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ON THE EXISTENCE OF SOLUTION OF A NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEM ARISING FROM A LIQUID METAL FLOW*

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Abstract *In this paper, we discuss the existence of solution of a nonlinear two-point boundary value problem with a positive parameter Q arising in the study of surface-tension-induced flows of a liquid metal or semiconductor. By applying the Schauder's fixed-point theorem, we prove that the problem admits a solution for* $0 \le Q \le 14.306$*. It improves the result of* $0 \le Q < 1$ *in* [2] *and* $0 \le Q \le 13.213$ *in* [3].

Key words *nonlinear two-point boundary value problem, Schauder fixed-point theorem, upper-lower estimate solution method.*
AMS(2000)subject classifications 34B

1 Introduction

In this paper, we discuss the following nonautonomous two-point boundary value problem

$$\begin{cases} \Big[x\big(\frac{f'}{x}\big)'\Big]' + Q\Big[f\big(\frac{f'}{x}\big)' - x\big(\frac{f'}{x}\big)^2\Big] = \beta x, & 0 < x < 1, \\ f(0) = f(1) = \big(\frac{f'}{x}\big)'|_{x=0} = \big(\frac{f'}{x}\big)'|_{x=1} - 1 = 0, \end{cases} \tag{1.1}$$

where $' = d/dx$. This problem arises from problems of surface-tension-induced flows of a liquid metal or semiconductor in a cylindrical floating zone of length $2L$ and radius R. The parameter $Q = 2L^3R^{-3}(Re)$ with the Reynolds number Re, and β is a constant to be determine.

Following [2,3], we obtain the following problem by differentiating (1.1) with respect to x,

$$\begin{cases} \Big[\big(\frac{f'}{x}\big)'\Big]'' + \Big[\frac{1+Qf}{x}\Big]\big(\frac{f'}{x}\big)'' - \Big[\frac{1+Q(xf)'}{x^2}\Big]\big(\frac{f'}{x}\big)' = 0, & 0 < x < 1, \\ f(0) = f(1) = \big(\frac{f'}{x}\big)'|_{x=0} = \big(\frac{f'}{x}\big)'|_{x=1} - 1 = 0. \end{cases} \tag{1.2}$$

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It is equivalent to the system

$$\begin{cases} (\dfrac{f'}{x})' = g, & 0 < x < 1, \\ f(0) = f(1) = 0 \end{cases} \tag{1.3}$$

and

$$\begin{cases} g'' + \Big[\dfrac{1+Qf}{x}\Big]g' - \Big[\dfrac{1+Q(xf)'}{x^2}\Big]g = 0, & 0 < x < 1, \\ g(0) = g(1) - 1 = 0. \end{cases} \tag{1.4}$$

Numerical solutions of (1.1) have been reported in [1] for $0 \le Q \le 32.7$ and $Q \ge 1749$. In [2], they have proved the existence of solutions theoretically for $0 \le Q < 1$ and the authors in [3] improved to $0 \le Q \le 13.213$. They used the upper-lower estimate solution method and Schauder fixed-point theorem on $f(x)$. In this paper, we will apply the Schauder fixed-point theorem on function $g(x)$ and obtain a slight better result for the existence of solutions. We prove that for $0 \le Q \le 14.306$, there exists at least one solution to equation (1.1).

2　The Existence of Solution

Denote the set

$$D = \{g |\, g \in C^2(0,\,1),\, x^3 \le g(x) \le x^{\frac{21}{50}},\, 0 \le x \le 1\,\}. \tag{2.1}$$

We first will prove the following result.

Theorem 2.1　For $0 \le Q \le 14.306$ and any $g \in D$, there exists a unique solution $g^* \in D$ satisfies the following equations

$$\begin{cases} (\dfrac{f'}{x})' = g, & 0 < x < 1, \\ f(0) = f(1) = 0, \end{cases} \tag{2.2}$$

and

$$\begin{cases} g^{*\prime\prime} + \Big[\dfrac{1+Qf}{x}\Big]g^{*\prime} - \Big[\dfrac{1+Q(xf)'}{x^2}\Big]g^* = 0, & 0 < x < 1, \\ g^*(0) = g^*(1) - 1 = 0. \end{cases} \tag{2.3}$$

Proof　It is easy to see from (2.2) that

$$f(x) = \frac{1}{2}(x^2-1)\int_0^x t^2 g(t)\,\mathrm{d}t - \frac{1}{2}x^2\int_x^1 (1-t^2)g(t)\,\mathrm{d}t. \tag{2.4}$$

$$f'(x) = x\int_0^x t^2 g(t)\,\mathrm{d}t - x\int_x^1 (1-t^2)g(t)\,\mathrm{d}t. \tag{2.5}$$

By (2.4)-(2.5), we get the equation

$$(xf)' = f(x) + xf'(x) = \frac{1}{2}(3x^2-1)\int_0^x t^2 g(t)\,\mathrm{d}t - \frac{3}{2}x^2\int_x^1 (1-t^2)g(t)\,\mathrm{d}t. \tag{2.6}$$

Notice $g \in D$ in (2.6), then $(xf)' \ge s(x)$ with

$$s(x) = \begin{cases} \dfrac{1}{2}(3x^2-1)\displaystyle\int_0^x t^2 t^3\,\mathrm{d}t - \dfrac{3}{2}x^2\int_x^1 (1-t^2)t^{\frac{21}{50}}\,\mathrm{d}t, & \dfrac{\sqrt{3}}{3} \le x \le 1, \\ \dfrac{1}{2}(3x^2-1)\displaystyle\int_0^x t^2 t^{\frac{21}{50}}\,\mathrm{d}t - \dfrac{3}{2}x^2\int_x^1 (1-t^2)t^{\frac{21}{50}}\,\mathrm{d}t, & 0 \le x \le \dfrac{\sqrt{3}}{3}. \end{cases} \tag{2.7}$$