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ENSEMBLE TIMESTEPPING ALGORITHMS FOR NATURAL CONVECTION

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Dedicated to Professor William J. Layton on the occasion of his 60^{th} birthday

Abstract. This paper presents two algorithms for calculating an ensemble of solutions to laminar natural convection problems. The ensemble average is the most likely temperature distribution and its variance gives an estimate of prediction reliability. Solutions are calculated by solving two coupled linear systems, each involving a shared coefficient matrix, for multiple right-hand sides at each timestep. Storage requirements and computational costs to solve the system are thereby reduced. Stability and convergence of the method are proven under a timestep condition involving fluctuations. A series of numerical tests, including predictability horizons, are provided which confirm the theoretical analyses and illustrate uses of ensemble simulations.

Key words. Natural convection, Ensemble calculation, Uncertainty quantification, Finite element method

1. Introduction

Ensemble calculations are essential in predictions of the most likely outcome of systems with uncertain data, e.g., weather forecasting [13], ocean modeling [15], turbulence [12], etc. Ensemble simulations classically involve J sequential, fine mesh runs or J parallel, coarse mesh runs of a given code. This leads to a competition between ensemble size and mesh density. We develop linearly implicit timestepping methods with shared coefficient matrices to address this issue. For such methods, it is more efficient in both storage and solution time to solve J linear systems with a shared coefficient matrix than with J different matrices.

Prediction of thermal profiles is essential in many applications [1, 8, 17, 18]. Herein, we extend [6] from isothermal flows to temperature dependent natural convection. We consider two natural convection problems enclosed in mediums with: **non-zero wall thickness** [3] and **zero wall thickness**; Figure 1 illustrates a typical setup. The latter problem is often utilized as a thin wall approximation.

Consider the **Thick wall problem**. Let $\Omega_f \subset \Omega$ be polyhedral domains in $\mathbb{R}^d(d = 2, 3)$ with boundaries $\partial\Omega_f$ and $\partial\Omega$, respectively, such that dist$(\partial\Omega_f,\partial\Omega)$ > 0. The boundary $\partial\Omega$ is partitioned such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $|\Gamma_1| > 0$. Given $u(x, 0; \omega_j) = u^0(x; \omega_j)$ and $T(x, 0; \omega_j) = T^0(x; \omega_j)$ for $j = 1, 2, ..., J$, let $u(x, t; \omega_j) : \Omega \times (0, t^*] \to \mathbb{R}^d$, $p(x, t; \omega_j) : \Omega \times (0, t^*] \to \mathbb{R}$, and

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$T(x, t; \omega_j) : \Omega \times (0, t^*] \to \mathbb{R}$ satisfy

$$u_t + u \cdot \nabla u - Pr\Delta u + \nabla p = PrRa\gamma T + f \;\; in \; \Omega_f, \tag{1}$$

$$\nabla \cdot u = 0 \;\; in \; \Omega_f, \tag{2}$$

$$T_t + u \cdot \nabla T - \nabla \cdot (\kappa \nabla T) = g \;\; in \; \Omega, \tag{3}$$

$$u = 0 \;\; on \; \partial\Omega_f, \;\; u = 0 \;\; in \; \Omega - \Omega_f, \;\; T = 0 \;\; on \; \Gamma_1 \;\; and \;\; n \cdot \nabla T = 0 \;\; on \; \Gamma_2. \tag{4}$$

Here n denotes the usual outward normal, γ denotes the unit vector in the direction of gravity, Pr is the Prandtl number, Ra is the Rayleigh number, and $\kappa = \kappa_f$ in Ω_f and $\kappa = \kappa_s$ in $\Omega - \Omega_f$ is the thermal conductivity of the fluid or solid medium. Further, f and g are the body force and heat source, respectively.

Let $< u >^n := \frac{1}{J} \sum_{j=1}^{J} u^n$ and $u'^n = u^n - < u >^n$. To present the idea, suppress the spatial discretization for the moment. We apply an implicit-explicit time-discretization to the system (1) - (4), while keeping the coefficient matrix independent of the ensemble members. This leads to the following timestepping method:

$$\frac{u^{n+1} - u^n}{\Delta t} + < u >^n \cdot \nabla u^{n+1} + u'^n \cdot \nabla u^n$$
$$- Pr \triangle u^{n+1} + \nabla p^{n+1} = PrRa\gamma T^{n+1} + f^{n+1}, \tag{5}$$

$$\nabla \cdot u^{n+1} = 0, \tag{6}$$

$$\frac{T^{n+1} - T^n}{\Delta t} + < u >^n \cdot \nabla T^{n+1} + u'^n \cdot \nabla T^n - \kappa \Delta T^{n+1} = g^{n+1}. \tag{7}$$

Consider the **Thin wall problem**. The main difference is a "u_1" term on the r.h.s of the temperature equation (10) absent in (3). This apparently small difference in the model produces a significant difference in the stability of the approximate solution. In particular, a discrete Gronwall inequality is used which allows for the loss of long-time stability; see Section 4 below. Consider:

$$u_t + u \cdot \nabla u - Pr\Delta u + \nabla p = PrRa\gamma T + f \;\; in \; \Omega, \tag{8}$$

$$\nabla \cdot u = 0 \;\; in \; \Omega, \tag{9}$$

$$T_t + u \cdot \nabla T - \nabla \cdot (\kappa \nabla T) = u_1 + g \;\; in \; \Omega, \tag{10}$$

$$u = 0 \;\; on \; \partial\Omega, \quad T = 0 \;\; on \; \Gamma_1, \quad n \cdot \nabla T = 0 \;\; on \; \Gamma_2, \tag{11}$$

where u_1 is the first component of the velocity. If we again momentarily disregard the spatial discretization, our timestepping method can be written as:

$$\frac{u^{n+1} - u^n}{\Delta t} + < u >^n \cdot \nabla u^{n+1} + u'^n \cdot \nabla u^n - Pr \triangle u^{n+1} + \nabla p^{n+1} = PrRa\gamma T^n + f^{n+1}, \tag{12}$$

$$\nabla \cdot u^{n+1} = 0, \tag{13}$$

$$\frac{T^{n+1} - T^n}{\Delta t} + < u >^n \cdot \nabla T^{n+1} + u'^n \cdot \nabla T^n - \kappa \Delta T^{n+1} = u_1^n + g^{n+1}. \tag{14}$$

By lagging both u' and the coupling terms in the method, the fluid and thermal problems uncouple and each sub-problem has a shared coefficient matrix for all ensemble members.

Remark: The formulation (12) - (14) arises, e.g., in the study of natural convection within a unit square or cubic enclosure with a pair of differentially heated vertical walls. In particular, the temperature distribution is decomposed into $\theta(x, t) =$