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AN EFFICIENT METHOD FOR COMPUTING HYPERBOLIC SYSTEMS WITH GEOMETRICAL SOURCE TERMS HAVING CONCENTRATIONS *1)

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Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday

Abstract

We propose a simple numerical method for calculating both unsteady and steady state solution of hyperbolic system with geometrical source terms having concentrations. Physical problems under consideration include the shallow water equations with topography, and the quasi one-dimensional nozzle flows. We use the interface value, rather than the cell-averages, for the source terms, which results in a well-balanced scheme that can capture the steady state solution with a remarkable accuracy. This method approximates the source terms via the numerical fluxes produced by an (approximate) Riemann solver for the homogeneous hyperbolic systems with slight additional computation complexity using Newton's iterations and numerical integrations. This method solves well the sub- or super-critical flows, and with a transonic fix, also handles well the transonic flows over the concentration. Numerical examples provide strong evidence on the effectiveness of this new method for both unsteady and steady state calculations.

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1. Introduction

Hyperbolic systems with geometric source terms arise in many physical applications, including the shallow water equations with bottom topography and the quasi one-dimensional nozzle flow equations with variable cross-sectional area. When the source terms in the system have concentrations, corresponding to a δ function in the source, the usual numerical method for source term approximation may give poor approximations to the steady state equations due to the first order numerical viscosity used at discontinuities [12]. A well accepted strategy for such problems is to design so called *well-balanced* scheme that balances the numerical flux with the source term such that the steady state solution is captured numerical with exactly or with at least a second order accuracy. Many well-balanced schemes have been proposed by many authors in recent years, including well-balanced scheme based on non-conservative product [12]

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and its extensions [13], [11], [5], [6], [9], [10], LeVeque's quasi-steady scheme [18], kinetic schemes [4], [19], [2], [25], relaxation schemes [20], central schemes [15]. Nonlinear extension of Roe's linear idea [23] was made in [3], [24], [14]. Most of these methods require the modification of the numerical flux.

The interface method of Jin [14] uses the numerical flux for the homogeneous hyperbolic systems in the source term, and was shown in [14] that for smooth solutions, it captures the steady state at *cell interfaces* with a second order accuracy, thus is well-balanced. Designed for Godunov [8] and Roe [22] type schemes, this method has the advantage that it does not require any modification of the numerical flux for the convection term. By using the numerical flux directly in the source term it needs almost no additional computation complexity to deal with source term.

In this paper we derive a new set of well-balanced scheme that can be viewed as an improvement of Jin's interface method. It is a hybridization of the conventional cell average method with a improved interface type method at concentration points. The main idea is based on finite volume approximation of hyperbolic systems, with a more accurate approximation of the volume average of the source term. While Jin's interface method can be viewed as the trapezoidal approximation of this source average, we found that more accurate approximation of the source average significantly improves the approximation of the steady state solutions. This involves more accurate numerical integrations and Newton's iterations, but the added computational compexity, compared to the interface method, is small. This new method can accurately capture both unsteady and steady state solutions. Moreover, with a simply fix, it is capable of handling the transonic flows at source concentrations.

In section 2, 3 and 4 we introduce our method for the shallow water equations, isothermal and non-isothermal nozzle flow equations respectively. The property of preservation of steady state equations is shown. Numerical examples show that the new method gives satisfactory unsteady and steady state solutions.

In the sequel we will use $x_{j+1/2}$ to denote the grid point, $\Delta x = x_{j+1/2} - x_{j-1/2}$ the mesh size, $w_{j+1/2} = w(x_{j+1/2})$ the interface value of a general quantity w, and $w_j = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} w(x)\,dx$ be the cell average of w over the cell $[x_{j-1/2}, x_{j+1/2}]$.

2. The Shallow Water Equations

Consider the one-dimensional shallow water equations with topography

$$h_t + (hv)_x = 0, \tag{2.1}$$

$$(hv)_t + (hv^2 + \frac{1}{2}gh^2)_x = -ghB_x\,, \tag{2.2}$$

where h is the depth of the water, u is the mean velocity, g is the gravitational constant, and $B(x)$ is the bottom topograph. The steady state solutions satisfy

$$hv = C_1, \tag{2.3}$$

$$\frac{1}{2}v^2 + gh + gB = C_2. \tag{2.4}$$

These steady state conditions are satisfied not only on smooth part of the solution, but also across a bottom discontinuity [1]. A numerical method is called *well-balanced* [12] if it satisfies the steady state conditions (2.3), (2.4) exactly or with at least second order accuracy even when the bottom contains discontinuities. In this section we design a well-balanced scheme for (2.1), (2.2) which can preserve these steady states even at cells containing discontinuity of $B(x)$.

2.1. A hybrid method