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Coefficient Estimates for a Class of m-fold Symmetric Bi-univalent Function Defined by Subordination

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Abstract: In this paper, we investigate the coefficient estimates of a class of m-fold bi-univalent function defined by subordination. The results presented in this paper improve or generalize the recent works of other authors.

Key words: analytic function, univalent function, coefficient estimate, m-fold symmetric bi-univalent function, subordination

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z\colon |z| < 1\}$. We denote by $\mathcal{S}$ the class of all functions $f(z) \in \mathcal{A}$ which are univalent in U.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

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$$f(f^{-1}(\omega)) = \omega \qquad \left(|\omega| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

The inverse functions $g = f^{-1}$ is given by

$$f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of all bi-univalent functions in unit disk U.

For each functions $f \in \mathcal{S}$, the function

$$h(z) = \sqrt{m} f(z^m) \qquad (z \in U,\ m \in \mathbf{N}^+)$$

is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [1] and [2]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in U,\ m \in \mathbf{N}^+). \tag{1.3}$$

Analogous to the concept of m-fold symmetric univalent functions, here we introduced the concept of m-fold symmetric bi-univalent functions. For the normalized form of f given by (1.3), Srivastava *et al.*[3] obtained the series expansion for f^{-1} as follows:

$$\begin{aligned} g(\omega) &= f^{-1}(\omega) \\ &= \omega - a_{m+1}\omega^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]\omega^{2m+1} \\ &\quad - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right]\omega^{3m+1} + \cdots . \end{aligned} \tag{1.4}$$

We denote by Σ_m the class of m-fold symmetric bi-univalent function in U. For $m = 1$, the formula (1.4) coincides with the formula (1.2) of the class Σ. Some m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1 - z^m}\right)^{\frac{1}{m}}, \qquad [-\log(1 - z^m)]^{\frac{1}{m}}, \qquad \left[\frac{1}{2}\log\left(\frac{1 + z^m}{1 - z^m}\right)\right]^{\frac{1}{m}}.$$

The class of bi-univalent functions was first introduced and studied by Lewin[4] and was showed that $|a_2| < 1.51$. Brannan and Clunie[5] improved Lewin's results to $|a_2| \leq \sqrt{2}$ and later Netanyahu[6] proved that $\max\{|a_2|\} = \dfrac{4}{3}$ if $f(z) \in \Sigma$. Recently, many authors investigated the estimates of the coefficients $|a_2|$ and $|a_3|$ for various subclasses of bi-univalent functions (see [7]–[9]). Not much is known about the bounds on general coefficient $|a_n|$ for $n \geq 4$. In the literature, only few works determine general coefficient bounds $|a_n|$ for the analytic bi-univalent functions (see [10]–[14]).

In this paper, let $\mathcal{P}$ denote the class of analytic functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots ,$$

and then

$$\mathrm{Re}\{p(z)\} > 0 \qquad (z \in U).$$

By [2], the m-fold symmetric function p in the class $\mathcal{P}$ is given of the form:

$$p(z) = 1 + p_m z + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots .$$