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Restarted FOM Augmented with Ritz Vectors for Shifted Linear Systems†

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Abstract. The restarted FOM method presented by Simoncini [7] according to the natural collinearity of all residuals is an efficient method for solving shifted systems, which generates the same Krylov subspace when the shifts are handled simultaneously. However, restarting slows down the convergence. We present a practical method for solving the shifted systems by adding some Ritz vectors into the Krylov subspace to form an augmented Krylov subspace. Numerical experiments illustrate that the augmented FOM approach (restarted version) can converge more quickly than the restarted FOM method.

Key words: Augmented Krylov subspace; FOM; restarting; shifted systems.

AMS subject classifications: 65F10, 65Y20

1 Introduction

Given a real large nonsymmetric matrix A, we are interested in solving the following shifted systems

$$Ax = b \tag{1}$$

and

$$\hat{A}x = b, \tag{2}$$

where $\hat{A} = A + \sigma I$ for several (say a few hundreds; see e.g. [7]) σ, $\sigma \in R$. These kinds of linear systems arise in many fields. For instance, in image restorations, numerical methods for integral equations, structural dynamics, and QCD problems.

The system (1) can be called seed system and (2) called add system. It is well known that the Krylov subspace $K_m(A, r_0) = \text{span}\{r_0, Ar_0, \cdots, A^{m-1}r_0\}$ is the same as $K_m(\hat{A}, \hat{r}_0) = \text{span}\{\hat{r}_0, \hat{A}\hat{r}_0, \cdots, \hat{A}^{m-1}\hat{r}_0\}$ with $\hat{r}_0 = \beta_0 r_0$. Therefore, if we apply a Krylov subspace method to

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solve (1) and (2) simultaneously, the basis has to be calculated only once. Then the iterative solution of the add system may be calculated at a very low extra expense.

However, the Krylov subspace required to satisfactorily approximate (1) and (2) appears to be too large, and the computational cost is expensive, so the method needs to be restarted, taking the current system residual as a new generating vector. If the new Krylov subspace is the same for all shifted systems, the computational efficiency can be maintained. We want to get the identical basis vectors for $K_m(A, r_m)$ and $K_m(\hat{A}, \hat{r}_m)$, so the collinearity of residuals r_m and $\hat{r}_m$ is required. Simoncini[7] gave the natural collinearity of r_m and $\hat{r}_m$ for FOM by computing

$$r_m = b - Ax_m = r_0 - AV_m d_m = -h_{m+1,m} v_{m+1} e_m^T d_m = \beta v_{m+1}, \quad (3)$$

$$\hat{r}_m = b - \hat{A}\hat{x}_m = \hat{r}_0 - \hat{A} V_m \hat{d}_m = -\hat{h}_{m+1,m} v_{m+1} e_m^T \hat{d}_m = \hat{\beta} v_{m+1}. \quad (4)$$

So restarting can also be employed in the shifted case. For GMRES, the natural collinearity of residuals are not satisfied. Frommer and Glassner [1] give a variant of GMRES by forcing the residual $\hat{r}_m$ to be collinear to r_m, so restarting can also be employed. However restarting will slow down the convergence since some information is lost when restarting.

In [3,4], we have presented restarted GMRES and restarted FOM augmented with exact eigenvectors to solve system (1) and (2) based on the Morgan's augmented method [5] and the idea of Frommer and Glassner [1]. However, these methods are generally impractical since the exact eigenvectors are difficult or impossible to obtain. In this paper, we show that FOM augmented by adding some Ritz vectors, instead of exact eigenvectors, can also be used to solve the shifted systems (1) and (2). This version of augmented FOM (denoted as shifted FOM-R) can practically be implemented since the Ritz vectors can be easily derived. Numerical experiments illustrate the efficiency of the method.

2 Review of some known facts

In [5], Morgan presented an augmented Krylov subspace method by adding some eigenvectors z_i associated with a few of the smallest eigenvalues of A into the standard Krylov subspace to form an augmented Krylov subspace

$$K_{m,l}(A, r_0, z_i) = \text{span}\{r_0, Ar_0, \cdots, A^{m-1}r_0, z_1, \cdots, z_l\}.$$

Let m be the dimension of the standard Krylov subspace$K_m(A, r_0)$ and $V_m = [v_1, \cdots, v_m]$ is the basis of the subspace. Suppose l eigenvectors $z_1, \cdots, z_l$ are added into the subspace. Let $W_s = [v_1, \cdots, v_m, z_1, \cdots, z_l]$, $Q_{s+1} = [v_1, \cdots, v_{m+1}, q_1, \cdots, q_l]$, where q_i is formed by orthogonalizing the vectors Az_i against the previous columns of Q_{s+1}. By the augmented Arnoldi process [5,6], we have the Arnoldi factorization

$$AW_s = Q_{s+1}\bar{H}_s \quad (s = m + l), \quad (1)$$

where $\bar{H}_s$ is an $(s + 1) \times s$ upper-Hessenberg matrix. An augmented Krylov subspace method is a project process on the augmented subspace $K_{m,l}(A, r_0, z_i)$.

In [4], we have shown that the restarted FOM method augmented with exact eigenvectors (denoted as shifted FOM-E) can be used to solve the shifted systems (1) and (2). In that case, we want to derive the approximate solution of the seed system and the add system in the augmented Krylov subspace $K_{m,l}(A, r_0, z_i)$. According to FOM , the approximate solution of the seed system $x_s = x_0 + W_s d_s$ can be obtained with $d_s = H_s^{-1}\|r_0\|_2 e_1$, and by forcing the residual of the add