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A FAST SINE TRANSFORM ALGORITHM FOR TOEPLITZ MATRICES AND ITS APPLICATIONS*

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Abstract *In this paper, a fast algorithm for the discrete sine transform(DST) of a Toeplitz matrix of order N is derived. Only* $O(N\log N)+O(M)$ *time is needed for the computation of M elements. The auxiliary storage requirement is O(N). An application of the new fast algorithm is also discussed.*

Key words *Toeplitz matrices; discrete sine transform; Jacobi rotation method.*

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1 Introduction

We denote by S_N the discrete sine transform matrix of order N

$$S_N = \left[\sqrt{\frac{2}{N+1}}\sin\frac{(m+1)(l+1)\pi}{N+1}\right]_{m,l=0}^{N-1},$$

for convenience, the indexes of S_N run from 0 to N-1. S_N is orthogonal and symmetric, which means $S_N^{-1} = S_N^T = S_N$.

A real $N\times N$ Toeplitz matrix can be denoted by

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{N-1} \\ a_{-1} & a_0 & a_1 & \cdots & a_{N-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{-N+1} & a_{-N+2} & a_{-N+3} & \cdots & a_0 \end{bmatrix}. \quad (1)$$

That is, $A = (a_{l,k}),\ a_{l,k} = a_{k-l},\ l,k = 0,1,\cdots,N-1.$

Toeplitz matrices and discrete sine transform(DST) arise in many applications of digital signal and image processing([1]-[10]). In this paper we propose a fast algorithm for the discrete

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sine transform of a Toeplitz matrix. The algorithm is much faster than only an FFT implementation of the DST(see [4] would yield). We give an application of the DST to the eigenvalue computation of symmetric Toeplitz matrices and some numerical examples.

2 Fast sine transform algorithm for Toeplitz matrices

Let $\hat{A} = (\hat{a}_{m,n}) = TAT$, $s_{m,l} = \sin\frac{(m+1)(l+1)\pi}{N+1}$, then

$$\hat{a}_{m,n} = \frac{2}{N+1}(\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} s_{m,l}a_{l,k}s_{n,k}). \tag{2}$$

Introducing the quantities

$$x_{m,k} = \sum_{l=0}^{N-1} w^{(m+1)(l+1)}a_{l,k} \text{ with } w = \exp\left(i\frac{\pi}{N+1}\right), \tag{3}$$

we get

$$\hat{a}_{m,n} = \frac{2}{N+1}\left(\sum_{k=0}^{N-1} Im\{x_{m,k}\}s_{n,k}\right), \tag{4}$$

where Im$\{\cdot\}$ denotes the image part of a complex entity, and we have used the fact that the matrix A is real.

Let

$$y_{m,n+1} = \sum_{k=0}^{N-1} x_{m,k}w^{(n+1)k}, \tag{5}$$

then we have

$$\hat{a}_{m,n} = \frac{2}{N+1}\text{Im}\left\{w^{n+1}\frac{y_{m,n+1} - \bar{y}_{m,-(n+1)}}{2i}\right\}, \tag{6}$$

here $\bar{z}$ denotes the complex conjugate of z. In fact,

$$\begin{aligned}
\hat{a}_{m,n} &= \frac{2}{N+1}(\sum_{k=0}^{N-1}\text{Im}\{x_{m,k}\}s_{n,k}) \\
&= \frac{2}{N+1}(\sum_{k=0}^{N-1}\text{Im}\{x_{m,k}\}\text{Im}\{w^{(n+1)(k+1)}\}) \\
&= \frac{2}{N+1}\text{Im}\{w^{n+1}\sum_{k=0}^{N-1}(\text{Im}\{x_{m,k}\}w^{(n+1)k})\}.
\end{aligned}$$

As

$$\begin{aligned}
y_{m,n+1} - \bar{y}_{m,-(n+1)} &= \sum_{k=0}^{N-1} x_{m,k}w^{(n+1)k} - \overline{(\sum_{k=0}^{N-1} x_{m,k}w^{-(n+1)k})} \\
&= \sum_{k=0}^{N-1}(w^{(n+1)k}x_{m,k} - \overline{w^{-(n+1)k}}\bar{x}_{m,k})
\end{aligned}$$