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Planar Cardinal Spline Curves with Minimum Strain Energy

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Abstract: In this paper, we focus on how to use strain energy minimization to obtain the optimal value of the free parameter of the planar Cardinal spline curves. The unique solution can be easily obtained by minimizing an appropriate approximation of the strain energy. An example is presented to illustrate the effectiveness of our method.

Key words: Cardinal spline, Catmull-Rom spline, minimization, strain energy, fair curve

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1 Introduction

In computer aided design (CAD) and related application fields, the construction of fair curves is an important issues (see [1]–[3]). Although the fairness of a curve is difficult to be expressed in a quantitative way, the general ways to construct the fair curves are achieved by minimizing some energy functions (see [4]). The strain energy (also called bending energy) and curvature variation energy are two widely adopted metrics to describe the fairness of a curve (see [1] and [5]). For example, a lot of works on planar G^1 or G^2 Hermite interpolation via strain energy minimization or curvature variation minimization have been proposed, see e.g. [4] and [6]–[9].

As we know, the Cardinal spline (see [10]) contains a free parameter. We can obtain different shapes of interpolating curves by altering the value of the free parameter even if the points are remained unchanged. Therefore, a natural idea drives us to find the optimal

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value of the free parameter by minimizing the strain energy or curvature variation energy to construct the fair Cardinal spline curves. In this paper, we apply the strain energy minimization to achieve this goal. We observe that the unique solution can be easily obtained by minimizing an appropriate approximation of the strain energy.

The rest of this paper is organized as follows. In Section 2, the main problem is presented. In Section 3, the strain energy minimization method is described. In Section 4, an example is outlined to show the effectiveness of the method. A short conclusion is given in Section 5.

2 Cardinal Spline

The Cardinal spline (see [10]) is joined by a sequence of individual curves to form a whole curve that is specified by an array of points and a free parameter. The planar Cardinal spline curves can be expressed by

$$\boldsymbol{b}_i(t) = f_0(t)\boldsymbol{p}_i + f_1(t)\boldsymbol{p}_{i+1} + f_2(t)\boldsymbol{p}_{i+2} + f^3(t)\boldsymbol{p}_{i+3}, \qquad i = 0, 1, \cdots, n-3, \tag{2.1}$$

where $0 \le t \le 1$, $\boldsymbol{p}_k \in \mathbf{R}^2$ $(k = 0, 1, \cdots, n;\ n \ge 3)$ are given points, and

$$\begin{cases} f_0(t) = -\alpha t^3 + 2\alpha t^2 - \alpha t, \\ f_1(t) = (2-\alpha)\, t^3 + (\alpha - 3)\, t^2 + 1, \\ f_2(t) = (\alpha - 2)\, t^3 + (3 - 2\alpha)\, t^2 + \alpha t, \\ f_3(t) = \alpha t^3 - \alpha t^2, \end{cases} \tag{2.2}$$

with α is a free real parameter.

By a simple deduction from (2.1) and (2.2), we have

$$\boldsymbol{b}_i(0) = \boldsymbol{p}_{i+1}, \qquad \boldsymbol{b}_i(1) = \boldsymbol{p}_{i+2}. \tag{2.3}$$

From (2.3) we know that each individual Cardinal spline curve interpolates the second and the third points, which means the whole curve passes all the given points except the first point $\boldsymbol{p}_0$ and the end point $\boldsymbol{p}_n$. Because the curves are usually required to pass all the given points, it is necessary to add an auxiliary point $\boldsymbol{p}_{-1}$ to the front of $\boldsymbol{p}_0$ and an auxiliary point $\boldsymbol{p}_{n+1}$ to the back of $\boldsymbol{p}_n$, which ensure that the whole curve can pass the first and the end points. Generally, we can repeat the first point and the end point as the two auxiliary points, *viz.* the two auxiliary points are taken as $\boldsymbol{p}_{-1} = \boldsymbol{p}_0$, $\boldsymbol{p}_{n+1} = \boldsymbol{p}_n$.

By the expression of the Cardinal spline curves, we can find that the free parameter α has a clear influence on the shape of the whole curve when all the points are remained unchanged. For example, suppose that the given points and the two auxiliary points are

$$\boldsymbol{p}_{-1} = (35, 47),\ \boldsymbol{p}_0 = (35, 47),\ \boldsymbol{p}_1 = (16, 40),\ \boldsymbol{p}_2 = (15, 15),\ \boldsymbol{p}_3 = (25, 36),\ \boldsymbol{p}_4 = (40, 15),$$
$$\boldsymbol{p}_5 = (65, 25),\quad \boldsymbol{p}_6 = (50, 40),\ \boldsymbol{p}_7 = (60, 42),\ \boldsymbol{p}_8 = (80, 37),\ \boldsymbol{p}_9 = (80, 37).$$

Fig. 2.1 shows the influence of the free parameter α on the shape of the whole curve, where the parameter of the dotted lines is taken as $\alpha = 0.3$, the parameter of the solid lines is taken as $\alpha = 0.8$.

Since the free parameter α has an obvious influence on the shape of the Cardinal spline curves, a natural idea derives us to find the optimal value of the free parameter to obtain the