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A SYSTEMATIC METHOD TO CONSTRUCT MIMETIC FINITE-DIFFERENCE SCHEMES FOR INCOMPRESSIBLE FLOWS

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Abstract. We present a general procedure to construct a non-linear mimetic finite-difference operator. The method is very simple and general: it can be applied for any order scheme, for any number of grid points and for any operator constraints.
In order to validate the procedure, we apply it to a specific example, the Jacobian operator for the vorticity equation. In particular we consider a finite difference approximation of a second order Jacobian which uses a 9x9 uniform stencil, verifies the skew-symmetric property and satisfies physical constraints such as conservation of energy and enstrophy. This particular choice has been made in order to compare the present scheme with Arakawa's renowned Jacobian, which turns out to be a specific case of the general solution. Other possible generalizations of Arakawa's Jacobian are available in literature but only the present approach ensures that the class of solutions found is the widest possible. A simplified analysis of the general scheme is proposed in terms of truncation error and study of the linearised operator together with some numerical experiments. We also propose a class of analytical solutions for the vorticity equation to compare an exact solution with our numerical results.

Key words. Mimetic schemes, Arakawa's Jacobian, finite-difference, non-linear instability.

Introduction

We consider the vorticity equation for two-dimensional incompressible inviscid flow on a biperiodic domain D in the variables x and y,

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot (\mathbf{v}\zeta) = 0 \tag{1a}$$

where

$$\nabla \cdot \mathbf{v} = 0 \tag{1b}$$

$$\mathbf{v} = \mathbf{k} \times \nabla \psi \tag{1c}$$

$$\zeta = \mathbf{k} \cdot \nabla \times \mathbf{v} = \nabla^2 \psi \tag{1d}$$

$\zeta = \zeta(x, y)$ is the vorticity, $\mathbf{v} = (u(x, y); v(x, y); 0)$ is the velocity field, $\psi = \psi(x, y)$ is the stream function and $\mathbf{k}$ is the unit vector normal to the plane of motion; $\mathbf{A} \cdot \mathbf{B}$, $\mathbf{A} \times \mathbf{B}$ and ∇ denote respectively the standard three-dimensional dot and cross product of two vectors $\mathbf{A} = (A_1; A_2; A_3)$ and $\mathbf{B} = (B_1; B_2; B_3)$ and the gradient operator i.e. $\mathbf{A} \cdot \mathbf{B} = \sum_{i=1}^{3} A_i B_i$, $\mathbf{A} \times \mathbf{B} = (A_2B_3 - A_3B_2; A_3B_1 - A_1B_3; A_1B_2 - A_2B_1)$ and $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$. Using eqs. (1b)-(1c) and recalling that we deal with a two-dimensional flow, eqs.(1a),(1d) simplifies to:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial x} = 0; \quad \zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}.$$

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We introduce the Jacobian operator

$$J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x} \tag{2}$$

with the following properties:

- Skew-symmetry:

$$J(a,b) = -J(b,a) \tag{3}$$

- Integral property:

$$\overline{aJ(b,c)} = \overline{cJ(a,b)} \tag{4}$$

where $\overline{f} = \int_D f dxdy$.

We can rewrite equation (1a) as:

$$\frac{\partial \zeta}{\partial t} = J(\zeta, \psi). \tag{5}$$

Conserved quantities

We start with two definitions.

Definition 1. *The mean kinetic energy for the equation (5) is defined as:*

$$K = \frac{1}{2}\overline{(\nabla\psi)^2}. \tag{6}$$

Definition 2. *The enstrophy (mean square vorticity) for the equation (5) is defined as:*

$$G = \frac{1}{2}\overline{\zeta^2}. \tag{7}$$

For any motion governed by equation (1) we have physical constraints such as conservation of energy,

$$\begin{aligned}\frac{\partial K}{\partial t} &\overset{(8a)}{=} \overline{\frac{1}{2}\frac{\partial(\nabla\psi)^2}{\partial t}} \overset{((8b)}{=} \overline{(\nabla\psi)\cdot\frac{\partial(\nabla\psi)}{\partial t}} \overset{((8c)}{=} \overline{-\psi\frac{\partial(\Delta\psi)}{\partial t}} \\ &\overset{((8d)}{=} \overline{-\psi\frac{\partial\zeta}{\partial t}} \overset{((8e)}{=} \overline{-\psi J(\zeta,\psi)} \overset{((8f)}{=} 0\end{aligned} \tag{8}$$

and conservation of enstrophy,

$$\frac{\partial G}{\partial t} \overset{((9a)}{=} \overline{\frac{1}{2}\frac{\partial(\zeta^2)}{\partial t}} \overset{(9b)}{=} \overline{\zeta\frac{\partial\zeta}{\partial t}} \overset{(9c)}{=} \overline{\zeta J(\zeta,\psi)} \overset{(9d)}{=} 0 \tag{9}$$

where the RHS of both equations is zero thanks to the skew-symmetric (3) and the integral (4) properties with, respectively, $a = c = \psi$ and $a = b = \zeta$.

It is well known that non-linear problems as system (1) require the correct modeling of sub-grid terms (see, for example, J. Smagorinsky 1963 [17], J. W. Deardorff 1970 [2]); in this context special attention has been given when considering large-eddy simulations (LES) to the interaction between truncation error of the underlying discretization and the sub-grid scale modeling ([26], [27], [28]). The main issue is that a false transfer of energy between different scales can occur depending on different forms of truncation error, corresponding to different forms of discretization. In 1959 Phillips [14], treating non-linear numerical instability, proposed to add a smoothing term to equation (1a), but his solution resulted to be physically incorrect and to compromise the simulation. To overcome this problem, Arakawa [1]