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GENERALIZED NEKRASOV MATRICES AND APPLICATIONS*

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Abstract

In this paper, the concept of generalized Nekrasov matrices is introduced, some properties of these matrices are discussed, obtained equivalent representation of generalized diagonally dominant matrices.

Key words: Nekrasov matrix, Generalized Nekrasov matrix, Generalized diagonally dominant matrix.

1. Introduction

In matrix computations, the investigation of Nekrasov matrices is both important in theory and applications. The concept of generalized Nekrasov matrices is introduced in this paper. Let the set of complex (real) $n \times n$ matrices be $C^{n\times n}(R^{n\times n})$, and denoted:

$$r_i(A) = \sum_{j\neq i} |a_{ij}|, \qquad \forall\, i \in <n>= \{1, 2, \ldots, n\}$$

$$R_1(A) = r_1(A), \qquad R_i(A) = \sum_{j<i} |a_{ij}| \frac{R_j(A)}{|a_{jj}|} + \sum_{j>i} |a_{ij}|, 2 \leq i \leq n$$

$$\alpha(A) = \{i \in <n> \mid |a_{ii}| = R_i(A)\}, \quad \beta(A) = \{i \in <n> \mid |a_{ii}| = r_i(A)\}$$

$$J_\alpha(A) = \{i \in <n> \mid |a_{ii}| > R_i(A)\}, \quad J_\beta(A) = \{i \in <n> \mid |a_{ii}| > r_i(A)\}$$

$\forall \alpha = \{i_1 < i_2 < \ldots < i_k\} \subseteq <n>$, denote $\alpha' = <n> \setminus \alpha$, $A[\alpha]$ is the principal submatrix whose rows and columns are indexed by α, and $A[\alpha'] = A(\alpha)$. Denote the directed graph of A by $\Gamma(A)$, the sets $V(A)$ and $E(A)$ are called the vertex set and arc set, respectively.

Definition 1.1. Suppose $A = (a_{ij}) \in C^{n\times n}$ satisfies

$$|a_{ii}| \geq R_i(A), \qquad \forall\, i \in <n> \tag{1}$$

then A is called the weak Nekrasov matrix and denote $A \in N_0$; if all inequalities in (1) are strict, then A is called the Nekrasov matrix and denote $A \in N$; if there exists a permutation matrix P such $PAP^T \in N$, then A is called the quasi-Nekrasov matrix and denote $A \in \widetilde{N}$; if there exists a positive diagonal matrix X such $AX \in N$, then A is called the generalized Nekrasov matrix and denote $A \in N^*$; if $<n>= \alpha(A) \cup J_\alpha(A)$, $J_\alpha(A) \neq \phi$ and for any $i \in \alpha(A)$ there exists a path in $\Gamma(A) : i \to i_1 \to \ldots \to i_p \to j$ such $j \in J_\alpha(A)$, then A is called the Nekrasov matrix with nonzero element chain and denote $A \in CN$.

Definition 1.2. Suppose $A = (a_{ij}) \in C^{n\times n}$ satisfies

$$|a_{ii}| \geq r_i(A), \qquad \forall\, i \in <n> \tag{2}$$

then A is called the diagonally dominant matrix and denote $A \in D_0$; if all inequalities in (2) are strict, then A is called strictly diagonally dominant matrix and denote $A \in D$; if $A \in D_0$, $J_\beta(A) \neq \phi$, and for any $i \in \beta(A)$ there exists a path in $\Gamma(A) : i \to i_1 \to \ldots \to i_p \to j$ such $j \in J_\beta(A)$, then A is called the diagonally dominant matrix with nonzero element chain and

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denote $A \in CD$; if there exists a positive diagonal matrix X such $AX \in D$,then A is called the generalized strictly diagonally dominant matrix and denote $A \in D^*$.

Clearly if $a_{ii} \neq 0 (i \in < n >)$ then $D_0 \subset N_0$, $D \subset N$.

2. Results

Lemma 2.1. Let $A = (a_{ij}) \in C^{n \times n} \cap N$, then there exist a positive diagonal matrix X and a matrix $B \in CD$ such $A = BX$.

Proof. Without loss of generality assume $r_i(A) > 0$ for $\forall i \in < n >$, (if not for example $r_1(A) = 0$, then only discuss $A(1) \in N$). Denote

$$X_1 = diag(r_1(A)/|a_{11}|, 1, \dots, 1), \qquad AX_1 = A^{(1)} = (a_{ij}^{(1)})$$

$$X_2 = diag(1, r_2(A^{(1)})/|a_{22}^{(1)}|, 1, \dots, 1), \qquad A^{(1)}X_2 = A^{(2)} = (a_{ij}^{(2)})$$

$$\dots \qquad \dots$$

$$X_{n-1} = diag(1, \dots, 1, r_{n-1}(A^{(n-2)})/|a_{n-1n-1}^{(n-2)}|, 1), \quad A^{(n-2)}X_{n-1} = A^{(n-1)} = (a_{ij}^{(n-1)})$$

Moreover denote $X^{-1} = X_1 X_2 \dots X_{n-1} = diag(d_1, d_2, \dots, d_{n-1}, 1)$. Then

$$d_i = r_i(A^{(i-1)})/|a_{ii}^{(i-1)}| = R(A)/|a_{ii}| < 1, \quad 1 \leq i \leq n-1$$

where $A^{(0)} = A$. Therefore $A^{(n-1)} = AX^{-1} = (a_{ij}^{(n-1)})$ satisfies

$$|a_{11}^{(n-1)}| = r_1(A) = R_1(A) \geq r_1(A^{(n-1)})$$

$$|a_{22}^{(n-1)}| = r_2(A^{(1)}) = R_2(A) \geq r_2(A^{(n-1)})$$

$$\dots$$

$$|a_{n-1n-1}^{(n-1)}| = r_{n-1}(A^{(n-2)}) = R_{n-1}(A) \geq r_{n-1}(A^{(n-1)})$$

$$|a_{nn}^{(n-1)}| = |a_{nn}| > R_n(A) = r_n(A^{(n-1)})$$

so $A^{(n-1)} \in D_0$ and $\beta(A^{(n-1)}) \neq < n >$. For first row of $A^{(n-1)}$, since $r_1(A^{(n-1)}) = \sum_{n>j>1} |a_{1j}| \ d_j + |a_{1n}|$, if there exists $j_0 \in < n-1 > \backslash \{1\}$ such $a_{1j_0} \neq 0$, then $r_1(A^{(n-1)}) < r_1(A) = |a_{11}^{(n-1)}|$, hence $1 \notin \beta(A^{(n-1)})$. If $a_{1j} = 0, \forall j \in < n-1 > \backslash \{1\}$, then $a_{1n} \neq 0$. Since $|a_{11}^{(n-1)}| = r_1(A) = |a_{1n}| > 0$ and $n \notin \beta(A^{(n-1)})$, so vertex $1 \in \beta(A^{(n-1)})$ and vertex n are adjoin.

For second row of $A^{(n-1)}$, since $r_2(A^{(n-1)}) = \sum_{j \neq 2} |a_{2j}| d_j + |a_{2n}|$, if there exists $j_0 \in < n-1 > \backslash \{1, 2\}$ such $a_{2j_0} \neq 0$, then $r_2(A^{(n-1)}) < R_2(A) = d_1|a_{21}| + \sum_{j>2} |a_{2j}| = |a_{22}^{(n-1)}|$, hence $2 \in \beta(A^{(n-1)})$. If $a_{2j} = 0, \forall j \in < n-1 > \backslash \{1, 2\}$, then $r_2(A^{(n-1)}) = d_1|a_{21}| + a_{2n} = R_2(A) = |a_{22}^{(n-1)}|$, i.e. $2 \in \beta(A^{(n-1)})$. If $a_{21} \neq 0$, then vertex 2 and vertex 1 are adjoin, and vertex 1 either satisfies $1 \notin \beta(A^{(n-1)})$ or $1 \in \beta(A^{(n-1)})$ but is adjoin with vertex $n \notin \beta(A^{(n-1)})$, thus vertex 2 is adjoin with vertex in set $< n > \backslash \beta(A^{(n-1)})$. If $a_{21} = 0$, then must be $a_{2n} \neq 0$, hence vertex 2 and vertex $n \notin \beta(A^{(n-1)})$ are adjoin. Therefore if vertex $2 \in \beta(A^{(n-1)})$, then there exists a path in $\Gamma(A)$ such vertex 2 and some vertex of set $< n > \backslash \beta(A^{(n-1)})$ are adjoin.

In general, for any $i \in < n-1 > \backslash \{1\}$, by above deduction we have that vertices $1, 2, \dots, i-1$ either not belong in $\beta(A^{(n-1)})$ or belong in $\beta(A^{(n-1)})$ but there exists a path in $\Gamma(A)$ such these vertices are adjoin with vertices of set $< n > \backslash \beta(A^{(n-1)})$. For i-th row of $A^{(n-1)}$, since $r_i(A^{(n-1)}) = \sum_{j \neq i}^{n-1} |a_{ij}| d_j + |a_{in}|$, if there exists $i < j_0 \leq n-1$ such $a_{ij_0} \neq 0$, then $r_i(A^{(n-1)}) < R_i(A) = \sum_{j<i} |a_{ij}| d_j + \sum_{j>i} |a_{ij}| = |a_{ii}^{(n-1)}|$, hence $i \notin \beta(A^{(n-1)})$. If $a_{ij} = 0$