Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2016-0105

Vol. **22**, No. 1, pp. 157-181
July 2017

Implicit Asymptotic Preserving Method for Linear Transport Equations

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Received 12 July 2016; Accepted (in revised version) 5 October 2016

Abstract. The computation of the radiative transfer equation is expensive mainly due to two stiff terms: the transport term and the collision operator. The stiffness in the former comes from the fact that particles (such as photons) travel at the speed of light, while that in the latter is due to the strong scattering in the optically thick region. We study the fully implicit scheme for this equation to account for the stiffness. The main challenge in the implicit treatment is the coupling between the spacial and angular coordinates that requires the large size of the to-be-inverted matrix, which is also ill-conditioned and not necessarily symmetric. Our main idea is to utilize the spectral structure of the ill-conditioned matrix to construct a pre-conditioner, which, along with an exquisite split of the spatial and angular dependence, significantly improve the condition number and allows a matrix-free treatment. We also design a fast solver to compute this pre-conditioner explicitly in advance. Our method is shown to be efficient in both diffusive and free streaming limit, and the computational cost is comparable to the state-of-the-art method. Various examples including anisotropic scattering and two-dimensional problems are provided to validate the effectiveness of our method.

AMS subject classifications: 65M06, 65M30, 65F08, 65F35

PACS: 02.60.Dc, 02.60.Cb, 02.70.Bf

Key words: Implicit method, asymptotic preserving, pre-conditioner, diffusive regime, free streaming limit.

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1 Introduction

The linear transport equation describes the physical process of interaction of radiation with background material such as radiative transfer, neutron transport, and etc. It often contains a diffusive scaling that accounts for the strong scattering effect and leads to diffusion equations. On the contrary, when the scattering is weak, the propagation of radiation is almost a free transport with the speed of light, which is named as free streaming limit. In practice, the material usually contains both strong and weak scattering regimes, and thereby it is desirable to design a numerical method that is uniformly accurate in both cases without resolving the small scales.

Asymptotic preserving (AP) schemes arise to serve this purpose. As the name states, it preserves the asymptotic limit at the discrete level. More specifically, such method, when applied to certain equations with small parameters, should automatically become a stable solver for the corresponding limit equations without resolving the mesh size and time step. In the context of steady neutron transport, AP scheme was first studied by Larsen and Morel [21] and then Jin and Levermore [14], and Klar [17]. A rigorous convergence analysis was subsequently carried out by Golse, Jin and Levermore [11]. For time-dependent transport problem, a decomposition of the distribution function is often performed, either via a macro-micro decomposition [20] or an even-odd decomposition [15, 16]. Upon such decomposition, the stiff and non-stiff terms get separate off and an implicit-explicit scheme is applied to treat two terms respectively. The idea was later extended to a higher order implementation by Boscarino et al. [5]. Another related approach is termed the unified gas kinetic scheme framework, which was first proposed by Mieussens for linear transport equation [22] and recently modified by Sun et al. to deal with nonlinear problem [27].

Nevertheless, most AP schemes mentioned above, still suffer from a restrictive parabolic CFL condition that comes from the diffusion limit. Additionally, they are designed for diffusive scaling and thus may fail to capture the free streaming limit at which the radiation travels at the speed of light. Therefore, a fully implicit method is desired to remedy both problems. However, the main challenge that prevents researchers from directly applying the fully implicit method is the inversion of a large size of matrix due to the high dimension and the coupling between the spatial and angular coordinates. Even worse, this matrix is often ill-conditioned in some regimes and makes the inversion impractical.

Constant efforts have been made in the past few decades towards developing efficient implicit solver, both deterministic and stochastic. Here we only mention a few revolutionary works but an exhaustive bibliography is out of reach. For deterministic method, the current state-of-the-art is the Krylov iterative method for the discrete-ordinate system preconditioned by diffusion synthetic acceleration (DSA) [19, 30]. Besides the ray-effects [23] that generates by the discrete ordinate method, this preconditioned Krylov iteration has a drawback, mainly due to the complication of the method. Since each time iteration includes a sub-iterations of sweeps and diffusion solvers for DSA pre-