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# The Equivalence of Ishikawa-Mann and Multistep Iterations in Banach Space†

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**Abstract.** Let $E$ be a real Banach space and $T$ be a continuous $\Phi-$strongly accretive operator. By using a new analytical method, it is proved that the convergence of Mann, Ishikawa and three-step iterations are equivalent to the convergence of multistep iteration. The results of this paper extend the results of Rhoades and Soltuz in some aspects.

**Key words**: $\Phi-$strongly accretive operator; $\Phi-$strongly pseudocontractive operator; continuous; Mann iteration; Ishikawa iteration; multistep iteration.

**AMS subject classifications:** 47H09, 47H10

## 1 Introduction

Let $E$ denote an arbitrary real Banach space and $E^*$ denote the dual space of $E$. The duality map $J\colon E \to 2^{E^*}$ is defined by

$$Jx := \{u^* \in E^* : \langle x, u^* \rangle = \|x\|^2; \|u^*\| = \|x\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between elements of $E$ and $E^*$. We first recall and define some concepts as follows:

**Definition 1.1.** Let $K$ be a nonempty subset of $E$ and let $T\colon K \to E$ be an operator.

(i). $T$ is said to be accretive if, for $\forall x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq 0. \tag{1}$$

(ii). $T$ is said to be strongly accretive if, for $\forall x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq k\|x-y\|^2, \tag{2}$$

where $k > 0$ is a constant. Without loss of generality, we assume that $k \in (0,1)$.

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(iii). $T$ is said to be $\Phi-$strongly accretive if, for $\forall x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq \Phi(\|x-y\|)\|x-y\|, \tag{3}$$

where $\Phi$: $[0,\infty) \to [0,\infty)$ is a function for which $\Phi(0)$=0, $\Phi(r)$>0 for all $r$>0, $\liminf\limits_{r\to\infty} \Phi(r)$>0 and the function $h(r) = r\Phi(r)$ is nondecreasing on $[0,\infty)$.

If $I$ denotes the identity operator, it follows from inequalities (1)-(3) that $T$ is pseudocontractive (respectively, strongly pseudocontractive, $\Phi-$strongly pseudocontractive) if and only if $(I-T)$ is an accretive (respectively, strongly accretive,$\Phi-$strongly accretive). It is shown in [1] that the class of single-valued strongly pseudocontractive operators is a proper subclass of the class of single-valued $\Phi-$strongly pseudocontractive operators. The classes of single-valued operators have been studied by many authors (see, for example [1]- [13]).

Now, we state concepts which will be needed in the sequel.

(a). The iteration (see [9])

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n^1, \\ y_n^1 = (1-\beta_n^1)x_n + \beta_n^1 T x_n, \quad n = 0,1,2,\cdots \end{cases} \tag{4}$$

is called the Ishikawa iteration sequence, where $\{\alpha_n\}, \{\beta_n^1\}$ are real sequences in $[0,1]$ satisfying some appropriate conditions.

(b). In particular, if $\beta_n^1 = 0$ for $n \geq 0$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n = 0,1,2,\cdots \tag{5}$$

is called the Mann iteration (see [10]).

(c). In [11], Noor introduced the three-step procedure

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + T y_n^1, \\ y_n^1 = (1-\beta_n^1)x_n + \beta_n^1 T y_n^2, \\ y_n^2 = (1-\beta_n^2)x_n + \beta_n^2 T x_n, \quad n = 0,1,2,\cdots, \end{cases} \tag{6}$$

where $\{\alpha_n\}, \{\beta_n^1\}, \{\beta_n^2\}$ are real sequences in $[0,1]$ satisfying some appropriate conditions.

(d). In [13], Rhoades and Soltuz introduced the multi-step procedure

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + T y_n^1, \\ y_n^i = (1-\beta_n^i)x_n + \beta_n^i T y_n^{i+1}, \quad i = 1,\cdots,p-2, \\ y_n^{p-1} = (1-\beta_n^{p-1})x_n + \beta_n^{p-1} T x_n, \quad n = 0,1,2,\cdots, \end{cases} \tag{7}$$

where $p \geq 2$ is a fixed order, $\{\alpha_n\}$ is a sequence in $(0,1)$ such that for all $n \in \mathbb{N}$

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty. \tag{8}$$

Moreover, for all $n \in \mathbb{N}$

$$\{\beta_n^i\} \subset [0,1), \quad 1 \leq i \leq p-1, \quad \lim_{n\to\infty} \beta_n^1 = 0. \tag{9}$$