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NEW ERROR ESTIMATES FOR LINEAR TRIANGLE FINITE ELEMENTS IN THE STEKLOV EIGENVALUE PROBLEM*

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Abstract

This paper is concerned with the finite elements approximation for the Steklov eigenvalue problem on concave polygonal domain. We make full use of the regularity estimate and the characteristic of edge average interpolation operator of nonconforming Crouzeix-Raviart element, and prove a new and optimal error estimate in $\|\cdot\|_{0,\partial\Omega}$ for the eigenfunction of linear conforming finite element and the nonconforming Crouzeix-Raviart element. Finally, we present some numerical results to support the theoretical analysis.

Mathematics subject classification: 65N25, 65N30.
Key words: Steklov eigenvalue problem, Concave polygonal domain, Linear conforming finite element, Nonconforming Crouzeix-Raviart element, Error estimates.

1. Introduction

Steklov eigenvalue problems have important physical background and many applications. For instance, they appear in the analysis of stability of mechanical oscillators immersed in a viscous fluid (see [12] and the references therein), in the study of surface waves (see [7]), in the study of the vibration modes of a structure in contact with an incompressible fluid (see [6]) and in the analysis of the antiplane shearing on a system of collinear faults under slip-dependent friction law (see [10]). Thus the numerical methods for solving these problems have attracted more and more scholars' attention. Till now, systematical and profound studies on the conforming finite elements approximation for Steklov eigenvalue problems have been made on polygonal domain such as [2–4, 6, 9, 15, 16, 21, 22]). Recently, the nonconforming finite elements for Steklov problems have also been considered, see, e.g., [1, 8, 17–19, 23]. The aim of this paper is to discuss the error estimates of linear triangle finite elements, including the linear conforming finite element and the nonconforming Crouzeix-Raviart element(C-R element), approximation for Steklov eigenvalue problems with variable coefficients on concave polygonal domain.

We consider the following Steklov eigenvalue problem

$$-\mathrm{div}(\alpha\nabla u) + \beta u = 0 \quad \text{in } \Omega, \qquad \alpha\frac{\partial u}{\partial n} = \lambda u \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset R^2$ is a polygonal domain with ω being the largest inner angle of Ω, and $\frac{\partial u}{\partial n}$ is the outward normal derivative.

Having in mind that $H^s(\Omega)$ denotes the Sobolev space with real order s on Ω, $\|\cdot\|_s$ is the norm on $H^s(\Omega)$ and $H^0(\Omega) = L_2(\Omega)$, and $H^s(\partial\Omega)$ denotes the Sobolev space with real order s on $\partial\Omega$ with the norm $\|\cdot\|_{s,\partial\Omega}$.

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Suppose that the coefficients $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are bounded by above and below by positive constants. We assume that $\alpha \in C^1(\bar{\Omega})$.

The weak form of (1.1) is given by: Find $\lambda \in R$, $u \in H^1(\Omega), \|u\|_{0,\partial\Omega} = 1$, such that

$$a(u,v) = \lambda b(u,v), \quad \forall v \in H^1(\Omega), \tag{1.2}$$

where

$$a(u,v) = \int_\Omega (\alpha \nabla u \cdot \nabla v + \beta uv)dx, \;\; b(u,v) = \int_{\partial\Omega} uvds.$$

It is easy to know that $a(\cdot,\cdot)$ is a symmetric, continuous and $H^1(\Omega)$-elliptic bilinear form on $H^1(\Omega) \times H^1(\Omega)$.

In the existing literatures, the error estimate of linear triangle elements eigenfunction, including conforming element and nonconforming Crouzeix-Raviart element (hereafter termed C-R element for simplicity), in $\|\cdot\|_{0,\partial\Omega}$ is all $O(h^{r+\frac{r}{2}})$ where r is the regularity exponent of the eigenfunction (see Lemma 2.1). It is obvious that this estimate is not optimal since it doesn't achieve the order of interpolation error. In this paper, we improve this estimate when eigenfunctions are singular (i.e., $r < 1$) and prove that in this case the error estimate of linear triangle elements eigenfunction can achieve $O(h^{r+\frac{1}{2}})$. Comparing the proof arguments of existing estimates (see, e.g., [3, 9, 17, 23]), we make full use of the regularity estimate and the characteristic of edge average interpolation operator of C-R element, especially in the analysis for conforming finite elements, and obtain the improved error estimates (2.25) and (3.6) which are optimal.

Throughout this paper, C denotes a positive constant independent of h, which may not be the same constant in different places.

2. The Nonconforming C-R Element Approximation for the Steklov Eigenvalue Problem

Consider the source problem (2.1) associated with (1.1): Find $w \in H^1(\Omega)$, such that

$$a(w,v) = b(f,v), \quad \forall v \in H^1(\Omega). \tag{2.1}$$

As for the source problem (2.1), from Proposition 4.1 in [1] and Lemma 2.1 in [19] we have the following regularity results.

Lemma 2.1. *If $f \in L_2(\partial\Omega)$, then $w \in H^{1+\frac{r}{2}}(\Omega)$ and*

$$\|w\|_{1+\frac{r}{2}} \le C_\Omega \|f\|_{0,\partial\Omega}; \tag{2.2}$$

if $f \in H^{\frac{1}{2}}(\partial\Omega)$, then $w \in H^{1+r}(\Omega)$ and

$$\|w\|_{1+r} \le C_\Omega \|f\|_{\frac{1}{2},\partial\Omega}; \tag{2.3}$$

if $f \in H^{\varepsilon}(\partial\Omega)$, $\varepsilon \in (0, r-1/2)$, then $w \in H^{\frac{3}{2}+\varepsilon}(\Omega)$ and

$$\|w\|_{\frac{3}{2}+\varepsilon} \le C_\Omega \|f\|_{\varepsilon,\partial\Omega}. \tag{2.4}$$

Here $r = 1$ when $\omega < \pi$, and $r < \frac{\pi}{\omega}$ which can be arbitrarily close to $\frac{\pi}{\omega}$ when $\omega > \pi$, and C_Ω is a priori constant.