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A LINEAR IMPLICIT L1-LEGENDRE GALERKIN CHEBYSHEV COLLOCATION METHOD FOR GENERALIZED TIME- AND SPACE-FRACTIONAL BURGERS EQUATION*

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Abstract

In this paper, a linear implicit L1-Legendre Galerkin Chebyshev collocation method for the generalized time- and space-fractional Burgers equation is developed. A linear implicit finite difference scheme based on the L1-algorithm for the Caputo fractional derivative is proposed for temporal discretization. And the Legendre Galerkin Chebyshev collocation method, based on the Legendre-Galerkin variational form, but the nonlinear term and the right-hand term are treated by Chebyshev-Gauss interpolation, is proposed for spatial discretization. Rigorous stability and convergence analysis are developed. Numerical examples are shown to demonstrate the accuracy, stability and effectiveness of the method.

Mathematics subject classification: 65M70, 65M12, 65M15, 26A33, 35R11.
Key words: Generalized fractional Burgers equation, Stability and convergence analysis, Legendre Galerkin Chebyshev collocation method, Finite difference method.

1. Introduction

In this paper, we consider the following generalized Burgers equation with time- and space-fractional derivatives of the form [12,27]:

$$\begin{cases} \frac{\partial^\alpha U}{\partial t^\alpha} + \varepsilon U \frac{\partial U}{\partial x} - \nu \frac{\partial^2 U}{\partial x^2} + \eta \frac{\partial^\beta U}{\partial x^\beta} = g(x,t), & (x,t) \in (-1,1) \times (0,T], \\ U(x,t) = 0, & (x,t) \in \mathbb{R}\backslash(-1,1) \times [0,T], \\ U(x,0) = U_0(x), & x \in (-1,1), \end{cases} \tag{1.1}$$

where $\varepsilon, \nu > 0$, $\eta > 0$ are parameters and $0 < \alpha, \beta < 1$ are parameters describing the order of the fractional time- and space-derivatives, respectively. The function $U(x,t)$ is assumed to be a causal function of time and space. The function $g(x,t)$ is a source term. Here, the fractional derivatives are considered in the Caputo sense, i.e.,

$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} := {}_0^C\mathcal{D}_t^\alpha U(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial U(x,s)}{\partial s} \frac{\mathrm{d}s}{(t-s)^\alpha}, \tag{1.2}$$

$$\frac{\partial^\beta U(x,t)}{\partial x^\beta} := {}_{-1}^{\,C}\mathcal{D}_x^\beta U(x,t) = \frac{1}{\Gamma(1-\beta)} \int_{-1}^x \frac{\partial U(y,t)}{\partial y} \frac{\mathrm{d}y}{(x-y)^\beta}, \tag{1.3}$$

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where $\Gamma(\cdot)$ is the Euler's Gamma function (see, e.g., [16]). (1.1) is referred to as the time-fractional Burgers and the space-fractional Burgers equation in the cases $0 < \alpha < 1, \eta = 0$ and $0 < \beta < 1, \alpha = 1$, respectively.

As suggested in [4], a well-posed fractional differential equation (FDE) with space fractional derivative operator on a bounded domain must also specify the value of the solution at points exterior to the domain, not just at the boundary. In (1.1), we adopt the so-called "absorbing boundary conditions" (see, e.g., [5]), that is we assume that the particles are "killed" whenever they leave the domain $(-1, 1)$.

The space-fractional Burgers equation describes the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe [36]. The fractional derivative results from the memory effect of the wall friction through the boundary layer [7, 15, 36]. The same form can be found in other systems such as shallow-water waves [14] and waves in bubbly liquids [26]. We refer to [10,28,31,34,36] for an incomplete list of references on the applications of the fractional Burgers equation. The approximate solution of time and/or space Burgers equations are obtained by several methods, such as Adomian decomposition method [17,27,29], homotopy analysis method [30,35], variational iteration method [12,32] and parametric spline functions method [8].

Besides, several numerical methods are used to solve the fractional Burgers equation, such as finite difference (FD) method [9,17,19,36,37,39], and spectral method [1,3,36,38]. Sugimoto [36] solved the space-fractional Burgers equation by FD method and Fourier spectral method in the numerical experiments, respectively. Fractional Burgers equation with fractional nonlinear term and diffusion term is introduced by Zayernouri et al. [38], in order to test the spectral collocation method based on the Jacobi polyfractonomials. The time-fractional Burgers equation is taken as a numerical example by Li [17] and Zhao [39], and solved by FD method. Esen et al. [9] developed a full discrete scheme for the time-fractional Burgers equation, based on FD method in time and Haar wavelet method in space. A space-time Legendre spectral collocation method is proposed to solve space-time fractional Burgers equation in [3]. No error analysis was explored in the above researches. Recently, a FD method is used to solve the time-fractional Burgers equation defined by a new generalized time fractional derivative [37], and stability analysis is provided. In [19], Li et al. proposed a linear implicit finite difference scheme for solving the time-fractional Burgers equation, and a convergence rate of $O(\tau + h^2)$ is established, where τ and h are the temporal and spatial step sizes, respectively. Asgari et al. [1] proposed two semi-implicit Fourier pseudospectral schemes for the solution of generalized time fractional Burgers type equations, with an analysis of consistency, stability, and convergence.

The aim of this study is to develop a linear implicit L1-Legendre Galerkin Chebyshev collocation method for the generalized Burgers equation with time- and space-fractional derivatives. First of all, a linear implicit L1 FD scheme [25] is proposed for temporal discretization. Thus, the advantage of the work [19] is preserved, i.e., iterative methods become dispensable when we solve the problem and the computational cost can be significantly reduced compare to the usual implicit FD schemes. Then, due to the existence of the space fractional derivative operator, local methods such as FD method and finite element method (FEM) loss a big advantage that they enjoy for usual PDEs. On the other hand, the main disadvantage of global methods such as spectral method is no longer an issue for fractional PDEs [6, 13]. Therefore, spectral method should be better suited for spatial discretization of the problem (1.1). In this paper, the Legendre Galerkin Chebyshev collocation (LGCC) method [20], based on the Legendre-Galerkin variational form, but the nonlinear term and the right-hand term are treated by