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THE MULTIGRID METHOD OF NONCONFORMING FINITE ELEMENTS FOR SOLVING THE BIHARMONIC EQUATION

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Abstract

An optimal order of the multigrid method is given in energy-norm for the nonconforming finite element for solving the biharmonic equation, by using the nodal interpolation operator as the transfer operator between grids.

1. Introduction

Several aspects of the nonconforming finite element method have been discussed in [1-5]. In this paper, we will introduce the multigrid method and prove that the multigrid method of nonconforming finite elements can attain the same optimal convergence order as the nonconforming finite element method in energy-norm.

The multigrid method of conforming finite elements has been studied[7−8]. For the multigrid method of nonconforming finite elements, because the finite element spaces associated with the nets are not nest ($V_{k-1} \not\subset V_K$), it is difficult to define the transfer operator or the prolongation operator , especially when the nonconforming element interpolation order is greater than 2, as in Fraeijs de Veubeke triangular elements, the Adini rectangular element, the Zienkiewicz triangular element, etc.[1,4]. For the Morley elements, the transfer operators between grids can be defined e.g. [10–12]. At present we try to use the nodal interpolation operator as the intergrid transfer operator. The error estimate of an optimal order convergence property of the multigrid method is given in energy-norm for the multigrid method of nonconforming finite elements such as the Morley element, Fraeijs de Veubeke triangular elements, the Adini rectangle element and the Zienkiewicz triangular element. Furthermore, the method presented in this paper is effective for other high order conforming or nonconforming element interpolations.

The remainder of the paper is organized as follows. In Section 2, the nonconforming finite elements and their properties of approximating the biharmonic equation as well as properties of the nodal interpolation operator as the intergrid transfer operator are stated. In Section 3, the multigrid method is given. In Section 4, the convergence

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properties of the multigrid method of nonconforming finite elements are analyzed in energy-norm.

2. The Biharmonic Equation and Nonconforming Finte Element

We consider the biharmonic equation:

$$\begin{cases} \Delta^2 u = f, & \text{in } \Omega, \\ U = \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where $\Omega \subset \Re^2$ is a bounded polygonal domain, $f \in H^{-l}$, $l = 0, 1$.

The boundary problem (2.1) has a unique solution $u \in H^{4-l}(\Omega) \cap H_0^2(\Omega)$, which satisfies the elliptic regularity[6]:

$$\| u \|_{H^{4-l}(\Omega)} \leq c(\Omega) \| f \|_{H^{-l}(\Omega)} . \tag{2.2}$$

Let $V = H_0^2(\Omega)$. A variational form of equation (2.1) can be written as: Find $u \in V$ such that

$$a(u, v) = \int_\Omega f v dx, \qquad \forall v \in V \tag{2.3}$$

where $a(u.v) = \int_\Omega D^2 u D^2 v dx = \int_\Omega \sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial x_i \partial y_j} \frac{\partial^2 v}{\partial x_i \partial y_j} dx.$

Let Γ_1 be an initial triangulation and satisfy the quasi-uniformity condition, namely, for $\forall \tau \in \Gamma_1$,

$$\frac{h(\tau)}{\rho(\tau)} \leq \lambda \tag{2.4}$$

where $h(\tau)$ denotes the diameter of triangle τ, $\rho(\tau)$ denotes the diameter of the inscribed circle for triangle τ, and λ is a constant independent of $h(\tau)$. If the following finite element space is the Zienkiewicz finite element space, then the above initial triangulation must be in such a way that the three sides of every triangle in Γ_1 are parallel to three given directions[1].

We now construct Γ_{k+1} for $k \geq 1$, inductively. For each $\tau \in \Gamma_k$, four triangles in Γ_{k+1} are obtained by connecting the midpoints of the edges of triangle τ. Thus Γ_{k+1} satisfies the quasi-uniform condition (2.4). For the Zienkiewicz finite element, three sides of all $\tau \in \Gamma_{k+1}$ are also parallel to three given directions. We denotes $h_k = \max\{\operatorname{diam}\tau, \tau \in \Gamma_k\}$. Then $h_{k+1} = \frac{1}{2} h_k$.

Let V_k be the nonconforming finite element space associated with triangulation Γ_k. Then $V_k \not\subset V$ and V_k is affine (such as the Adini element and the Zienkiewicz element) or almost-affine (such as the Morley element and Fraeijs de Veubeke elements). Thus the interpolation polynomials of nonconforming elements satisfy the inverse inequality[4], namely, there exists a constant c independent of h_k, such that

$$| v_k |_{H^2(\tau)} \leq c h_k^{-2} \| v_k \|_{L^2(\tau)}, \quad for \ \forall v_k \in V_k, \tau \in \Gamma_k, \tag{2.5}$$