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MACRO STOKES ELEMENTS ON QUADRILATERALS

MICHAEL NEILAN AND DUYGU SAP

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Dedicated to Professor William J. Layton on the occasion of his 60^{th} birthday

Abstract. We construct a pair of conforming and inf–sup stable finite element spaces for the two–dimensional Stokes problem yielding divergence–free approximations on general convex quadrilateral partitions. The velocity and pressure spaces consist of piecewise quadratic and piecewise constant polynomials, respectively. We show that the discrete velocity and a locally post–processed pressure solution are second–order convergent.

Key words. Finite element analysis, divergence–free, quadrilateral mesh.

1. Introduction

In this paper we construct a low–order, conforming, inf–sup stable, and divergence–free yielding finite element method for the Stokes problem on quadrilateral partitions of $\Omega \subset \mathbb{R}^2$. Schemes satisfying these criteria, in particular the divergence–free one, have several desirable properties, for example, a decoupling of the velocity and pressure errors (cf. (22a)), the exact enforcement of several conservation laws [14], and improved long–time stability and accuracy of time–stepping schemes [4].

In more detail we propose a finite element pair $\boldsymbol{V}_h \times W_h$ consisting of piecewise polynomials with respect to a quadrilateral partition satisfying the inf–sup condition:

$$\sup_{\boldsymbol{v}\in \boldsymbol{V}_h\setminus\{0\}} \frac{\int_\Omega (\operatorname{div} \boldsymbol{v}) q\,dx}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}} \ge \beta \|q\|_{L^2(\Omega)} \qquad \forall q \in W_h, \tag{1}$$

as well as the divergence–free property:

$$\int_\Omega (\operatorname{div} \boldsymbol{v}) q\,dx = 0 \quad \forall q \in W_h \quad \Longleftrightarrow \quad \operatorname{div} \boldsymbol{v} \equiv 0 \text{ in } L^2(\Omega). \tag{2}$$

We note that these two properties are antithetical to each other in the sense that (2) is equivalent to the inclusion $\operatorname{div} \boldsymbol{V}_h \subseteq W_h$, whereas (1) requires $W_h \subseteq \mathbb{P}_W(\operatorname{div} \boldsymbol{V}_h)$, where $\mathbb{P}_W$ denotes the L^2–projection onto W_h.

The construction of our finite element pairs is motivated by a smooth de Rham complex (or Stokes complex [15]) given by the sequence of mappings

$$0 \xrightarrow{\subset} H^2_0(\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}^1_0(\Omega) \xrightarrow{\operatorname{div}} L^2_0(\Omega) \longrightarrow 0, \tag{3}$$

where $\mathbf{curl} = (\partial/\partial x_2, -\partial/\partial x_1)^T$. If the domain is simply connected, then this complex is exact, i.e., the range of each map is the kernel of the succeeding map. The exactness property implies that the divergence operator is surjective from $\boldsymbol{H}^1_0(\Omega)$ onto $L^2_0(\Omega)$, and in addition, implies the existence of a stream function for incompressible flows. To ensure the stability of our finite element method and to construct

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divergence-free approximations, we build an exact subsequence of (3):

$$0 \xrightarrow{\subset} \Sigma_h \xrightarrow{\mathbf{curl}} \boldsymbol{V}_h \xrightarrow{\text{div}} W_h \longrightarrow 0, \tag{4}$$

where $\Sigma_h \subset H_0^2(\Omega)$, $\boldsymbol{V}_h \subset \boldsymbol{H}_0^1(\Omega)$ and $W_h \subset L_0^2(\Omega)$ are finite dimensional spaces consisting of piecewise polynomials. Note that the implied inclusion div $\boldsymbol{V}_h \subseteq W_h$ in (4) yields pointwise divergence–free approximations. In addition, if the subcomplex (4) is exact, then the mapping div : $\boldsymbol{V}_h \to W_h$ is surjective, and thus div $\boldsymbol{V}_h = W_h$. Along with a uniform bound of the right-inverse, this result implies the inf-sup condition (1). A key feature of this methodology is that the complex provides a guiding tool to develop a pair $\boldsymbol{V}_h \times W_h$ satisfying inf–sup stability and the divergence–free criterion. In particular, the H^2–conforming relative Σ_h dictates both the local and global properties of these spaces. As far as we are aware, all divergence-free yielding Stokes pairs follow this program, i.e., all finite element pairs $\boldsymbol{V}_h \times W_h$ satisfying div $\boldsymbol{V}_h = W_h$ have an H^2–conforming relative satisfying the exact sequence (4) (see, e.g., [3, 22, 15, 6, 17, 14, 20]).

A disadvantage of divergence–free yielding and conforming finite element pairs is that they tend to be high–order or require certain meshes to ensure stability and conformity. For example, on general triangular partitions, and for piecewise polynomial spaces, the minimal polynomial degree for the velocity space is four [22, 15]. For tensor product meshes, the smallest local velocity space in two dimensions is $\mathcal{Q}_{3,2} \times \mathcal{Q}_{2,3}$ [3, 6, 14, 20, 25], and the construction of these elements does not extend to general convex quadrilaterals defined by bilinear mappings. On the other hand, a *nonconforming* finite element method that imposes the divergence–free constraint pointwise on each quadrilateral element has recently been done in [26]. The method given there is low–order and is applicable to convex quadrilaterals. However, due to the nonconformity, the error estimates of this method are still coupled with a negative scaling of the viscosity.

We address some of these shortcomings by introducing a conforming finite element pair that yields divergence–free approximations, and in addition, is relatively low–order and stable on general shape–regular quadrilateral partitions. In our approach we take the H^2 finite element relative Σ_h in (4) to be the de Verbeke–Sanders macro element, a globally C^1 piecewise cubic spline [11, 10, 18, 21]. Via the subcomplex (4) we are then led to a piecewise quadratic (macro) velocity space and a piecewise constant pressure space. The global dimension of the spaces is comparable to the lowest–order Taylor–Hood pair [23], and furthermore, because the velocity error is decoupled from the pressure, the method still enjoys second–order accuracy. We also show that a locally computed post-processed pressure solution has second order accuracy. We mention that the use of macro elements on simplicial partitions has recently been done in [1, 7]. The work presented here complements and extends these results to quadrilateral meshes.

The rest of the paper is organized as follows. In Section 2 we set the notation and give some preliminary results. We define the finite element spaces and provide a unisolvent set of degrees of freedom in Section 3. In Section 4 we prove that the Stokes pair is inf–sup stable, and carry out a convergence analysis for the discrete problem. In addition we propose a local post–processed pressure solution that is second–order accurate.