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An Artificial Boundary Condition for a Class of Quasi-Newtonian Stokes Flows

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Abstract. An artificial boundary condition method, derived in terms of infinite Fourier series, is applied to solve a class of quasi-Newtonian Stokes flows. Based on the natural boundary reduction involving an artificial condition on the artificial boundary, the coupled variational problem and its numerical solution are obtained. The unique solvability of the continuous and discrete formulations are discussed, and the error analysis for the problem is also considered. Finally, an *a posteriori* error estimate for the corresponding problem is provided.

AMS subject classifications: 65H05, 65B99
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1. Introduction

Interior and exterior nonlinear transmission problems often arise in elasticity [4, 10] and fluid mechanics [11]. The coupled finite element method (FEM) and artificial boundary condition method [8, 9], often called the natural boundary element method [5, 20] or DtN method [6, 13], can be one of the most effective methods to solve exterior nonlinear-linear transmission problems — cf. [2–4, 7, 10, 12, 15, 19] and references therein for more details.

There are several investigations for incompressible materials on bounded domains using finite or mixed finite element methods (e.g. [1, 2, 14, 16–18]), and some on unbounded domains (e.g. [3, 10, 12]), using coupling methods. The purpose of this work is to investigate a class of quasi-Newtonian Stokes flows where the kinematic viscosity is a nonlinear monotone function of the fluid velocity gradient in the plane.

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We consider the following configuration. Let Ω_0 be a bounded and simply connected domain in $\mathbb{R}^2$ with a sufficiently smooth boundary $\partial\Omega_0 = \Gamma_0$; and let Ω_1 be the annular region with the boundaries Γ_0 and Γ_1, where Γ_1 is another sufficiently smooth boundary with an interior region that contains Ω_0, and $\Omega^c = \mathbb{R}^2 \backslash (\overline{\Omega}_0 \cup \overline{\Omega}_1)$. In what follows, $\mathbb{R}^{2\times 2}$ denotes the space of square matrices of order 2 with real entries, $\mathbf{I} \triangleq (\delta_{ij})$ is the identity matrix of $\mathbb{R}^{2\times 2}$, and given $\boldsymbol{\tau} \triangleq (\tau_{ij})$, $\boldsymbol{\sigma} \triangleq (\boldsymbol{\sigma}_{ij}) \in \mathbb{R}^{2\times 2}$ we write

$$\mathrm{tr}(\boldsymbol{\tau}) \triangleq \sum_{i=1}^{2} \tau_{ii}\,, \qquad \boldsymbol{\sigma} : \boldsymbol{\tau} \triangleq \sum_{i,j=1}^{2} \boldsymbol{\sigma}_{ij}\tau_{ij}\,,$$

where $\boldsymbol{\sigma}(\boldsymbol{u},p) \triangleq (\boldsymbol{\sigma}_{ij}(\boldsymbol{u},p)) \in \mathbb{R}^{2\times 2}$ is the Cauchy stress tensor and $\boldsymbol{\varepsilon}(\boldsymbol{u}) \triangleq (\varepsilon_{ij}(\boldsymbol{u})) \in \mathbb{R}^{2\times 2}$ denotes the strain tensor of small deformations with representation $\varepsilon_{ij}(\boldsymbol{u}) \triangleq \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. The constitutive equation in Ω_1 is then given by

$$\boldsymbol{\sigma}(\boldsymbol{u},p) = \psi(|\nabla \boldsymbol{u}|)\nabla \boldsymbol{u} - p\mathbf{I}\,, \tag{1.1}$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is the nonlinear kinematic viscosity function of the fluid that satisfies the Carreau law for viscoelastic flows $\psi(t) \triangleq \kappa_0 + \kappa_1(1+t^2)^{(\beta-2)/2}$, $\forall\ t, \kappa_0 \in \mathbb{R}^+$, $\beta \in [1,2]$ — cf. [18]. In passing, we note that Eq. (1.1) reduces to the usual linear model when $\beta = 2$, and that the extension of our approach to kinematic viscosity functions not satisfying Eq. (1.2) or Eq. (1.3) below (which includes the Carreau law with $\kappa_0 = 0$ or $\beta > 2$) will be reported elsewhere.

Let $\psi_{ij} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be the mapping defined by $\psi_{ij}(\boldsymbol{r}) \triangleq \psi(|\boldsymbol{r}|)r_{ij}$ for all $\boldsymbol{r} \triangleq (r_{ij}) \in \mathbb{R}^{2\times 2}$ with $i,j \in \{1,2\}$, and let the mapping $\boldsymbol{\Phi} : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be defined by $\boldsymbol{\Phi}(\boldsymbol{r}) \triangleq (\psi_{ij}(\boldsymbol{r}))$ for all $\boldsymbol{r} \in \mathbb{R}^{2\times 2}$. Then it is easy to check that ψ is of class C^1, and there exists $C_1, C_2 > 0$ such that for all $\boldsymbol{r} \triangleq (r_{ij})$, $\boldsymbol{s} \triangleq (s_{ij}) \in \mathbb{R}^{2\times 2}$ we have

$$|\psi_{ij}(\boldsymbol{r})| \le C_1 \|\boldsymbol{r}\|_{\mathbb{R}^{2\times 2}}\,, \quad \left|\frac{\partial}{\partial r_{kl}}\psi_{ij}(\boldsymbol{r})\right| \le C_1\,, \quad \forall\ i,j,k,l \in \{1,2\} \tag{1.2}$$

and

$$\sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial r_{kl}}\psi_{ij}(\boldsymbol{r}) s_{ij} s_{kl} \ge C_2 \|\boldsymbol{s}\|^2_{\mathbb{R}^{2\times 2}}\,. \tag{1.3}$$

Furthermore, Eq. (1.1) can be rewritten as

$$\boldsymbol{\sigma}(\boldsymbol{u},p) = \boldsymbol{\Phi}(\nabla \boldsymbol{u}) - p\mathbf{I}\,; \tag{1.4}$$

and for a linear elastic material in Ω^c this reduces to

$$\boldsymbol{\sigma}(\boldsymbol{u},p) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\mathbf{I}\,, \tag{1.5}$$

where μ is the familiar Lamé constant.

We now take $[H^1(\Omega_1)]^2 \cap [H^1_{loc}(\Omega^c)]^2$ as the space of functions $\boldsymbol{v} \triangleq \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ defined in $\Omega_1 \cup \Gamma_1 \cup \Omega^c$ such that $\boldsymbol{v}|_{\Omega_1} \in [H^1(\Omega_1)]^2$ and $\boldsymbol{v}|_{\Omega^c} \in [H^1_{loc}(\Omega^c)]^2$. For given $\boldsymbol{f} \in [L^2(\Omega_1)]^2$,