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THE DUAL MIXED METHOD FOR AN UNILATERAL PROBLEM *1)

Lie-heng Wang
(*LSEC, ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

In this paper, the dual mixed method for an unilateral problem, which is the simplified modelling of scalar function for the friction-free contact problem, is considered. The dual mixed problem is introduced, the existence and uniqeness of the solution of the problem are presented, and error bounds $O(h^{\frac{3}{4}})$ and $O(h^{\frac{3}{2}})$ are obtained for the dual mixed finite element approximations of Raviart-Thomas elements for $k = 0$ and $k = 1$ respectively.

Key words: Dual Mixed Method, Unilateral Problem.

1. Introduction

In the early work [2], the dual mixed finite element method for the unilateral problem with a lower order term was considered, in which the divergence constraint is naturally incorporated into the unilateral formulation (the details can be found in the Remark 2.1 in section 2). In this paper , we consider an unilateral problem in the absence of the lower order term and involving the complex boundary conditions, which can be considered as the simplified modelling of scalar function for the friction-free contact problem (c.f.[5], [6] and [7]).

Let Ω be a bounded domain in R^2, with boundary $\partial\Omega = \Gamma_D \cup \Gamma_F \cup \Gamma_C, \Gamma_D \cap \Gamma_F = \phi, \Gamma_F \cap \Gamma_C = \phi, \Sigma = \partial\Omega \setminus \Gamma_D$ and $\bar{\Gamma}_C \subset \Sigma$. For a given $f \in L^2(\Omega), t \in L^2(\Gamma_F), g \in H_{00}^{\frac{1}{2}}(\Gamma_C)$, we consider the following unilateral problem:

$$\left\{ \begin{array}{l} find \;\; u \in \mathbf{C}, \;\; such \;\; that \\ (\nabla u, \nabla(v-u)) \geq (f, v-u) + < t, v-u >_{\Gamma_F} \quad \forall \;\; v \in C, \end{array} \right. \tag{1.1}$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product, $< \cdot,\cdot >$ denotes the duality product, and

$$\mathbf{C} = \{v \in H^1_{\Gamma_D}(\Omega) : v \geq g \;\; on \;\; \Gamma_C\}, \tag{1.2}$$

and

$$H^1_{\Gamma_D}(\Omega) = \{v \in H^1(\Omega) : v = 0 \;\; on \;\; \Gamma_D\}. \tag{1.3}$$

It can be easily seen that the variational inequality (1.1) is equivalent to the following minimization problem:

$$\left\{ \begin{array}{l} find \;\; u \in \mathbf{C}, \;\; such \;\; that \\ F(u) = \min F(v) \;\; \forall \;\; v \in \mathbf{C}, \end{array} \right. \tag{1.4}$$

where

$$F(v) = \frac{1}{2}(\nabla v, \nabla v) - (f, v) - < t, v >_{\Gamma_F} . \tag{1.5}$$

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By the general variational calculus, problem (1.1) (or (1.4)) is equivalent to the following differential problem:

$$\begin{cases} -\Delta u = f \quad in \quad L^2(\Omega), \\ u = 0 \quad on \quad \Gamma_D \quad in \quad H^{\frac{1}{2}}(\Gamma_D), \partial_\nu u = t \quad on \quad \Gamma_F \quad in \quad H^{-\frac{1}{2}}(\Gamma_F), \\ u - g \geq 0 \quad on \quad \Gamma_C \quad in \quad H^{\frac{1}{2}}_{00}(\Gamma_C), \partial_\nu u \geq 0 \quad on \quad \Gamma_C, \\ < \partial_\nu u, u - g >_{\Gamma_C} = 0. \end{cases} \tag{1.6}$$

Here and what follows, the general notations of Sobolev spaces (c.f. [1], [8]) are used. And we introduce the following notations for sequence use:

$$H_{div}(\Omega) = \{\mathbf{q} \in (L^2(\Omega))^2 = \mathbf{L}^2(\Omega) : div\mathbf{q} \in L^2(\Omega)\}, \tag{1.7}$$

with norm

$$\|\mathbf{q}\|_{div} = \{\|\mathbf{q}\|^2_{0,\Omega} + \|div\mathbf{q}\|^2_{0,\Omega}\}^{\frac{1}{2}}. \tag{1.8}$$

The paper is organized as follows: In Sect.2 we derive a saddle-point problem with Lagrangian multiplier to relax the constraint: $div\mathbf{p} + f = 0$ in Ω, which is refered to as the dual mixed problem; And in Sect.3 the existence and uniqueness of the solution of the dual mixed problem are presented. Finally, in Sect.4, Raviart-Thomas finite element approximation ($k \geq 0$) to the dual mixed problem is considered, and the error bounds $O(h^{\frac{3}{4}})$ (for $k = 0$) and $O(h^{\frac{3}{2}})$ (for $k = 1$) are obtained respectively. Additionally, in Appendix we present the equivalence of $\|\mu\|_{H^{\frac{1}{2}}_{00}(\Gamma_C)}$ and $\inf_{v \in H^1(\Omega), \gamma v|_{\Gamma_C} = \mu} \|v\|_{1,\Omega} \quad \forall \quad \mu \in H^{\frac{1}{2}}_{00}(\Gamma_C)$, which is used in the context.

2. Derivation of the Dual Mixed Problem

Along the lines of [3], we now derive the dual mixed formulation for the unilateral problem (1.1).

Lemma 2.1. $\forall \quad v \in \mathbf{C}$

$$F(v) = \sup_{\mathbf{q} \in K}\{-\frac{1}{2}(\mathbf{q}, \mathbf{q}) + < q_\nu, g >_{\Gamma_C}\}, \tag{2.1}$$

where

$$K = \{\mathbf{q} \in Q : q_\nu \geq 0 \quad on \quad \Gamma_C\}, \tag{2.2}$$

$$Q = \{\mathbf{q} \in H_{div}(\Omega) : div\mathbf{q} + f = 0 \quad in \quad \Omega, q_\nu = t \quad on \quad \Gamma_F\}, \tag{2.3}$$

and ν denotes the outer unit normal vector on $\partial\Omega$, and $q_\nu = \mathbf{q} \cdot \nu$ the outer normal component of $\mathbf{q}$ on $\partial\Omega$.

Proof. $\forall \quad v \in \mathbf{C}, \mathbf{q} \in K$, by Green's integration formula and (1.5) we have

$$\begin{aligned} F(v) &= \tfrac{1}{2}(\nabla v, \nabla v) + (div\mathbf{q}, v) - < q_\nu, v >_{\Gamma_F} \\ &= \tfrac{1}{2}(\nabla v, \nabla v) - (\mathbf{q}, \nabla v) + < q_\nu, v >_{\Gamma_C} \\ &\geq \tfrac{1}{2}(\nabla v, \nabla v) - (\mathbf{q}, \nabla v) + < q_\nu, g >_{\Gamma_C} \\ &\geq -\tfrac{1}{2}(\mathbf{q}, \mathbf{q}) + < q_\nu, g >_{\Gamma_C}, \end{aligned} \tag{2.4}$$

and the equalities hold iff $\mathbf{q} = \nabla v \quad in \quad \Omega$ and $v = g \quad on \quad \Gamma_C$. Thus the lemma is proved.

From (2.1) and the problem (1.1), we have the following dual problem:

$$\begin{aligned} &\inf_{v \in \mathbf{C}} F(v) = \inf_{v \in \mathbf{C}} \sup_{\mathbf{q} \in K}\{-\tfrac{1}{2}(\mathbf{q}, \mathbf{q}) + < q_\nu, g >_{\Gamma_C}\} \\ &= -\inf_{\mathbf{q} \in K}\{\tfrac{1}{2}(\mathbf{q}, \mathbf{q}) - < q_\nu, g >_{\Gamma_C}\}, \end{aligned} \tag{2.5}$$

which is equivalent to the following problem:

$$\begin{cases} find \quad \mathbf{p} \in K, \quad such \quad that \\ (\mathbf{p}, \mathbf{q} - \mathbf{p}) \geq < g, q_\nu - p_\nu >_{\Gamma_C} \quad \forall \quad \mathbf{q} \in K. \end{cases} \tag{2.6}$$