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ERROR ANALYSIS OF AN IMMERSED FINITE ELEMENT METHOD FOR EULER-BERNOULLI BEAM INTERFACE PROBLEMS

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Abstract. This article presents an error analysis of a Hermite cubic immersed finite element (IFE) method for solving interface problems of the differential equation modeling a Euler-Bernoulli beam made up of multiple materials together with suitable jump conditions at material interfaces. The analysis consists of three essential groups. The first group is about IFE functions including bounds for the IFE shape functions and inverse inequalities. The second group is about error bounds for IFE interpolation derived with a multi-point Taylor expansion technique. The last group, and perhaps the most important group, is for proving the optimal convergence of the IFE solution generated by the usual Galerkin scheme based on the Hermite cubic IFE space considered in this article.

Key words. Error estimation, interface problem, interface independent mesh, Euler-Bernoulli beam, Hermite cubic finite element, multi-point Taylor expansion, optimal convergence.

1. Introduction

This article presents an error analysis of an immersed finite element (IFE) method that can solve interface problems with interface independent meshes for the differential equation modeling a Euler-Bernoulli beam formed with multiple materials. Without loss of generality, we consider an Euler-Bernoulli beam of length 1 formed with two materials such that the flexural rigidity of this beam is a piecewise positive constant function

$$\beta(x) = \begin{cases} \beta^-, & x \in (0, \alpha), \\ \beta^+, & x \in (\alpha, 1), \end{cases} \tag{1}$$

where $\alpha \in \Omega = (0, 1)$ is the interface of the two materials whose flexural rigidity are β^- and β^+, respectively. It is well known, see such as [8], that the deflection $u(x)$ of an Euler-Bernoulli beam at a point $x \in \Omega$ corresponding to a given load $f(x)$ satisfies the following so called beam equation:

$$(\beta u''(x))'' = f(x), \; x \in \Omega^- \cup \Omega^+, \tag{2a}$$

where $\Omega^- = (0, \alpha)$ and $\Omega^+ = (\alpha, 1)$. Boundary conditions are required to uniquely determine a solution from (2a), and we will consider the clamped boundary conditions at the ends of the beam:

$$u(0) = w_0, \; u'(0) = w_1, \; u(1) = w_3, \; u'(1) = w_4, \tag{2b}$$

other boundary configurations, such as the popular cantilevered boundary conditions can also be considered. At the material interface α, the deflection $u(x)$ is

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required to satisfy the following rigid connection conditions:

$$
\begin{cases} u(\alpha-) = u(\alpha+), & \text{(continuity in the deflection)},\\ u'(\alpha-) = u'(\alpha+), & \text{(continuity in the bending angle)},\\ \beta^- u''(\alpha-) = \beta^+ u''(\alpha+), & \text{(continuity of the bending moment)},\\ \beta^- u'''(\alpha-) = \beta^+ u'''(\alpha+), & \text{(continuity of the shear)}. \end{cases} \tag{2c}
$$

Specifically, the interface problem to be discussed for the Euler-Bernoulli beam equation is to find the deflection function $u(x)$ such that all the equations in (2) are satisfied.

Immersed finite element methods are a group of finite element methods that can solve interface problems with meshes independent of material interfaces where the coefficients in the differential equations are discontinuous. Instead of simple polynomials, IFE methods use Hsieh-Clough-Tocher type macro finite element functions [5, 7] which are piecewise polynomials on each interface element constructed according to interface jump conditions. In other words, IFE methods employ functions that already partially solve interface problems locally on interface elements rather than generic polynomials having nothing to do with a specific interface problem to be solved. Of course, on all the non-interface elements which are not cut by material interfaces, i.e., which are occupied by one of the materials, IFE methods just use polynomials of choice as usual finite element methods. In a certain sense, the fundamental concept for IFE methods has a trace in the generalized finite element method proposed in the 1980s [3, 4] which employs shape functions on an element constructed by locally solving the problem in that element, even though these shape functions may be non-polynomials, they possess key features of the solution to a boundary value problem.

An IFE method was introduced in [13] for solving an interface problem of a two point boundary value problem. Afterwards, IFE methods have been developed for solving elliptic interface problems [2, 10, 12, 14, 16, 19, 25, 26], some time dependent interface problems [11, 22, 18, 20], interface problems for linear elasticity [9, 15, 21, 23], Stokes interface problems [1], and interface problems for some 4-th order differential equations [17]. In particular, a Hermite cubic IFE space has been developed in [27] for solving interface problems of the differential equations modeling a Euler-Bernoulli beam. Fundamental features of this IFE space are presented in [27], such as the unisolvence for the Hermite cubic IFE shape functions on every interface element, the consistence of the IFE shape functions with their corresponding Hermite cubic finite element shape functions. More importantly, it is reported by numerical examples in [27] that this IFE space can be used to produce an optimally convergent approximation to the interface problem of the beam equation. Our goal in this article is to carry out an error analysis for the Hermite cubic IFE space developed in [27] and to theoretically prove that the interpolation in this IFE space has the optimal approximation capability, and the numerical solution produced in this IFE space by the usual Galerkin finite element scheme can converge optimally to the exact solution to the beam interface problem.

In the error analysis to be presented later in this article, we will use the standard Sobolev space on an open subinterval D of Ω: for every integer $m \geq 0$,

$$
H^m(D) = \{u \mid u^{(j)} \in L^2(D), 0 \leq j \leq m\} \tag{3}
$$