ , by the nonnegative of  $R_{i,\delta}$ , we have

$$\begin{cases} 0 - \phi^{k,i_\delta} = R_{i,\delta}(\bar{u} - \phi) \geq 0, \\ R_{i,\delta}F(\bar{u} + R_{i,\delta}^T 0) = R_{i,\delta}F(\bar{u}) \geq 0. \end{cases} \tag{3.2}$$

Multiplying these two inequalities and noting  $R_{i,\delta}^T R_{i,\delta} = E_{i,\delta} \leq I$ , we have

$$0 \leq (0 - \phi^{k,i_\delta})^T R_{i,\delta}F(\bar{u} + R_{i,\delta}^T 0) = (\bar{u} - \phi)^T R_{i,\delta}^T R_{i,\delta}F(\bar{u}) \leq (\bar{u} - \phi)^T F(\bar{u}) = 0.$$

*Proof.* This together with (3.2) and the unique solution of (2.8), we have that  $u_{n_{i,\delta}}^{k,i_\delta} = 0$  is the solution of (2.8).  $\square$

**Lemma 3.4.** Let  $u_{\pi_i}^k = [u_{i,\delta}^k, u_{i_c,\delta}^k]^T$  with  $u_{i,\delta}^k = R_{i,\delta}u^k \in R^{n_{i,\delta}}$ , where the sequence  $\{u^k\}$  is generated by Algorithm 2.1. Let  $y_{\pi_i}^k = [y_{i,\delta}^k, y_{i_c,\delta}^k]^T$  with  $y_{i,\delta}^k = R_{i,\delta}y^k \in R^{n_{i,\delta}}$ , where  $y_{i_c,\delta}^k \in R^{n_{i_c,\delta}}$  is given by  $y_{i,\delta}^k = u_{i,\delta}^k + u_{n_{i,\delta}}^{k,i_\delta}$ ,  $y_{i_c,\delta}^k = u_{i_c,\delta}^k$ . Then  $y^k$  is the solution of the following NCP:

$$\begin{cases} y_{i,\delta} \geq \phi_{i,\delta}, \\ R_{i,\delta}F(y) \geq 0, \\ (y_{i,\delta} - \phi_{i,\delta})^T R_{i,\delta}F(y) = 0, \end{cases} \tag{3.3}$$

where  $\phi_{i,\delta} = R_{i,\delta}\phi$ .

*Proof.* By the definition of  $y_{i,\delta}^k$ , we have

$$y_{i,\delta}^k - \phi_{i,\delta} = u_{i,\delta}^k + u_{n_{i,\delta}}^{k,i_\delta} - \phi_{i,\delta} = u_{n_{i,\delta}}^{k,i_\delta} - R_{i,\delta}(\phi - u^k) = u_{n_{i,\delta}}^{k,i_\delta} - \phi^{k,i_\delta}.$$

By the definition of  $y$ , we have  $R_{i,\delta}F(y) = R_{i,\delta}F(y^k) = R_{i,\delta}F(u^k + R_{i,\delta}^T u_{n_{i,\delta}}^{k,i_\delta})$ . Consequently, (3.3) follows from (2.8). This completes the proof.  $\square$

By Lemmas 3.3 and 3.4, we can easily conclude the following result.

**Lemma 3.5.** Let  $\bar{u}$  be the solution of (1.1). Then  $\bar{u}_{i,\delta} = R_{i,\delta}\bar{u}$  is the unique solution of the following NCP:

$$\begin{cases} y_{i,\delta} \geq \phi_{i,\delta}, \\ R_{i,\delta}F(y) \geq 0, \\ (y_{i,\delta} - \phi_{i,\delta})^T R_{i,\delta}F(y) = 0, \end{cases}$$

where  $\phi_{i,\delta} = R_{i,\delta}\phi$ ,  $y_{\pi_i} = \begin{bmatrix} y_{i,\delta} \\ y_{i_c,\delta} \end{bmatrix}$  and  $y_{i,\delta} = R_{i,\delta}y$ ,  $y_{i_c,\delta} = \bar{u}_{i_c,\delta}$ .

**Lemma 3.6.** Let the sequence  $\{u^k\}$  be generated by Algorithm 2.1 and  $\bar{u}$  be the unique solution of (1.1) with  $\bar{u}_{\pi_i} = [\bar{u}_{i,\delta}, \bar{u}_{i_c,\delta}]^T$ ,  $i = 1, 2, \dots, m$ . Denote  $y_{\pi_i}^k = [y_{i,\delta}^k, y_{i_c,\delta}^k]^T$ , where  $y_{i,\delta}^k = u_{i,\delta}^k + u_{n_{i,\delta}}^{k,i_\delta}$ ,  $y_{i_c,\delta}^k = u_{i_c,\delta}^k$ . Then for any  $\delta \geq 0$ , we have

$$A_{i,\delta}|y_{i,\delta}^k - \bar{u}_{i,\delta}| \leq -G_{i,\delta}|u_{i_c,\delta}^k - \bar{u}_{i_c,\delta}|. \tag{3.4}$$

*Proof.* We verify (3.4) by componentwise. Consider an arbitrary index  $j$ . We first assume that

$$|y_{i,\delta}^k - \bar{u}_{i,\delta}|_j = (y_{i,\delta}^k - \bar{u}_{i,\delta})_j,$$

which means that

$$(y_{i,\delta}^k - \bar{u}_{i,\delta})_j \geq 0.$$

Thus, if  $(y_{i,\delta}^k)_j = \phi_j$ , then  $(\bar{u}_{i,\delta})_j = \phi_j$ . Hence, (3.4) holds for the  $j$ th component, since the left-hand side is nonpositive while the right-hand side is nonnegative.

If  $(y_{i,\delta}^k) > \phi_j$ , then by Lemma 3.4, we have

$$F_j(y^k) = 0. \tag{3.5}$$

Since  $\bar{u}$  is the solution of (1.1), we have

$$F_j(\bar{u}) \geq 0. \tag{3.6}$$

Thus, by subtracting (3.6) from (3.5), we get

$$(A_{i,\delta}(y_{i,\delta}^k - \bar{u}_{i,\delta}))_j + (G_{i,\delta}(u_{i_c,\delta}^k - \bar{u}_{i_c,\delta}))_j \leq F_j(y^k) - F_j(\bar{u}) \leq 0,$$

where the first inequality comes from the Lemma 3.1 and (2.4). Hence, we have

$$(A_{i,\delta}|y_{i,\delta}^k - \bar{u}_{i,\delta}|)_j \leq -(G_{i,\delta}|u_{i_c,\delta}^k - \bar{u}_{i_c,\delta}|)_j, \tag{3.7}$$

since  $A$  is an  $M$ -matrix. We next assume that

$$|y_{i,\delta}^k - \bar{u}_{i,\delta}|_j = (\bar{u}_{i,\delta} - y_{i,\delta}^k)_j.$$

In this case, we have

$$(y_{i,\delta}^k - \bar{u}_{i,\delta})_j \leq 0.$$

Similarly, we can establish (3.7). Hence inequality (3.4) holds. □

The following lemma gives an estimate of the iteration errors.

**Lemma 3.7.** *Let the sequence  $\{u^k\}$  be generated by Algorithm 2.1 and  $\varepsilon^k := u^k - \bar{u}$ , where  $\bar{u}$  is the unique solution of (1.1). Then,*

$$|\varepsilon^{k+1}| \leq (I - \sum_{i=1}^m E_{i,0} E_{i,\delta} M_{i,\delta}^{-1} A) |\varepsilon^k|.$$

*Proof.* We deduce from (2.10) that

$$\begin{aligned} 0 &\leq |\varepsilon^{k+1}| = |u^{k+1} - \bar{u}| \\ &= |u^k + \sum_{i=1}^m u_e^{k,i\delta} - \bar{u}| = |\varepsilon^k + \sum_{i=1}^m \bar{R}_{i,\delta}^T u_{n_{i,\delta}}^{k,i\delta}| \\ &= \left| \sum_{i=1}^m E_{i,0} \pi_i^T \left( \begin{bmatrix} u_{n_{i,\delta}}^{k,i\delta} \\ O \end{bmatrix} + \begin{bmatrix} u_{i,\delta}^k - \bar{u}_{i,\delta} \\ u_{i_c,\delta}^k - \bar{u}_{i_c,\delta} \end{bmatrix} \right) \right| \\ &\leq \sum_{i=1}^m \left| E_{i,0} \pi_i^T \begin{bmatrix} y_{i,\delta}^k - \bar{u}_{i,\delta} \\ u_{i_c,\delta}^k - \bar{u}_{i_c,\delta} \end{bmatrix} \right| \\ &\leq \sum_{i=1}^m \left| E_{i,0} \pi_i^T \begin{bmatrix} -A_{i,\delta}^{-1} G_{i,\delta} |u_{i_c,\delta}^k - \bar{u}_{i_c,\delta}| \\ u_{i_c,\delta}^k - \bar{u}_{i_c,\delta} \end{bmatrix} \right|, \end{aligned}$$

where  $y_{i,\delta}^k = u_{i,\delta}^k + u_{n_{i,\delta}}^{k,i\delta}$ , the third equality follows from (2.7), and the last inequality follows from (3.4). We deduce from the above formula that

$$\begin{aligned} 0 \leq |\varepsilon^{k+1}| &\leq \sum_{i=1}^m |E_{i,0}\pi_i^T \begin{bmatrix} O & -A_{i,\delta}^{-1}G_{i,\delta} \\ O & I_{i_c,\delta} \end{bmatrix}| |\varepsilon_{\pi_i}^k| \\ &= |\varepsilon^k| - \sum_{i=1}^m |E_{i,0}\pi_i^T| |\varepsilon_{\pi_i}^k| + \sum_{i=1}^m |E_{i,0}\pi_i^T \begin{bmatrix} O & -A_{i,\delta}^{-1}G_{i,\delta} \\ O & I_{i_c,\delta} \end{bmatrix}| |\varepsilon_{\pi_i}^k| \\ &= |\varepsilon^k| + \sum_{i=1}^m |E_{i,0}\pi_i^T \begin{bmatrix} -I_{i,\delta} & -A_{i,\delta}^{-1}G_{i,\delta} \\ O & O \end{bmatrix}| |\varepsilon_{\pi_i}^k| \\ &= |\varepsilon^k| - \sum_{i=1}^m |E_{i,0}R_{i,\delta}^T A_{i,\delta}^{-1} [A_{i,\delta} \quad G_{i,\delta}]| |\varepsilon_{\pi_i}^k| \\ &= |\varepsilon^k| - \sum_{i=1}^m |E_{i,0}R_{i,\delta}^T A_{i,\delta}^{-1} [I_{i,\delta} \quad O] A_{\pi_i}| |\varepsilon_{\pi_i}^k| \\ &= |\varepsilon^k| - \sum_{i=1}^m |E_{i,0}R_{i,\delta}^T A_{i,\delta}^{-1} [I_{i,\delta} \quad O] \pi_i A \pi_i^T \pi_i| |\varepsilon^k| \\ &= |\varepsilon^k| - \sum_{i=1}^m |E_{i,0}R_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta} A| |\varepsilon^k| \\ &= \left( I - \sum_{i=1}^m |E_{i,0}E_{i,\delta}M_{i,\delta}^{-1}A| \right) |\varepsilon^k|, \end{aligned}$$

where the sixth equality follows from (2.2) and the last equality follows from (2.6). □

**Theorem 3.1.** *Let the sequence  $\{u^k\}$  be generated by Algorithm 2.1 and  $\varepsilon^k := u^k - \bar{u}$ , where  $\bar{u}$  is the unique solution of (1.1). Then we have*

$$0 \leq |\varepsilon^{k+1}| \leq T_\theta |\varepsilon^k|,$$

where  $T_\theta = I - \sum_{i=1}^m |E_{i,0}E_{i,\delta}M_{i,\delta}^{-1}A|$  is a nonnegative matrix. Moreover, for any vector  $\omega = A^{-1}e$  with  $e > 0$ , there exists a scalar  $\gamma \in (0, 1)$  such that

$$\|T_\theta\|_\omega \leq \gamma. \tag{3.8}$$

Furthermore,  $\{u^k\}$  converges to the solution of (1.1) for any initial point  $u^0$ .

*Proof.* By Lemma 3.7, we have

$$|\varepsilon^{k+1}| = \left( I - \sum_{i=1}^m |E_{i,0}E_{i,\delta}M_{i,\delta}^{-1}A| \right) |\varepsilon^k| := T_\theta |\varepsilon^k|.$$

In order to show  $\|T_\theta\|_\omega \leq \gamma$ , we only need to show that  $T_\theta \geq 0$  and  $T_\theta\omega < \omega$ . Clearly,  $T_\theta \geq 0$ ,

since

$$\begin{aligned} & I - \sum_{i=1}^m E_{i,0} M_{i,\delta}^{-1} A \\ &= I - \sum_{i=1}^m E_{i,0} E_{i,\delta} + \sum_{i=1}^m E_{i,0} E_{i,\delta} \left( I - M_{i,\delta}^{-1} A \right) \\ &= I - \sum_{i=1}^m E_{i,0} E_{i,\delta} + \sum_{i=1}^m E_{i,0} E_{i,\delta} M_{i,\delta}^{-1} |N_{i,\delta}| \\ &\geq 0, \end{aligned}$$

where the inequality follows from  $\sum_{i=1}^m E_{i,0} E_{i,\delta} \leq I$ , and  $M_{i,\delta}^{-1} |N_{i,\delta}| \geq 0$ .

Next, we show that  $T_\theta \omega < \omega$  with  $\omega = A^{-1}e$ , where  $e > 0$ . It is easy to verify that

$$T_\theta \omega = \left( I - \sum_{i=1}^m E_{i,0} M_{i,\delta}^{-1} A \right) \omega = \omega - \sum_{i=1}^m E_{i,0} M_{i,\delta}^{-1} e.$$

Noting that  $M_{i,\delta}^{-1} e \geq 0$  since  $A = M_{i,\delta} - N_{i,\delta}$  is an  $M$ -splitting, we have  $T_\theta \omega < \omega$ . Hence, there exists a scalar  $\gamma \in (0, 1)$ , such that  $\|T_\theta\|_\omega \leq \gamma$ . Since  $\rho(T_\theta) \leq \|T_\theta\|_\omega$ , Algorithm 2.1 converges for any initial vector  $u^0$ . This completes the proof.  $\square$

### 4. The Effect of Overlap on Algorithm 2.1

In this section, we discuss the effect of overlap on Algorithm 2.1. By the definition of weighted max norm, we immediately have the following result, see also [9].

**Lemma 4.1.** *Let  $T, \bar{T}$  be nonnegative matrices. Assume that  $T\omega \leq \bar{T}\omega$  for some vector  $\omega > 0$ . Then  $\|T\|_\omega \leq \|\bar{T}\|_\omega$ .*

Now, we present the following theorem about the effect of overlap on Algorithm 2.1, whose proof is similar to [8, Theorem 5.2].

**Theorem 4.1.** *Let  $\omega > 0$  be any positive vector such that  $A\omega > 0$  and  $T_{RAS,\delta} = I - \sum_{i=1}^m E_{i,0} M_{i,\delta}^{-1} A$  and  $T_{RAS,\bar{\delta}} = I - \sum_{i=1}^m E_{i,0} E_{i,\delta} M_{i,\bar{\delta}}^{-1} A$ . If  $\delta \geq \bar{\delta}$ ,*

$$\|T_{RAS,\delta}\|_\omega \leq \|T_{RAS,\bar{\delta}}\|_\omega. \tag{4.1}$$

Moreover, if the Perron vector  $\omega_{\bar{\delta}}$  of  $T_{RAS,\bar{\delta}}$  satisfies  $\omega_{\bar{\delta}} > 0$  and  $A\omega_{\bar{\delta}} > 0$ , then we also have

$$\rho(T_{RAS,\delta}) \leq \rho(T_{RAS,\bar{\delta}}). \tag{4.2}$$

*Proof.* Since  $V_{i,\bar{\delta}} \subseteq V_{i,\delta}$ ,  $i = 1, 2, \dots, m$ , we have  $A \leq M_{i,\delta} \leq M_{i,\bar{\delta}} \leq \text{diag}(A)$ . Noting that  $A$  is an  $M$ -matrix, this yields

$$M_{i,\delta}^{-1} \geq M_{i,\bar{\delta}}^{-1} \quad \text{for } i = 1, 2, \dots, m.$$

For all  $\omega > 0$  such that  $v = A\omega > 0$ , we have

$$\begin{aligned} 0 \leq T_{RAS,\delta} \omega &= \left( I - \sum_{i=1}^m (E_{i,0} E_{i,\delta} M_{i,\delta})^{-1} A \right) \omega = \omega - \sum_{i=1}^m (E_{i,0} E_{i,\delta} M_{i,\delta})^{-1} v \\ &\leq \omega - \sum_{i=1}^m E_{i,0} (E_{i,\bar{\delta}} M_{i,\bar{\delta}})^{-1} v = T_{RAS,\bar{\delta}} \omega. \end{aligned}$$

Hence we get (4.1) by Lemma 4.1. If the Perron vector  $\omega_{\bar{\delta}}$  of  $T_{RAS,\bar{\delta}}$  can be chosen as  $\omega$ , we have  $\|T_{RAS,\bar{\delta}}\|_{\omega_{\bar{\delta}}} = \rho(T_{RAS,\bar{\delta}})$ . So (4.1) yields  $\|T_{RAS,\bar{\delta}}\|_{\omega_{\bar{\delta}}} \leq \rho(T_{RAS,\bar{\delta}})$ . Since the spectral radius is never larger than any operator norm, thus we have (4.2).  $\square$

### 5. Numerical Result

In this section, we give some numerical experiments to investigate the behavior of the RAS method presented in this paper. The program is coded in Matlab and run on a personal computer. We test three problems as follows. Except comparing with the PSOR and classic additive Schwarz method as in [18], we will discuss the effect of overlap and different number of subdomains on the Algorithm 2.1.

**Problem 1.** We consider the following problem which similar to the problem presented in [18]. Let  $\Omega = (0, 1) \times (0, 1)$  and consider the following test problem:

$$u \geq 0, \quad -\Delta u + f(u, x, y) \geq 0, \quad u(-\Delta u + f(u, x, y)) = 0,$$

where  $f(u, x, y) = u/(1 + u) + xy - 2$ . We discretize the problem by using five-point difference scheme with a constant mesh step size:  $h = 1/(m + 1)$ , where  $m$  denotes the number of mesh nodes in  $x$ - or  $y$ -direction ( $n = m^2$  is the total number of unknowns). Then Problem 1 can be transformed to problem (1.1).

**Problem 2 ([14]).** Let  $\Omega = (0, 1)$  and consider the following test problem:

$$u \geq 0, \quad -\Delta u + f(u, x) \geq 0, \quad u(-\Delta u + f(u, x)) = 0,$$

where  $f(u, x) = u^2 - [\sin(x - 0.1) - x(\sin(0.1) + \sin(0.9)) + \sin(0.1)]^2 - \sin(x - 0.1)$ . We discretize the problem by the Lagrange linear finite-element scheme with a constant mesh step  $h = 1/(n + 1)$ . Then Problem 2 can be transformed to problem (1.1).

**Problem 3 ([19]).** We consider the following nonlinear complementarity problem

$$u \geq 0, \quad F(u) \geq 0, \quad u^T F(u) = 0.$$

Here  $F(u) = Au + D(u) + f$ , where,

$$A = \frac{1}{h^2} \begin{pmatrix} H & -I & & & \\ -I & H & \ddots & & \\ & \ddots & \ddots & -I & \\ & & & -I & H \end{pmatrix}, \quad H = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & & -1 & 4 \end{pmatrix},$$

$h = \frac{1}{\sqrt{n+1}}$ ,  $D(u) = (D_i) : R^n \rightarrow R^n$  being a given diagonal mapping with  $D_i : R \rightarrow R$ , for  $i = 1, 2, \dots, n$ , that is, component  $D_i$  of  $D$  is a function of the  $i$ th variable  $u_i$  only. Set  $D_i(u_i) = \lambda e^{u_i}$  and obtain a diagonal mapping  $D(u) = (D_i(u_i))$ . In our test, we fix  $\lambda = 0.8$ , and let  $f_i = \max\{0, v_i - 0.5\} \times 10^{w_i - 0.5}$ , where  $w_i$  and  $v_i$  are random numbers in  $[0, 1]$ ,  $i = 1, 2, \dots, n$ .

We compare different algorithms from the point of view of iteration numbers and CPU times. Here, we consider three algorithms: projected SOR method (denoted by PSOR), additive Schwarz algorithm (denoted by AS, see for example [17]) and restricted additive Schwarz algorithm (Algorithm 2.1, denoted by RAS). In AS and RAS algorithms, the corresponding

Table 5.1: The methods with different dimension for Problem 1.

$N$	PSOR		AS		RAS	
	iter	cpu	iter	cpu	iter	cpu
100	210	0.48	26	1.66	6	0.45
225	290	2.59	26	6.81	5	1.64
400	398	9.16	25	16.64	5	4.69
625	482	26.55	25	39.77	4	11.33
900	565	64.66	25	87.75	4	28.73
1225	655	132.36	26	158.02	4	48.91

Table 5.2: The methods with different dimension for Problem 2.

$N$	PSOR		AS		RAS	
	iter	cpu	iter	cpu	iter	cpu
100	46	0.28	32	0.64	14	0.38
225	123	3.75	28	3.91	12	1.81
400	243	30.91	26	29.52	10	11.84
625	420	168.75	25	130.78	10	51.53
900	606	649.55	25	445.22	10	175.67
1225	902	2025.94	25	1162.2	10	534.59

subproblems are solved by PSOR method and the relaxation parameter  $\omega = 1.6$ . The tolerance in the subproblems of the algorithms is chosen to be equal to  $10^{-4}$  in  $\|\cdot\|_2$ - norm, while in the outer iterative processes is chosen to be equal to  $10^{-6}$  in  $\|\cdot\|_2$ - norm.

First, we choose initial value  $u^0 = 1$  for Problem 1, and  $u^0 = 0$  for Problems 2 and 3. We change the dimension from  $n = 100$  to  $n = 1225$ . In AS and RAS algorithms, we decompose  $S$  into two equal parts with the overlapping size 20%, The results are presented in Tables 1-3. We can see from the tables, RAS needs less time and fewer iteration numbers to converge than those of AS and PSOR for most cases. What's more, with the same initial value, we may conclude that RAS has the good property just as AS that the convergence rate will not be deteriorated with the refinement of the mesh.

Next, for Problem 1, we fix the dimension  $n = 100$  and choose different initial values to test whether the algorithms converge. The results are presented in Table 4. We can see from the table, all algorithms globally converge to the solution and the behavior of RAS is better than AS and PSOR.

Then, for Problem 1, we discuss the effect of overlap on Algorithm 2.1. We change the overlap from 5% to 30%, and fix the dimension  $n = 400$ , the initial value  $u^0 = 1$ . The result

Table 5.3: The methods with different dimension for Problem 3.

$N$	PSOR		AS		RAS	
	iter	cpu	iter	cpu	iter	cpu
100	17	0.02	12	0.11	2	0.08
225	16	0.06	10	0.22	2	0.06
400	16	0.2	10	1.08	2	0.19
625	16	0.45	10	2.86	2	0.44
900	17	0.84	9	5.38	3	0.89
1225	17	1.77	9	10.14	2	1.66

Table 5.4: The methods with different initial value for Problem 1.

$u^0$	PSOR		AS		RAS	
	iter	cpu	iter	cpu	iter	cpu
-10	93	2.11	28	15.77	2	1.42
-5	93	2.16	27	14.98	2	1.11
-1	93	2.02	24	12.61	2	1
0	135	3.08	21	11.17	3	1.48
1	398	8.7	25	17.14	5	4.48
5	734	16.78	31	30.25	7	9.91
10	904	19.94	34	36.84	9	13.59
20	1081	25.03	37	44.39	10	17.69

Table 5.5: RAS with different overlapping for Problem 1.

ovp	iter	cpu	ovp	iter	cpu
5	9	7.92	10	7	5.81
15	5	5.53	20	5	4.92
25	4	4.58	30	4	6.09

are listed in Table 5, where ovp denotes the overlap, iter denotes the iterative number and cpu denotes the CPU times. From the table, we can see the more overlap there is, the faster the convergence of Algorithm 2.1.

In the last experiment, we discuss the effect of numbers of the subdomains for Problem 1. In this experiment, we fix the dimension  $n = 400$ , the initial value  $u^0 = 1$ , and fix the overlap between adjacent subdomains to be 10%. The result is listed in Table 6. As we can see, the iteration number of AS and RAS increase when the numbers of the subdomains increases. However, the cpu time only increase slightly, since the dimension of the subproblems decreases when the number of the subdomains increases.

Table 5.6: AS and RAS with different number of subdomains for Problem 1.

num	RAS		AS	
	iter	cpu	iter	cpu
2	7	6.47	27	19.11
3	8	5.22	48	24.2
4	9	4.28	71	25.86
5	11	4.63	97	31.63
6	13	5.78	126	45.77
7	16	7.78	154	56.95

## 6. Conclusion

In this paper, we proposed a restricted additive Schwarz method for nonlinear complementarity problem. The algorithm belongs to domain decomposition methods and can be considered as an extension of restricted additive Schwarz method for system of linear equations. We analyzed its convergence by using weighted max norm and compared the algorithm with classical additive Schwarz method and PSOR method. As we can see from the preliminary numerical results, the restricted additive Schwarz method spends less CPU time and needs fewer iteration

numbers. Moreover, we tested the effect of overlapping size on the restricted additive Schwarz method. From the numerical results, we may obtain an interesting conclusion: the convergence rate of RAS is mesh-independent.

There are still some future works interested needed to be done. For example, it is interesting for us to extend restricted additive Schwarz method to solve other variational inequality and mathematical programming problem. As we can see from Algorithm 2.1, the restricted additive Schwarz algorithm is nonsymmetric. It is also interesting for us to modify restricted additive Schwarz algorithm to be symmetric. Furthermore, we can study the independent of mesh for convergence rate of RAS.

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